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# THE MATHEMATICS OF FINANCE

BY

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# EDITOR'S INTRODUCTION

COURSES in the Mathematics of Finance (or Investment) have of recent years become a recognized unit of collegiate instruction in mathematics. A considerable number of well-organized textbooks designed for such courses are already on the market. There is a very definite feeling among the teachers of this subject, however, that all hitherto existing texts suffer from a serious fault: They are all burdened with a mass of formulas and rules which not only impose a severe strain on the memory of the student, but (and this is pedagogically far worse) they lead almost inevitably to mere mechanical substitution and to a lack of understanding of the fundamental principles and a consequent absence of any real power of analysis. Some teachers have perhaps felt that such a mass of formulas is inevitable, that it is inherent in the enormous variety of applications of simple and compound interest and discount.

The present text proves, however, that such is not the case. One of its most gratifying features lies in the fact that the authors have succeeded in showing how all the varied standard types of problems that arise in the mathematics of finance can be solved by the application of a small number of fundamental formulas and their simple transformations. This is a gain that will, I feel confident, profoundly affect the future teaching of this subject. It places the emphasis of the instruction where it should be, on a thorough understanding of principles. The latter once mastered, the rest is comparatively easy; without such mastery, however, any apparent success attained in the teaching of this subject is sure to be to a large extent illusory.

The feature above referred to will recommend the present text to every conscientious teacher who has the real welfare of his students at heart. He will find, on examining the text further, that the authors have succeeded in so organizing their material that, after mastering certain fundamental principles in the first chapter, the

student is led gradually from simple applications to the more complete — another obviously desirable feature.

Finally, he will find a wealth of carefully graded problems that are real, taken from actual business transactions.

J. W. YOUNG

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# AUTHORS' PREFACE

The outstanding features of this book are the following:

(*a*) General formulas are derived for the values of single sums of money, annuities certain, life annuities, and life insurances.

(*b*) Line diagrams are extensively used as an aid to the analysis of problems.

(*c*) Solutions of many typical examples are given in the text.

(*d*) The best method of computation is shown in the solution of examples.

(*e*) The exercises are chosen to illustrate practical problems of business.

(*f*) An excellent collection of tables is given, and

(*g*) The articles are so arranged as to make lesson assignments convenient, and an abridged course easily planned.

The authors have found, from their experience in teaching the elements of the mathematics of finance, that students learn to derive and to use general formulas for the values of annuities certain, life annuities, and life insurances as easily as they learn to derive and to use special formulas. The verbal statements which are given for most of these formulas are helpful to some students both in memorizing and in using them. Since the number of these general formulas is small, students are able to solve all problems in finance by the use of only a few principles. A good illustration of the advantage of the general formula is afforded by formula (1), Chapter IV, for the value of any life annuity based on a single life. By means of this formula, the values of ordinary, deferred, and forborne life annuities immediate and life annuities due can be written at once. The use of a small number of general principles permits also a greater amount of attention to the practical applications than is usually given in an elementary text.

The authors wish to express their appreciation of the kindness of Prof. C. H. Forsyth of Dartmouth College, who read the book in manuscript and made helpful suggestions and criticisms. They

are also indebted to Mr. J. C. Rietz, actuary of the Midland Mutual Life Insurance Company, and to Mr. H. C. Fetsch, actuary of the Ohio State Life Insurance Company, for their assistance in the preparation of the manuscript, Chapters IV and V. Prof. J. W. Glover of the University of Michigan, and his publisher, George Wahr, have kindly permitted the authors to include portions of his *Tables of Applied Mathematics in Finance, Insurance and Statistics*, a book which is invaluable to a student of these subjects.

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# MATHEMATICS OF FINANCE

## CHAPTER I

### INTEREST AND DISCOUNT

**1. The problem of finding the value of a sum of money.** If a principal of \$100 is invested for one year at 6% simple interest, the amount of the investment at the end of the year is \$106. If \$106 is the amount of an investment for one year at 6% simple interest, the value of the investment at the beginning of the year is \$100. These simple examples from arithmetic illustrate a problem whose solution is of primary importance in the mathematics of finance. If the value of a sum of money is known at a given time, this problem is to find its value at any other time. If $n$ denotes the number of years, called the *term in years*, between the given time and the time at which the value is to be found, and if $S$ denotes the larger and $P$ the smaller of the values at the two times, then the problem may be stated in the form: to find $S$ when $P$ and $n$ are given and to find $P$ when $S$ and $n$ are given. $S$ is called the *amount* or the *accumulated value* of $P$ for $n$ years and $P$ is called the *present value* or the *discounted value* of $S$ for $n$ years. The operation by which $S$ is found is called *accumulating;* that by which $P$ is found is called *discounting*. The accumulating operation increases the value of a sum of money, the discounting operation decreases its value. These operations are based on the fundamental assumption that money is constantly productive.

The value found by accumulating or discounting a given sum of money depends on the rate as well as on the term $n$. The rate may be either an interest rate or a discount rate. In this chapter four methods of accumulating and four methods of discounting are presented. These methods are based on the principles of simple

interest, simple discount, compound interest, and compound discount. The primary importance of these operations lies in the fact that by means of them the equations needed to find the unknowns in various types of problems in the elements of finance can be readily determined. A thorough mastery of them is essential. Before presenting them it is necessary to define interest and discount, and interest and discount rates. This is done in Art. 2.

**2. Interest and discount; interest and discount rates.** In the numerical examples in Art. 1, $P$ is \$100, $S$ is \$106, $n$ is 1, and $S - P$ is \$6; this difference, \$6, is called the interest on \$100 for one year and also the discount on \$106 for one year. In general $S - P$ is called the *interest on* $P$ for $n$ years and the *discount on* $S$ for $n$ years. When $S - P$ represents the interest on $P$, it will be denoted by $I$; when it represents the discount on $S$, it will be denoted by $D$. That is,

$$\text{Interest on } P = I = S - P = D = \text{discount on } S.$$

Interest and discount rates are defined with respect to some definite period of time. In practice the period chosen is a year or a part of a year. The interest rate per period is the number by which any sum must be multiplied to find the interest on it for the period; the discount rate per period is the number by which any sum must be multiplied to find the discount on it for the period. If \$1 is taken for the sum, these definitions become:

*The interest rate per period equals in numerical value the interest on* \$1 *for the period; the discount rate per period equals in numerical value the discount on* \$1 *for the period.*

In the above examples the interest rate per year is $\frac{6}{100}$ and the discount rate per year is $\frac{6}{106}$.

An interest or a discount rate may be expressed as a per cent, as a decimal fraction, or as a common fraction. For example, $6\% = .06 = \frac{6}{100}$. In computations the decimal fraction form is usually the more convenient; in some cases, however, the common fraction form may be used to advantage. In verbal statements the per cent form is generally employed.

### EXERCISES

1. The interest on \$100 for one year is \$4.25. Find the interest rate as a per cent, as a decimal fraction, and as a common fraction.

Ans. $4\frac{1}{4}\%$; .0425; $\frac{17}{400}$.

2. The discount on \$130 for one year is \$5.20. Find the discount rate as a per cent, as a decimal fraction, as a common fraction. Ans. 4%; .04; $\frac{4}{100}$.

3. If \$325 at simple interest amounts in one year to \$347.75, express the interest rate in each of the three ways. Ans. 7%; .07; $\frac{7}{100}$.

4. If $P = \$125$, $S = \$132.50$, and $n = 1$, find the interest and discount rates as common fractions. Ans. $\frac{3}{80}$; $\frac{3}{83}$.

5. The present value of \$125, due in one year, is \$117.98. Find the interest and discount rates each to four decimals.[1] Ans. .0595; .0562.

**3. Simple interest formulas.** In defining a simple interest rate, one year is the period chosen. If $i$ denotes the simple interest rate per year, then, by Art. 2, $Pi$ is the simple interest on $P$ for one year. In general, if $I$ represents the simple interest on $P$ for $n$ years, the value of $I$ is defined by

$$I = Pni \tag{1}$$

This equation shows that $I$ varies as $P$, $n$, and $i$.

By Art. 2, $S = P + I$. Replacing $I$ by $Pni$ gives

$$S = P(1 + ni) \tag{2}$$

or, solving for $P$,

$$P = \frac{S}{(1 + ni)} \tag{2'}$$

Formulas (2) and (2′) may be stated verbally in the form: *To find the amount of any sum for n years at the simple interest rate i, multiply this sum by* $1 + ni$; *to find the present value of any sum due in n years at the simple interest rate i, divide this sum by* $1 + ni$.

### EXERCISES

1. Find the simple interest on \$125.65 at $4\frac{1}{4}\%$ for 1 year, for $\frac{1}{2}$ year, for $\frac{1}{4}$ year. What is the amount in each case?

2. Find the simple interest on \$250 at $5\frac{1}{2}\%$ for 9 months. What is the amount? Ans. \$10.31; \$260.31.

[1] The phrase "to four decimals" should be interpreted to mean to the nearest digit in the fourth decimal place.

**3.** If \$325.50 amounted to \$345.03 in 1 year at simple interest, find the interest rate used. Ans. 6%.

**4.** How long will it take \$100 to accumulate to \$125 at 5% simple interest? How long to accumulate to \$200? Ans. 5 years; 20 years.

**5.** What principal invested at $5\frac{3}{4}\%$ will amount to \$1235.50 in 1 year and 9 months? Ans. \$1122.54.

**6.** Find the present value of a non-interest-bearing note for \$1100 due in 9 months, if a simple interest rate of 6% is used. Ans. \$1052.63.

**7.** Same as Exercise 6 except that the note bears simple interest at 5%. Ans. \$1092.11.

**8.** How long will it take any sum $P$ to accumulate to $2P$ at a simple interest rate $i$?

**4. Simple discount formulas.** In defining a simple discount rate, one year is the period usually chosen. If $d$ denotes the simple discount rate per year, then, by Art. 2, $Sd$ is the simple discount on $S$ for one year. In general, if $D$ represents the simple discount on $S$ for $n$ years, the value of $D$ is defined by

$$D = Snd \tag{3}$$

This equation shows that $D$ varies as $S$, $n$, and $d$.

By Art. 2, $P = S - D$. Replacing $D$ by $Snd$ gives

$$P = S(1 - nd) \tag{4}$$

or, solving for $S$,

$$S = \frac{P}{(1 - nd)} \tag{4'}$$

Formulas (4) and (4') may be stated verbally in the form: *To find the present value of any sum due in n years at the simple discount rate d, multiply this sum by* $(1 - nd)$; *to find the amount of any sum for n years at the simple discount rate d, divide this sum by* $(1 - nd)$.

### EXERCISES

**1.** Find the simple discount on \$150.60 at $4\frac{1}{4}\%$ for 1 year, for $\frac{1}{2}$ year, for $\frac{1}{4}$ year. What is the present value in each case?

**2.** Find the simple discount on \$250 at $5\frac{1}{2}\%$ for 9 months. What is the present value? Ans. \$10.31; \$239.69.

**3.** If the present value of \$125 due in 1 year is \$118.75, find the discount rate used. Ans. 5%.

**4.** The present value of \$200 is \$192.50. If the rate of discount is 5%, find the time in months for which the discount is calculated. Ans. 9 months.

**5.** What sum discounted at $5\frac{3}{4}\%$ for 1 year and 9 months has a present value of \$1235.50? Ans. \$1373.73.

**6.** If a wholesaler gives a discount of 15% and 10% off the list price, find the equivalent simple discount rate. Ans. 23.5%.

**7.** Same as Exercise 6, except that an additional discount of 2% for cash is allowed.

**8.** How long will it take any sum $S$ to become $\frac{S}{2}$ at a simple discount rate $d$? How long to become zero?

**5. Computation of simple interest and simple discount, exact and ordinary.** In formulas (1) and (3) for computing simple interest and simple discount, $n$ represents the number of years in the term. In business transactions $n$ is usually equal to or less than unity. When the number of months in the term is given, $n$ is found by dividing this number by 12. When the number of days in the term is given, $n$ is found by dividing this number by the number of days in a year. In some transactions a year is regarded as made up of twelve months of thirty days each or 360 days; in others, of twelve calendar months or 365 days. These two ways of regarding a year lead to two values for $n$ for any definite number of days $b$, namely, $\frac{b}{360}$ and $\frac{b}{365}$. When $n$ in formulas (1) and (3) is replaced by $\frac{b}{365}$, they become

$$I = \frac{Pbi}{365}, \qquad D = \frac{Sbd}{365}$$

and the values of $I$ and $D$ so determined are called *exact simple interest* and *exact simple discount* for a term of $b$ days. When $n$ is replaced by $\frac{b}{360}$ the formulas become

$$I = \frac{Pbi}{360}, \qquad D = \frac{Sbd}{360}$$

and the values of $I$ and $D$ so determined are called *ordinary simple interest* and *ordinary simple discount* for a term of $b$ days.

Excellent tables giving the ordinary and exact interest and discount when $P = S = 1$ have been constructed for values of $b$ ranging from 1 to 360 or 365 and for various values of $i$ and $d$. Such tables greatly facilitate the computation of simple interest and simple discount.

There is variation in practice, even among banks, as to when the exact or the ordinary formulas are used in determining interest or discount.

In short-term loans made by banks ordinary simple discount is frequently used. In such cases the borrower executes a note for a stated amount which does not bear interest and the bank immediately discounts this note at a quoted rate. For example, if the rate quoted is 7% and the note states that the borrower is to pay \$100 to the bank at the end of one year, the bank gives the borrower $100 - 7 = \$93$. The amount deducted from the face of the note in such cases, here \$7, is sometimes spoken of as interest in advance; it is in reality, however, the discount on the face of the note.

Exact simple discount is used by the Federal Reserve banks in rediscounting notes presented by those banks which are members of the Federal Reserve System.

Exact simple interest is customarily used by banks which pay interest on daily balances.

Ordinary simple interest is used by banks on money loaned by them on demand notes.

When the term during which interest or discount is to be computed is stated in days, $b$ has a unique value in the above formulas. For this reason banks are more and more adopting the practice of stating the term in days. When the term is the interval between two given dates, the value of $b$ depends usually on whether calendar months or months of thirty days are used in its determination. When calendar months are used, $b$ is the exact number of days between the two dates. This exact number may be found by adding the number of days in each month between the given dates or by use of Table II, which gives the number of each day of the year. When thirty-day months are used, the number of days can be found by the method presented in arithmetic. By Table II the exact

number of days from November 22, 1923, to February 10, 1924, is $39 + 41 = 80$ days. When thirty-day months are used, the number of days between these two dates is found as follows:

| Year | Month | Day | |
|---|---|---|---|
| 1924 | 2 | 10 | |
| 1923 | 11 | 22 | |
| | 2 | 18 | $= 60 + 18 = 78$ days. |

In general if $b_1$ denotes the number of days in any term when calendar months are used and $b_2$ the number when thirty-day months are used, then $b_1$ may be greater than, equal to, or less than $b_2$. There are four possible values for $n$ when $b_1$ and $b_2$ are different, namely, $\frac{b_1}{365}, \frac{b_2}{360}, \frac{b_1}{360}, \frac{b_2}{365}$. In the first of these, calendar months are used in determining both the numerator and the denominator; in the second, thirty-day months are used in determining both. The last two do not possess this uniformity. Financial institutions ordinarily use $\frac{b_1}{360}$ when calculating interest on accounts receivable and $\frac{b_1}{365}$ on accounts payable. In transactions not involving financial institutions, such as those between individuals, $\frac{b_2}{360}$ is frequently used.

### EXERCISES*

**1.** Find the number of days from November 22, 1923, to March 10, 1924, by calendar months and by thirty-day months. Ans. 109; 108.

**2.** A note dated February 5, 1923 is due in 90 days. Find its date of maturity. If it is due in 3 months, find its date of maturity.

Ans. May 6; May 5.

**3.** A note dated March 30 is paid September 5. Find the time in days for which the interest on this note is calculated, first, if 30-day months are used, and second, if the exact number of days are used. Ans. 155; 159.

**4.** In some states a note which falls due on Sunday or a holiday is legally payable on the next business day, the interest or discount being computed to the day of payment. In one of these states a note due in 30 days is given to

* In these exercises use the ordinary simple interest and discount formulas and the exact number of days between dates unless otherwise specified.

a bank on August 4. If September 3 falls on Sunday, for how many days will the discount be calculated; for how many days if September 4 is Labor Day?
Ans. 31; 32.

**5.** A note, dated January 3, 1920, fell due in 90 days. What was the date of maturity? It was discounted by a bank on February 1. For how many days was the discount calculated? Ans. April 2; 61.

**6.** A note was given to a bank on March 30 for 90 days. On April 5 it was rediscounted by a Federal Reserve bank. Find the number of days for which each discount was calculated. When was the note due?
Ans. 90; 84; June 28.

**7.** Find the ordinary simple interest on a note, dated January 12, 1923, for $1250.75 at 6% for 108 days. When was it due? Ans. $22.51; April 30, 1923.

**8.** Find the numbers to fill the blanks, using the ordinary simple interest formula:

| $P$ | $i$ | Term | $S$ |
|---|---|---|---|
| 1250 | | 65 days | 1265 |
| 1250 | .0425 | | 1350 |
| 1000 | .045 | 25 days | |
| | .04375 | 60 days | 1250 |
| 750.50 | | 195 days | 785.25 |
| 65.35 | .05 | | 71.55 |
| 75.25 | .05 | 151 days | |
| | .035 | 12 days | 65.15 |

**9.** On August 3, 1924, A gave B a note for $500. It was paid March 2, 1925. Find the ordinary simple interest due if 30-day months and a 6% interest rate are used. Ans. $17.42.

**10.** A note for $100 due in 30 days bore interest at 8% from date of maturity. The note was paid 58 days after its date of issue. Find the ordinary simple interest due. Ans. $.62.

**11.** A note for $100 due in 30 days is given to a bank which at once discounts it at 7%. How much does the maker of the note receive for it?
Ans. $99.42.

**12.** A note dated January 31, 1923, and due in 30 days is discounted by a bank on February 15. If the face of the note is $150 and it bears 6% interest from date, how much does the holder of the note receive for it if the bank's rate of discount is 7%? Ans. $150.31.

**13.** A non-interest-bearing note for \$100 was sold for \$98.95, 50 days before it was due. Find the rate of discount correct to four decimals. Ans. .0756.

**14.** A note for \$1275 was dated July 3. It bore interest at 6% from date and was due in 90 days. It was discounted August 31. If the holder of the note received \$1282.35, find the rate of discount correct to four decimals. Ans. .1057.

**15.** Find the face of a note which was due in 30 days, bore 6% interest, and which when discounted by a bank on the day it was made at 7% yielded the holder \$126.76. Ans. \$126.87.

**16.** The holder of a non-interest-bearing note dated January 15, 1921, and due in 6 months discounted it at a bank on March 1 at 7%. The bank's discount on the note was \$85.36. What was the face of the note? Ans. \$3227.90.

**17.** Solve Exercise 16 on the hypothesis that the note bore 6% simple interest from its date of issue. Ans. \$3133.88.

**18.** A note for \$125 due in 50 days bore simple interest at 7% from date of issue. It was discounted 35 days before its date of maturity. Find the bank's rate of discount, correct to four decimals, if the holder of the note received \$125.30. Ans. .0750.

**19.** A note for \$450, which bore simple interest from date and which was due in 30 days, was discounted 20 days before maturity at $7\frac{1}{2}$%. If the proceeds of the note was \$450.37, find to four decimals the note's rate of interest. Ans. .0600.

**20.** A certain bank pays 2% exact simple interest on daily balances of even hundreds when they amount to \$1000 or more. On a day that a man's balance was \$1250.67 find his interest credit.

**21.** A bank book showed the following balances for the first six days of February:

| | |
|---|---|
| Feb. 1, \$ 875.35 | Feb. 4, \$2635.28 |
| Feb. 2, 1872.38 | Feb. 5, 3625.24 |
| Feb. 3, 1555.80 | Feb. 6, 2732.86 |

If the bank pays $1\frac{1}{2}$% exact simple interest on daily balances on even hundreds when they are \$500 or more, find the interest credit for each day.

**22.** Find the numbers to fill the blanks:

| $P$ | $i$ | Term | Ordinary Simple Interest | Exact Simple Interest |
|---|---|---|---|---|
| 1256.35 | .05 | 100 days | | |
| 675.27 | .055 | 60 days | | |
| 2367.85 | .0475 | 108 days | | |

**23.** A Federal Reserve bank bought a \$1000 non-interest-bearing note 20 days before it was due. Find the purchase price of the note if exact simple discount at $4\frac{1}{4}\%$ was used. Ans. \$997.67.

**24.** A note for \$1000 due in 60 days from date of issue with interest at 5% was bought by a Federal Reserve bank 25 days before its maturity. Find the purchase price of the note if exact simple discount at $4\frac{1}{4}\%$ was used.

Ans. \$1005.39.

**25.** A non-interest-bearing note for \$12,000 due in 30 days was purchased by a bank on its day of issue. It was rediscounted by a Federal Reserve bank 20 days before it was due. Find each purchase price if ordinary simple discount at 7% was used in the first transaction and exact simple discount at $4\frac{1}{4}\%$ in the second.

**26.** A note dated January 10, 1922, for \$750 bore 7% interest from date and was due in 90 days. It was discounted by a bank at $7\frac{1}{2}\%$ on February 1 and by a Federal Reserve bank on February 15 at $5\frac{1}{2}\%$. How much did the bank pay for the note and how much did it receive from the Reserve bank, if ordinary simple discount was used in the first transaction and exact simple discount in the second? Ans. \$752.32; \$756.83.

**27.** A note for \$175, due in 30 days, bore 5% interest. It was discounted the day it was made at 7% by a bank which rediscounted it the same day at $5\frac{1}{4}\%$ with a Federal Reserve bank. Find the ordinary simple discount involved in the first transaction and the exact simple discount in the second.

Ans. \$1.03; \$.76.

**28.** A note for \$1000 bearing interest at 7% fell due in 90 days. It was discounted by a bank at $5\frac{1}{4}\%$ on the day it was made, and rediscounted on the same day by a Federal Reserve bank at the same rate. How much did the bank gain by the transaction if it uses ordinary simple discount and the Federal Reserve bank uses exact simple discount? Ans. \$.19.

**29.** If $I_o$ represents the ordinary simple interest on $P$ at rate $i$ for a term $n$ and $I_e$ represents the exact simple interest for the same $P$, $i$, and $n$, show that

$$I_o = \tfrac{73}{72} I_e$$
$$I_e = \tfrac{72}{73} I_o$$

**30.** Use the formulas in Exercise 29 to check the results in Exercise 22.

**31.** Derive formulas analogous to those in Exercise 29 for ordinary simple discount and exact simple discount.

**6. Problems based on the simple interest and the simple discount formulas.** The simple interest formulas

$$I = Pni \tag{1}$$

$$S = P(1 + ni) \tag{2}$$

contain five letters, and when the values of three are known, the values of the other two can be found by solving equations (1) and (2) simultaneously for them. The problem of finding $I$ and $S$ when $P$, $n$, and $i$ are given is the one of most frequent occurrence among problems of this type. When these formulas are used in business transactions, the term $n$ is usually one year or less.

The simple discount formulas

$$D = Snd \tag{3}$$

$$P = S(1 - nd) \tag{4}$$

lead to the solution of analogous problems in discount. Here also the term is usually one year or less in business transactions. An important exception is found in the straight line or simple discount method of computing depreciation (Art. 62, Chapter III).

Additional problems can be solved by using any two or more of the equations (1), (2), (3), and (4). An important one of this type relates to corresponding rates. *The simple interest rate $i$ and the simple discount rate $d$ are said to be corresponding for a given value of $n$ when each leads to the same accumulated value, $S$, of $P$ in $n$ years.* Equating the values of $S$ given by formulas (2) and (4), and dividing by $P$, gives the following equation connecting the corresponding rates $i$ and $d$:

$$1 + ni = \frac{1}{1 - nd} \tag{5}$$

or, solving for $d$,

$$d = \frac{i}{1 + ni} \tag{5'}$$

or, solving for $i$,

$$i = \frac{d}{1 - nd} \tag{5''}$$

Equations (5′) and (5″) show that, when $i$ and $d$ are corresponding rates, $d$ is the present value of $i$ for $n$ years at the simple interest rate $i$, and $i$ is the amount of $d$ for $n$ years at the simple discount rate $d$. In particular, when $n = 1$, $d = \frac{6}{106}$ corresponds to $i = \frac{6}{100}$ and $i = \frac{6}{94}$ corresponds to $d = \frac{6}{100}$. This is as would be expected, as the borrower of a dollar may pay for the use of it $d$ dollars at the beginning of the term or $i$ dollars at the end of the term. Clearly, then, $d$ is the discounted value of $i$ and $i$ is the accumulated value of $d$.

## EXERCISES

**1.** Find the present value of \$500 due in 9 months without interest if the discount rate is 5%; if the interest rate is 5%; if the discount rate is $5\frac{1}{2}\%$; if the interest rate is $5\frac{1}{2}\%$.

**2.** Find the accumulated value of \$125 for 1 year and 3 months, if the interest rate is 6%; find the accumulated value, if the discount rate is 6%.

**3.** Show that if $i$ and $d$ are corresponding and $n = 1$,

$$(a)\quad i = \frac{d}{1-d} = d + d^2 + d^3 + \cdots$$

$$(b)\quad d = \frac{i}{1+i} = i - i^2 + i^3 - \cdots$$

**4.** If $d$ is $4\frac{1}{2}\%$ and $P = \$100$, find $S$ if $n = 1$; find the corresponding $i$.

**5.** Find the numbers to fill the blanks, $i$ and $d$ being corresponding rates:

| $d$ | $i$ | $n$ (years) |
|---|---|---|
| .04 | | 1 |
| .04 | | $\frac{1}{2}$ |
| | .06 | $\frac{1}{2}$ |
| | .07 | 1 |
| .05 | | 1 |

**6.** Find the numbers to fill the blanks, $i$ and $d$ being corresponding rates:

| $P$ | $i$ | $S$ | $d$ | $n$ (years) |
|---|---|---|---|---|
| 100 | | 103 | | $\frac{1}{2}$ |
| | .035 | 112 | | 1 |
| 125 | .045 | | | $\frac{1}{4}$ |
| | | 118 | .035 | $\frac{3}{4}$ |
| 135 | .06 | 140 | | |
| | .05 | 125 | .0475 | |
| 110 | .05 | | .0475 | |
| 120 | | 126 | .045 | |
| 175 | | | .055 | 1 |

**7.** If a bank discounts a 30-day note at 7%, find the corresponding simple interest rate earned by the bank. Use a year of 360 days.

**8.** Same as Exercise 7 except that the note is a 60-day note; a 90-day note.

**9.** The interest on \$56.75 for one year is \$4.25; find the corresponding interest and discount rates correct to four decimals. Ans. .0697; .0749.

**10.** The discount on \$56.75 for 6 months is \$2.10; find the corresponding simple interest and discount rates correct to four decimals.

Ans. .0769; .0740.

**11.** A mortgage note for \$1000 due in 1 year is executed in favor of a second mortgage loan company. The company discounts the face of the note at 7% and in addition charges a commission of 10% on the face of the note for making the loan. Find the simple interest rate earned by the company.

**12.** Solve $S = P(1 + ni)$ for each of the four letters.

**13.** Solve $P = S(1 - nd)$ for each of the four letters.

**14.** Show that the following relations hold where $i$ and $d$ are corresponding rates for a term of $n$ years:

$$(a)\quad Pi = Sd$$

$$(b)\quad S = \frac{P}{1 - nd} = P(1 + ni)$$

$$(c)\quad P = \frac{S}{1 + ni} = S(1 - nd)$$

State verbally the properties of corresponding rates expressed by relations ($a$) and ($c$).

**15.** The accumulated value of $P$ for $n$ years may be found by using the interest rate or the discount rate; the present value of $S$, by using either the interest rate or the discount rate. Verify these statements by the use of ($b$) and ($c$) of Exercise 14.

**7. Graphs of the simple interest and the simple discount formulas.** When $P$ and $i$ are given, equations (1) and (2) are of the first degree and hence have straight lines for their graphs. Similarly, when $S$ and $d$ are given, equations (3) and (4) have straight-line graphs. Figure 1 shows the graphs of $S = P(1 + ni)$ for $P = 1$, $i = .04, .05$, and $.06$, and of $P = S(1 - nd)$ for $S = 1$, $d = .04, .05$, and $.06$ with respect to the axes $ON$ and $OA$. The values of $I$ and $D$ are represented by the distances from $O'N'$. For example, $MP_1$ represents the amount and $M_1P_1$ the simple interest on \$1 for 10 years at 4% simple interest, and $MP_2$ represents the present value and $P_2M_1$ the simple discount on \$1 for 10 years at 4% simple discount.

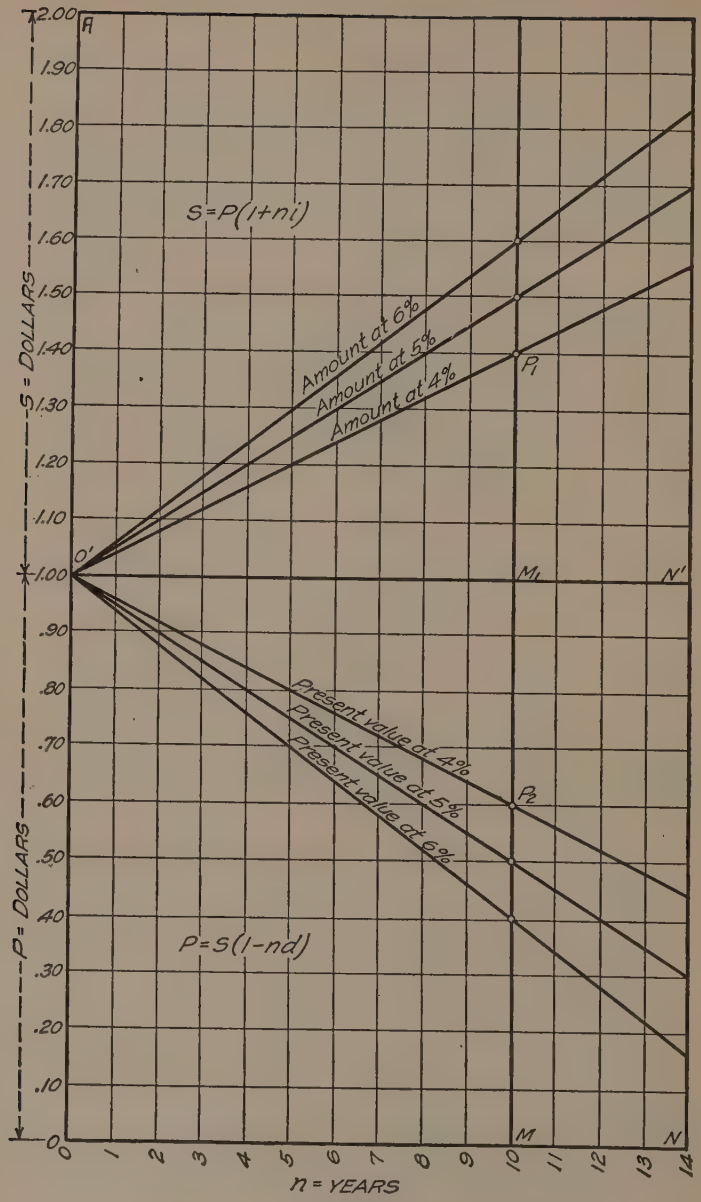

A
2.00
1.90
1.80
1.70
1.60
1.50
1.40
1.30
1.20
1.10
1.00
.90
.80
.70
.60
.50
.40
.30
.20
.10
0
S = DOLLARS
P = DOLLARS
S=P(1+ni)
Amount at 6%
Amount at 5%
Amount at 4%
P1
O'
M1
N'
Present value at 4%
Present value at 5%
Present value at 6%
P2
P=S(1-nd)
M
N
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
n = YEARS

FIGURE 1

It may be noted that the graphs of $S = P(1 + ni)$ when $S$ and $i$ are given and of $P = S(1 - nd)$ when $P$ and $d$ are given are not straight lines, since the equations are of the second degree in the other letters. In these cases the graphs are hyperbolas.

EXERCISES

1. Construct the graph of $S = P(1 + ni)$ if $P = 200$ and $i = 4\frac{1}{2}\%$.
2. Construct the graph of $P = S(1 - nd)$ if $S = 1000$ and $d = 10\%$.
3. Construct the graph of $S = P(1 + ni)$ if $S = 100$ and $i = 5\%$.

**8. Compound interest; compound discount.** In computing simple interest or simple discount, the principal on which interest or discount is computed is constant during the term. In computing compound interest, the principal on which interest is computed is increased by the amount of the interest payment each time the interest due is paid during the term. The *compound interest*, $S - P$, for the term $n$ on an original principal $P$, is the total amount of interest computed in this way. Similarly, in computing compound discount, the principal on which discount is computed is decreased by the amount of the discount payment each time the discount due is paid during the term, and the *compound discount*, $S - P$, for the term $n$ on $S$, is the total amount of discount computed in this way.

In compound interest or discount, the interest or discount payments are converted into principal annually, semi-annually, quarterly, or at some other periodic interval, the interval between successive conversions being called a *conversion* period. The number of conversion periods per year will be denoted by $m$, and the interest rate and the discount rate per conversion period will be denoted by $\frac{j}{m}$ and $\frac{f}{m}$ respectively. It follows that $j$ is $m$ times the interest rate per conversion period and $f$ is $m$ times the discount rate per period. The number, $j$, is called the *nominal interest rate per year*, and the number, $f$, the *nominal discount rate per year*, since in business transactions it is customary to quote or name these rates with the number of conversions rather than the rates per conversion period. When $m = 1$, the nominal interest rate,

$j$, is called the *effective interest rate* and is denoted by $i$. Likewise when $m = 1$, the nominal discount rate, $f$, is called the *effective discount rate* and is denoted by $d$. For example, a nominal interest rate of 6% converted quarterly means an interest rate of $1\frac{1}{2}\%$ per quarter year, while an effective interest rate of 6% means an interest rate of 6% per year. A nominal interest rate, $j$, converted $m$ times per year is sometimes denoted by the symbol, $j_{(m)}$, and a nominal discount rate, $f$, converted $m$ times per year, by $f_{(m)}$. Numerical rates may be denoted more conveniently by enclosing them within parentheses. For example, $(j = .06, m = 4)$ denotes a nominal interest rate of 6% converted four times per year.

It should be noted that the nominal rates per year are not the rates per year as defined in Art. 2 except when $m$ is unity. For example, at $(j = .06, m = 2)$, \$1 amounts to \$1.03 in a half year and this amounts to $1.03 + 1.03\ (.03) = \$1.0609$ in another half year, so that the total interest earned on \$1 in one year is \$.0609 and the rate per year is 6.09%. Two compound interest rates with different conversion periods which lead to the same amount of $P$ in one year are said to be corresponding. In the example given 6% converted semi-annually corresponds to 6.09% converted annually; in other words, the nominal rate $(j = .06, m = 2)$ corresponds to the effective rate $i = .0609$. A general definition of corresponding rates and a method for determining them will be given in Art. 13.

If an investment is of such a nature that the interest or discount payments cannot be used to change the principal as they are in compound interest or compound discount, these payments may be put into other investments. When they are put into investments having the same conversion periods and the same rate per period as the original investment, the total amount of interest or discount realized is the same as that given by compound interest or compound discount. When, however, the payments of interest or discount are put into investments bearing rates differing from that of the original investment, the results are not the same in general. The compound interest and discount formulas are developed in the next two articles. The more general case involving different interest rates will be considered in Art. 31, Chapter II.

**9. Compound interest formulas.** By Art. 8, 1 at the beginning of any interest conversion period amounts to $1 + \frac{j}{m}$ at the end of the period, and hence $A$ at the beginning amounts to $A\left(1 + \frac{j}{m}\right)$ at the end. It follows that

$P$ amounts to $P\left(1 + \frac{j}{m}\right)$ in *one* conversion period,

$P$ amounts to $P\left(1 + \frac{j}{m}\right)^2$ in *two* conversion periods

[for $P\left(1 + \frac{j}{m}\right)$ at the beginning of the second period amounts to $P\left(1 + \frac{j}{m}\right) \cdot \left(1 + \frac{j}{m}\right)$ at the end. Here $A = P\left(1 + \frac{j}{m}\right)$]

$P$ amounts to $P\left(1 + \frac{j}{m}\right)^3$ in *three* conversion periods

. . . . . .

$P$ amounts to $P\left(1 + \frac{j}{m}\right)^{mn}$ in $mn$ conversion periods, that is, in $n$ years. Hence, when $mn$ is an integer

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \tag{6}$$

or, solving for $P$,

$$P = \frac{S}{\left(1 + \frac{j}{m}\right)^{mn}} = S\left(1 + \frac{j}{m}\right)^{-mn} \tag{6'}$$

In the derivation of formulas (6) and (6′) $mn$ is an integer; these formulas, however, determine positive values for $S$ and $P$ when $mn$ is not an integer. For example, when $P = 1$, $j = .06$, $m = 2$, $n = \frac{1}{4}$, formula (6) gives $S = 1.0148892$ approximately. In all cases, whether $mn$ is integral or fractional, the positive value of $S$ determined by (6) is called the *amount* or *accumulated value* of $P$ for $n$ years, and the positive value of $P$ determined by (6′) is called the *present* or *discounted value* of $S$ due in $n$ years.

It should be noted that formula (6′), just as formula (2′), Art. 3, for discounting $S$, does not involve explicitly a rate of discount. Formula (2′) involves a simple interest rate, while formula (6′)

involves a compound interest rate. In Art. 6 it was seen that there is a definite simple discount rate which corresponds to a given simple interest rate. Likewise in Art. 13, it will be seen that there is a definite compound discount rate which corresponds to a given compound interest rate.

If $$v = \frac{1}{\left(1 + \frac{j}{m}\right)} = \left(1 + \frac{j}{m}\right)^{-1}$$

then formulas (6) and (6′) can be written

$$S = Pv^{-mn} \tag{6}$$

$$P = Sv^{mn} \tag{6′}$$

If $\left(1 + \frac{j}{m}\right)$ be called an *accumulation factor* and $v$ be called a *discount factor*, formulas (6) and (6′) may be stated verbally in the form: *To find the amount of any sum for n years, or mn periods at the nominal rate j converted m times per year, multiply this sum by the appropriate accumulation factor raised to the power mn; to find the present value of any sum due in n years multiply this sum by the discount factor to the same power.* If $m = 1$, the accumulation factor is $(1 + i)$, and the discount factor is $(1 + i)^{-1}$.

Table III gives values of integral powers of the accumulation factor correct to eight decimals. Table IV gives values of integral powers of the discount factor. Table VIII gives the values of fractional powers of the accumulation factor. Values of powers not found in these tables can be computed by use of logarithms.

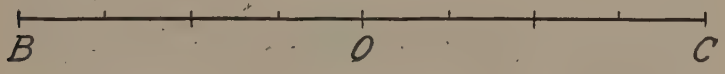
Formulas (6) and (6′) may be combined into a single formula by writing $V$ in place of $S$ or $P$ on the left, and $A$ in place of $P$ or $S$ on the right, and by writing $t$ for both $n$ and $-n$. This single formula is

$$V = A\left(1 + \frac{j}{m}\right)^{mt} \tag{7}$$

In formula (7), $t$ is *positive* when the amount or accumulated value of $A$ is to be found and *negative* when its present or discounted value is to be found. In other words, if $A$ represents the value of a sum of money at a given time, formula (7) determines its value $t$

years later than this time when $t$ is positive and its value $t$ years earlier when $t$ is negative.

Problems based on formula (7) can be visualized by the use of a diagram. In the following diagram each section of the line represents a half year:




The point $O$ denotes any given time, any point to the right of $O$ a later time, any point to the left of $O$ an earlier time. By formula (7) the value at $C$ of \$100 at $O$ is $100\ (1.03)^4 = \$109.27$ if ($j = .06$, $m = 2$), and the value at $B$ of \$100 at $O$ is $100\ (1.03)^{-4} = \$88.85$ if the same rate is used. In the first instance $t = 2$; in the second $t = -2$.

In all similar diagrams in this book each point is associated with a definite time. Any point to the right of a fixed point $O$ denotes a later time than that associated with $O$; any point to the left denotes an earlier time.

If $I$ denotes the compound interest on $P$ for $n$ years, and $D$ denotes the compound discount on $S$ for $n$ years, then by formulas (6) and (6′)

$$I = S - P = P\left[\left(1 + \frac{j}{m}\right)^{mn} - 1\right] \tag{8}$$

$$D = S - P = S\left[1 - \left(1 + \frac{j}{m}\right)^{-mn}\right] \tag{9}$$

In computing interest on $P$ for a term $n$ when $mn$ is not integral, it is customary in many transactions to compute the compound interest on $P$ for the term $n'$ where $mn'$ is the largest number of conversion periods contained in the term $n$ and to add the simple interest on the amount of $P$ at that time for the term $n - n'$; in computing discount on $S$ for a term $n$ when $mn$ is not integral, it is likewise customary to compute the compound discount on $S$ for the term $n''$ where $mn''$ is the smallest number of conversion periods containing the term $n$ and to add the simple interest on the present value of $S$ at that time for the term $n'' - n$.

EXAMPLE 1. Find the amount of \$1000 for a term of 4 years and 3 months at $(j = .06,\ m = 2)$.

SOLUTION BY LOGARITHMS. By formula (6)

$$S = 1000\ (1.03)^{\frac{17}{2}}$$
$$= \$1285.63$$

$$\log 1.03 = 0.012837$$
$$\log 1000 = 3.000000$$
$$\log (1.03)^{\frac{17}{2}} = 0.109115$$
$$\log S = 3.109115$$

EXERCISE 1. Compute $S$ by use of Tables III and VIII. [Hint. $(1.03)^{\frac{17}{2}} = (1.03)^8(1.03)^{\frac{1}{2}}$.]

EXERCISE 2. Solve Example 1 by finding the amount of \$1000 for 4 years at compound interest, and then finding the amount of this sum for 3 months at 6% simple interest.

EXAMPLE 2. Find the present value of \$1000 due in 2 years and 3 months at $(j = .06,\ m = 4)$.

SOLUTION. By formula (6′)

$$P = 1000\ (1.015)^{-9}$$
$$= \$874.59 \quad \text{(Table IV)}.$$

EXERCISE. Compute $P$ by use of logarithms.

EXAMPLE 3. Find the amount of \$1000 for a term of 35 years at $(j = .06,\ m = 4)$.

SOLUTION. By formula (6)

$$S = 1000\ (1.015)^{140} = 1000\ (1.015)^{70}(1.015)^{70}$$
$$= \$8039.81$$

## EXERCISES

**1.** Find, without using the tables, the compound interest on \$100 at $(j = .05,\ m = 2)$ if $n = 2$.

**2.** Same as Exercise 1 except that $(j = .06,\ m = 4)$ and $n = 1$.

**3.** Interpret the following:

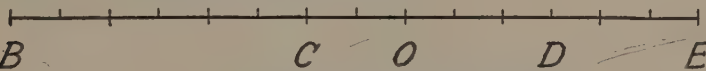
(*a*) $V = 1000\ (1 + .03)^{-6}$ when $m = 2$,
(*b*) $V = 1000\ (1 + .02)^{8}$ when $m = 4$.

**4.** Use Tables III and IV to find the values of

(*a*) $(1.025)^{10}$ (*c*) $(1.01125)^{48}$ (*e*) $(1.0275)^{-30}$
(*b*) $(1.035)^{25}$ (*d*) $(1.0275)^{-100}$ (*f*) $(1.0125)^{-80}$

Give at least two interpretations for each expression, using appropriate values for $j$, $m$, and $n$.

**5.** In the following diagram each section of the line represents a half year: the point $O$ denotes any given time; any point to the right of $O$, a later time; any point to the left of $O$, an earlier time.




If ($j = .06$, $m = 2$), what value does \$100 at $O$ have at $B$; at $C$; at $D$; at $E$? What is the value of $t$ in each case if formula (7) is used?

**6.** Find the numbers to fill the blanks:

| $S$ | $P$ | $D$ | $I$ | $j$ | $m$ | $n$ |
|---|---|---|---|---|---|---|
| | 125.00 | | | .05 | 4 | 6 |
| | 62.50 | | | .07 | 2 | $4\frac{1}{3}$ |
| 917.28 | | | | .06 | 1 | $2\frac{1}{2}$ |
| 815.25 | | | | .05 | 4 | 3 |

**7.** Find the amount of \$1200 for $4\frac{1}{4}$ years, using first the compound interest rate ($j = .05$, $m = 2$) for the whole term; second, this compound interest rate for the first four years, and then the simple interest rate, .05, for one fourth of a year. Ans. \$1480.25; \$1480.36.

**8.** Find the present value of \$1200 due in $4\frac{1}{4}$ years, using first the compound interest rate ($j = .05$, $m = 2$) for the whole term; second, this compound interest rate for $4\frac{1}{2}$ years, and then the simple interest rate, .05, for one fourth of a year. Ans. \$972.81; \$972.88.

**9.** Evaluate $(1.03)^{123}$, using Table III and the identity,*

$$(1.03)^{123} = (1.03)^{61}\,(1.03)^{62}$$

**10.** Evaluate $(1.03)^{-120}$, using Table IV and the identity

$$(1.03)^{-120} = (1.03)^{-60}\,(1.03)^{-60}$$

**11.** Evaluate $100(1.04)^{\frac{5}{2}}$, using Tables III and VIII.

**12.** Evaluate $(1.025)^{-\frac{17}{2}} = (1.025)^{-9}\,(1.025)^{\frac{1}{2}}$, using Tables IV and VIII.

* In calculating powers of $(1 + i)$ not found in the table, due regard should be given to the error involved in the product. If $n$ represents the exponent, it can be shown by methods established in the calculus that the error is a minimum for even values of $n$ if

$$(1 + i)^n = (1 + i)^{\frac{n}{2}}(1 + i)^{\frac{n}{2}} \text{ is used,}$$

and for odd values of $n$ if

$$(1 + i)^n = (1 + i)^{\frac{n-1}{2}}\,(1 + i)^{\frac{n+1}{2}} \text{ is used.}$$

**10. Compound discount formulas.** By Art. 8, 1 at the beginning of any discount conversion period discounts to $\left(1 - \frac{f}{m}\right)$ at the end of the period, and hence $A$ at the beginning discounts to $A\left(1 - \frac{f}{m}\right)$ at the end. It follows that

$S$ discounts to $S\left(1 - \frac{f}{m}\right)$ in *one* period,

$S$ discounts to $S\left(1 - \frac{f}{m}\right)^2$ in *two* periods,

. . . . . .

$S$ discounts to $S\left(1 - \frac{f}{m}\right)^{mn}$ in $mn$ periods, that is, in $n$ years. Hence, when $mn$ is an integer,

$$P = S\left(1 - \frac{f}{m}\right)^{mn} \tag{10}$$

or, solving for $S$, $S = \dfrac{P}{\left(1 - \frac{f}{m}\right)^{mn}} = P\left(1 - \frac{f}{m}\right)^{-mn}$ (10′)

In the derivation of formulas (10) and (10′) $mn$ is an integer; these formulas, however, determine positive values for $P$ and $S$ when $mn$ is not an integer. For example, when $S = 1$, $f = .06$, $m = 2$, $n = \frac{1}{2}$, formula (10) gives $P = 0.9848859$ approximately. In all cases, whether $mn$ is integral or fractional, the positive value of $P$ determined by (10) is called the *present* or *discounted* value of $S$ due in $n$ years, and the positive value of $S$ determined by (10′) is called the *amount* or *accumulated value* of $P$ for $n$ years.

A discussion of the compound discount formulas may now be given which is entirely analogous to that given in Art. 9 of the compound interest formulas. This will not be given, however, since most of the transactions in finance are based on the principles of compound interest rather than on those of compound discount. An important application of the compound discount formula is found in the constant percentage or compound discount method of computing depreciation (Art. 63, Chapter III).

### EXERCISES

**1.** Find the value at the end of 5 years of a house which cost \$6000 and which depreciated 3% each year. Use formula (10). Ans. \$5152.40.

**2.** If the cost of a house was \$6000 and its value at the end of 5 years was \$5250, find the compound discount rate, $f$, if $m = 1$.

**3.** Solve formula (10) for each of the letters involved in it except $m$.

**11. Graphs of the compound interest and compound discount formulas.** When $P$, $j$, and $m$ are known, formula (6) expresses the relation between $S$ and $n$, and when $S$, $f$, and $m$ are known, formula (10) expresses that between $P$ and $n$. The graphs of the corresponding simple interest and simple discount relations were found in Art. 7 to be straight lines. The graphs of these relations, however, are not straight lines since the variable $n$ occurs as an exponent; they are exponential curves.

Table III may be used in constructing graphs of the compound interest formula. In constructing graphs of the compound discount formula a table of corresponding values for $P$ and $n$ must be computed. Figure 2 shows the graphs of

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \text{ for } P = m = 1, \text{ and } \frac{j}{m} = .03, .04, .05, .06, .07, \text{ and } .08$$

and of

$$P = S\left(1 - \frac{f}{m}\right)^{mn} \text{ for } S = m = 1 \text{ and } \frac{f}{m} = .04, .05, .06, .10, .15, \text{ and } .20$$

with respect to the axes $ON$, $OA$. The values of the compound interest $I$ are represented by the vertical segments above $O'N'$ and those of the compound discount $D$, by the vertical segments between $O'N'$ and the discount curves. For example, $MP_1$ represents the amount and $M_1P_1$ the compound interest on 1 for 10 years at an interest rate of 3% per year and $MP_2$ represents the present value and $P_2M_1$ the compound discount on 1 for 10 years at a discount rate of 4% per year.

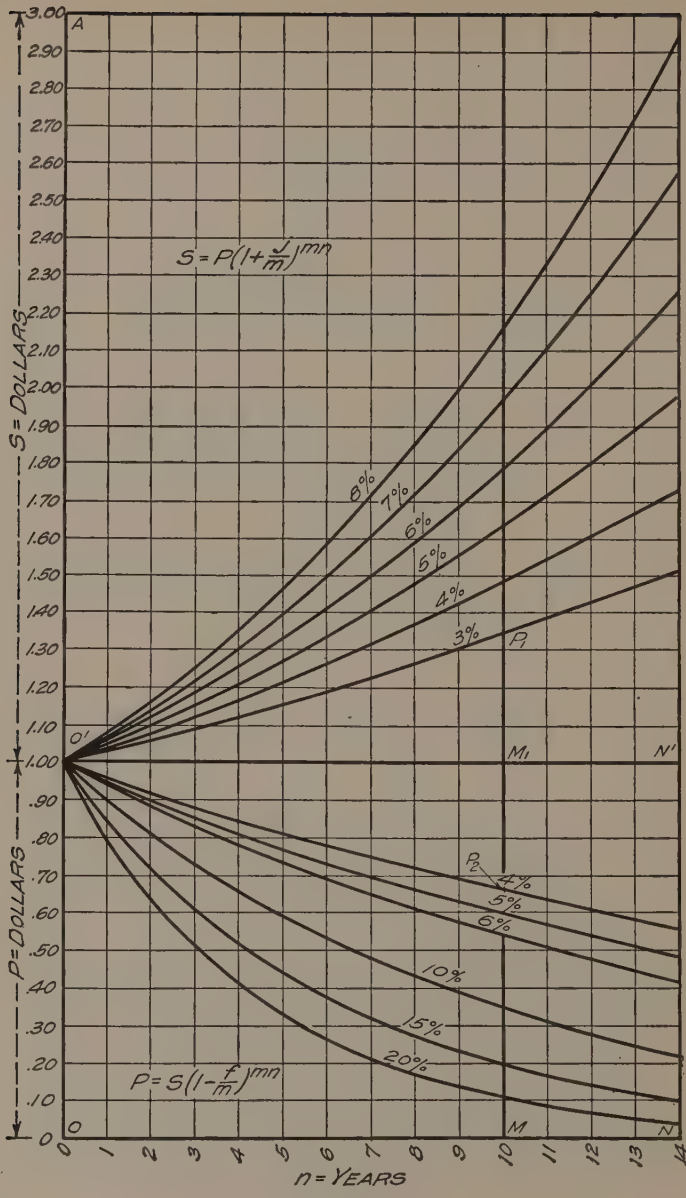

S = DOLLARS
P = DOLLARS
n = YEARS
S = P(1+r/m)^mn
P = S(1-r/m)^mn
8%
7%
6%
5%
4%
3%
P1
P2
4%
5%
6%
10%
15%
20%
A
A'
O
O'
M
M1
N
N'

FIGURE 2

The values of $P$ used in constructing the compound discount graphs shown in Figure 2 are given in the following table:

$$P = (1 - f)^n.$$

| $n$ | $f = .04$ | $f = .05$ | $f = .06$ | $f = .10$ | $f = .15$ | $f = .20$ |
|---|---|---|---|---|---|---|
| 1 | .9600 | .9500 | .9400 | .9000 | .8500 | .8000 |
| 2 | .9216 | .9025 | .8836 | .8100 | .7225 | .6400 |
| 3 | .8847 | .8573 | .8306 | .7290 | .6141 | .5120 |
| 4 | .8493 | .8145 | .7807 | .6561 | .5220 | .4096 |
| 5 | .8153 | .7738 | .7339 | .5905 | .4437 | .3277 |
| 6 | .7827 | .7351 | .6899 | .5314 | .3772 | .2621 |
| 7 | .7514 | .6983 | .6485 | .4783 | .3206 | .2097 |
| 8 | .7214 | .6634 | .6096 | .4305 | .2725 | .1678 |
| 9 | .6925 | .6303 | .5730 | .3874 | .2370 | .1342 |
| 10 | .6648 | .5987 | .5386 | .3487 | .1969 | .1074 |
| 11 | .6382 | .5688 | .5063 | .3138 | .1673 | .0859 |
| 12 | .6127 | .5403 | .4759 | .2824 | .1422 | .0687 |
| 13 | .5882 | .5133 | .4474 | .2452 | .1209 | .0550 |
| 14 | .5646 | .4877 | .4205 | .2288 | .1028 | .0440 |

The graphs in Figure 3 show the amounts of \$100 at 6% simple interest and at ($j = .06, m = 1$) compound interest for terms ranging from 0 to 25 years. These graphs show that the amounts at compound interest exceed those at simple interest for all terms greater than unity. For terms less than unity the amounts at simple interest exceed those at compound interest, as is seen in Figure 4.

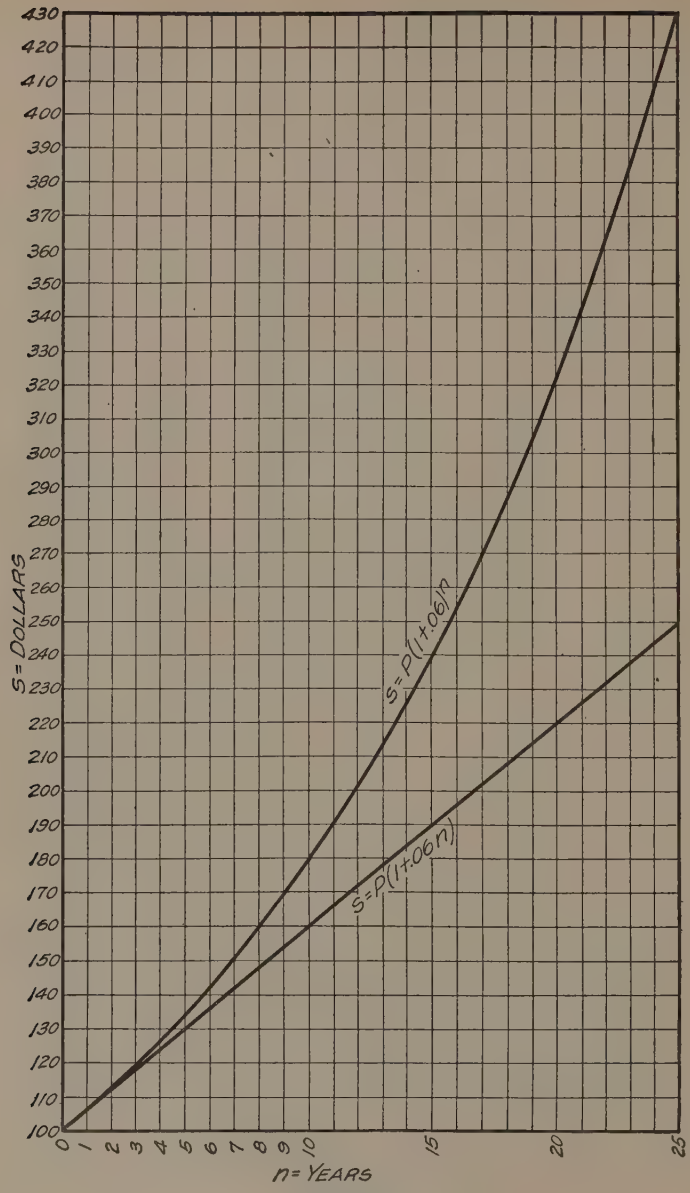



FIGURE 3

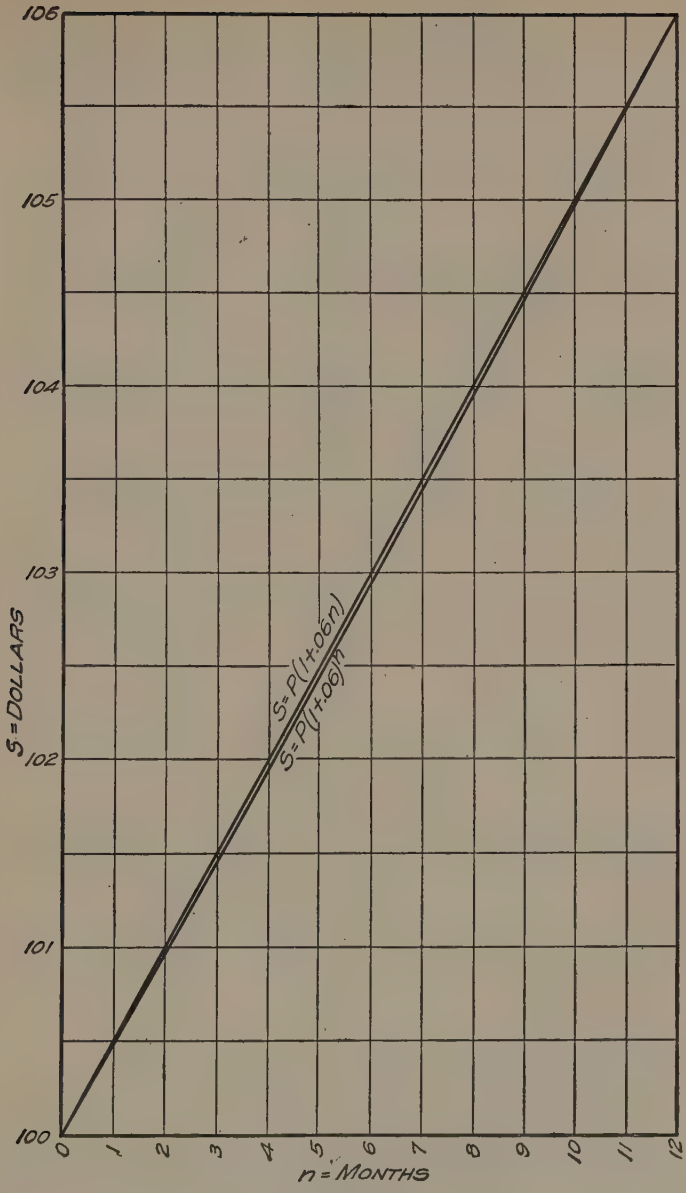

106
105
104
103
102
101
100
S = DOLLARS
S = P(1+.06n)
S = P(1+.06)^n
0
1
2
3
4
5
6
7
8
9
10
11
12
n = MONTHS

FIGURE 4

The values of $S$ used in constructing the graphs in Figure 4 are given in the following table where $P = 100$, $j = .06$, and $m = 1$ and $i = .06$:

| Term in Months | Compound Amount | Simple Amount |
|---|---|---|
| 0 | 100.00 | 100.00 |
| 1 | 100.49 | 100.50 |
| 2 | 100.98 | 101.00 |
| 3 | 101.47 | 101.50 |
| 4 | 101.96 | 102.00 |
| 5 | 102.46 | 102.50 |
| 6 | 102.96 | 103.00 |
| 7 | 103.46 | 103.50 |
| 8 | 103.96 | 104.00 |
| 9 | 104.47 | 104.50 |
| 10 | 104.98 | 105.00 |
| 11 | 105.49 | 105.50 |
| 12 | 106.00 | 106.00 |

On account of the small scale used, the two graphs in Figure 3 seem to coincide for the range 0 to 1 year. In Figure 4 a sufficiently large scale is used to show the graphs distinct for this range.

## EXERCISES

1. Construct the graph of $S = P\left(1 + \frac{j}{m}\right)^{mn}$ if $P = 100$, $\frac{j}{m} = 4\%$, and $m = 1$.
2. Construct the graph of $P = S\left(1 - \frac{j}{m}\right)^{mn}$ if $S = 100$, $\frac{j}{m} = 8\%$, and $m = 1$.
3. Construct the graph of $S = P\left(1 + \frac{j}{m}\right)^{mn}$ if $S = 110$, $P = 100$, and $m = 1$.
4. Construct the graphs of (*a*) $P = (1 - .06)^n$ and (*b*) $P = (1 - n\ .06)$.
5. On a large scale construct the graphs of the two equations in Exercise 4 for values of $n$ by months for the first year. For what value of $n$ do the curves intersect?

**12. Problems based on the compound interest formulas.** The compound interest formulas (Art. 9),

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \tag{6}$$

$$I = P\left[\left(1 + \frac{j}{m}\right)^{mn} - 1\right] \tag{8}$$

contain six letters, and when four are known, the other two can be found by solving the two equations simultaneously for them. The solutions of these equations can be accomplished by the processes learned in elementary algebra together with the additional process of taking the logarithm of both members of an equation. Some of the computations that arise may be performed most easily by means of the fundamental operations of arithmetic, some by means of tables, and some by means of computing machines. The computations to be performed should be noted carefully before choosing the means to be used.

In Art. 9 some examples have been solved in which $S$ or $P$ is the unknown. In this article some examples will be solved in which $n$ or $j$ is the unknown.

EXAMPLE 1. Find the term required for \$1000 to amount to \$3290.64 at $(j = .06, m = 4)$.

SOLUTION. Formula (6) becomes, on dividing both members by 1000,

$$3.29064 = (1.015)^{4n}$$

Taking logarithms of both members and solving for $n$ gives

$$n = \frac{\log 3.29064}{4 \log 1.015} = \frac{0.517280}{0.025864} = 20 \text{ years}$$

EXERCISE 1. Find the value of $n$ by use of Table III.

EXERCISE 2. By taking logarithms of both members of formula (6) and solving for $n$ show that

$$n = \frac{\log S - \log P}{m \log\left(1 + \frac{j}{m}\right)}$$

EXAMPLE 2. Find the time required for any sum to double itself at $(j = .06, m = 1)$.

SOLUTION. In this example $\frac{S}{P} = 2$, and formula (6) becomes $2 = (1.06)^n$.

Taking logarithms of both members and solving for $n$ gives

$$n = \frac{\log 2}{\log 1.06} = \frac{0.301030}{0.025306} = 11.89 \text{ years}$$

EXERCISE. Find $n$ by use of Table III.

EXAMPLE 3. If \$450 amounts to \$576.04 in 5 years, find $j$ if $m = 2$.

SOLUTION. By formula (6) $576.04 = 450\left(1 + \frac{j}{2}\right)^{10}$

Taking logarithms and solving for $\log\left(1 + \frac{j}{2}\right)$ gives

$$\log\left(1 + \frac{j}{2}\right) = \frac{\log 576.04 - \log 450}{10} = 0.000724$$

$$1 + \frac{j}{2} = 1.025$$

$$j = .05$$

EXERCISE 1. Find $j$ by means of Table III.

EXERCISE 2. By taking logarithms of both members of formula (6) and solving for $\log\left(1 + \frac{j}{m}\right)$, show that $\log\left(1 + \frac{j}{m}\right) = \frac{\log S - \log P}{mn}$

EXERCISE 3. By dividing both members of formula (6) by $P$ and solving for $\frac{j}{m}$, show that

$$\frac{j}{m} = \sqrt[mn]{\frac{S}{P}} - 1$$

## EXERCISES

**1.** Find the numbers to fill the blanks.

| $P$ | $j$ | $m$ | $n$ | $S$ |
|---|---|---|---|---|
| 750 | .05 | 2 | 2 | |
| 750 | .055 | 4 | 2 | |
| 1000 | .0475 | 2 | | 2000.00 |
| 2500 | .06 | 12 | | 3755.85 |
| 2750 | | 2 | $4\frac{1}{2}$ | 3492.50 |
| 293.65 | | 1 | 5 | 381.74 |
| | .05 | 12 | 5 | 285.32 |
| | .0425 | 1 | 2 | 962.35 |

**2.** The sum of \$100 is placed in a building and loan company to the credit of a child at the time of its birth. If the company's rate of interest is ($j = .04$, $m = 4$), how much will be due the child when it becomes 21 years old?

**3.** Would it be more to the advantage of a capitalist to loan money at ($j = .0575$, $m = 4$) or at ($j = .06$, $m = 1$)?

**4.** The sum of \$2000 is placed in a savings bank whose interest rate is ($j = .045$, $m = 4$). At the end of 3 years \$1000 is withdrawn. Find the amount in the bank 5 years from the date the deposit was made.

**5.** The sum of \$1000 is deposited in a savings bank whose rate is ($j = .05$, $m = 2$). If at the end of 4 years \$500 is withdrawn and \$600 at the end of an additional 6 years, find the value of the account 15 years from the date of the deposit.

**6.** A willed \$4939.38 to a library. The will provided that the legacy be invested until it amounted to \$10,000, and thereafter the income on this amount be used to purchase books. The legacy was invested at ($j = .055$, $m = 2$). In how many years did it accumulate to \$10,000? If the \$10,000 is invested at $5\frac{1}{2}\%$ with semi-annual payments of interest, and if the first interest payment each year is invested at 5% simple interest for six months, how much is available at the end of each year for the purchase of books?
Ans. 13 years; \$556.88.

**7.** A buys a house for \$5600, pays one-half in cash and gives a mortgage note bearing the rate ($j = .065$, $m = 2$) for the balance. If at the end of 4 years he pays \$3000, how much will be due on the note at the end of 5 additional years? Ans. \$848.75.

**8.** Find the amount to which \$2500 will accumulate in $4\frac{1}{3}$ years at ($j = .05$, $m = 2$), first by using the compound interest formula for the whole term, and second by using this formula for the largest integral value of $mn$ contained in the term and the simple interest formula for the remaining part. What is the difference between the two amounts? Why does the second method give the larger amount? Ans. \$3096.57; \$3096.77.

**9.** To what amount will \$1000 accumulate in $2\frac{1}{3}$ years if it is invested at ($j = .055$, $m = 2$)? Solve in two ways as in Exercise 8.

**10.** Discount \$3000 due in 2 years and 2 months at ($j = .06$, $m = 4$) by methods analogous to those used in Exercise 8. Ans. \$2636.83; \$2636.90.

**11.** Find the present value of \$3500 for 3 years and 5 months at ($j = .06$, $m = 2$) by each of the two methods of Exercise 10.

**12.** Find the nominal interest rate $j$ at which \$1000 will amount to \$1590.55 in 10 years if $m = 2$; if $m = 4$.

**13.** If a war savings stamp was sold in the month of January, 1918, for \$4.12, show that at ($j = .04$, $m = 4$) the stamp was worth \$5.00 January 1, 1923.

**14.** A treasury savings certificate was sold January 5, 1924, for \$800. It is redeemable at \$1000 in five years. Find $j$ if $m = 4$.

**15.** If the population of a certain city increased from 100,000 to 150,000 in 10 years, and if it is assumed that the population increased according to the

compound interest law, find the annual rate of increase on the assumption that this rate was constant.

**13. Corresponding rates.** The definitions of corresponding rates given in Arts. 6 and 8 may be generalized as follows: *Any two rates are said to be corresponding for a given value of n when each leads to the same accumulated value of P in n years.* By means of this definition and the accumulation formulas, the equation connecting any two corresponding rates can be written at once. For example, the equation connecting the interest rate $j$ converted $m$ times per year and its corresponding effective rate, $i$, for $n$ years is $P(1+i)^n = P\left(1+\frac{j}{m}\right)^{mn}$; dividing both members of this equation by $P$ and extracting the $n$th root gives

$$1+i=\left(1+\frac{j}{m}\right)^{m} \tag{11}$$

Solving for $j$ and writing $j_{(m)}$ in place of $j$ to show the number of conversions, gives

$$j_{(m)} = m[(1+i)^{\frac{1}{m}} - 1] \tag{11'}$$

It may be noted that in this case the corresponding rates, $i$ and $j$, are independent of $n$. Table IX gives the values of $j_{(m)}$ corresponding to various values of $i$ and $m$, and Table III may be used to find the values of $i$ corresponding to various values of $j$ and $m$.

In a similar way if the interest rate $j'$ converted $m'$ times per year corresponds to the interest rate $j''$ converted $m''$ times per year, the definition leads to the equation

$$\left(1+\frac{j'}{m'}\right)^{m'}=\left(1+\frac{j''}{m''}\right)^{m''}$$

A few examples will now be solved to illustrate further the method of finding the rate corresponding to a given rate.

EXAMPLE 1. Find the interest rate $j$ converted quarterly which corresponds to an interest rate of 6% converted annually.

SOLUTION. Equating accumulated values of $P$ for $n$ years, dividing by $P$, and extracting the $n$th root lead to

$$\left(1+\frac{j}{4}\right)^{4}=1.06$$

SOLVING, using Table VIII, $j = .0587$ approximately.

EXAMPLE 2. Find the interest rate, $i$, converted annually which corresponds to an interest rate of 6% converted quarterly.

SOLUTION. Equating accumulated values leads to

$$1 + i = 1.015^4$$
$$i = .0614 \text{ approximately.} \quad \text{(Table III)}$$

EXAMPLE 3. Find the discount rate $f$ converted semi-annually which corresponds to an interest rate of 6% converted quarterly.

SOLUTION. Equating accumulated values leads to

$$\frac{1}{\left(1 - \frac{f}{2}\right)^2} = (1.015)^4$$
$$f = .0587 \text{ approximately.} \quad \text{(Table IV)}$$

EXAMPLE 4. Find the effective discount rate $d$ which corresponds to the nominal discount rate ($f = .08$, $m = 2$).

SOLUTION. Equating accumulated values leads to

$$1 - d = (1 - .04)^2$$
$$d = .0784$$

Corresponding rates are useful in comparing investments bearing different rates.

## EXERCISES

**1.** Use the following table to find the values of $i$ corresponding to $j = .06$, $m = 1, 2, 3, 4, 6, 12, 365, 1000$, and $\infty$ :

| $m$ | $1 + i = \left(1 + \frac{.06}{m}\right)^m$ |
|---|---|
| 1 | 1.06000000 |
| 2 | 1.06090000 |
| 3 | 1.06120800 |
| 4 | 1.06136355 |
| 6 | 1.06152015 |
| 12 | 1.06167781 |
| 365 | 1.06183130 |
| 1000 | 1.06183462 |
| $\infty$ | 1.06183654 |

Verify the first six values of $(1 + i)$ by use of Table III, and the next two by use of the binomial formula. (The value of $1 + i$ when $m \to \infty$ is here included for the sake of completeness. See Art. 14 for a discussion of this case.)

**2.** Find the numbers to fill the blanks, using the formulas for corresponding rates:

| $j_{(m)}$ | $i$ | $m$ | $d$ | $f_{(m)}$ |
|---|---|---|---|---|
| .05 | | 4 | | |
| | | 2 | .04 | |
| | .03 | 4 | | |
| | | 2 | | .035 |

**3.** Find the numbers to fill the blanks, using the formula

$$\left(1 + \frac{j'}{m'}\right)^{m'} = \left(1 + \frac{j''}{m''}\right)^{m''}:$$

| $j'$ | $m'$ | $j''$ | $m''$ |
|---|---|---|---|
| .05 | 2 | | 4 |
| .05 | 2 | | 1 |
| | 1 | .06 | 2 |
| | 4 | .07 | 12 |

Express verbally each of these four problems, using the solutions obtained. Notice that if $m' > m''$, then $j' < j''$.

**4.** If the effective interest rate of income from an investment is 5%, find the two equivalent nominal rates if $m = 2$ in one case and $m = 4$ in the other.

**5.** Find the present value of \$1239 due in 2 years if $(j = .05, m = 1)$.

**6.** Find the discounted value of \$1239 due in 2 years if the effective rate of discount is 4.7618%.

**7.** Show that two rates are corresponding for a given value of $n$ when each leads to the same discounted value of $S$ in $n$ years. Why do Exercises 5 and 6 have the same answer?

**8.** Show by use of Exercise 7 that an effective rate of interest of 5% corresponds to an effective rate of discount of 4.7618%.

**9.** Find the discount rate converted quarterly which corresponds to $(j = .05, m = 2)$.

**10.** What effective interest rate corresponds to a discount rate of $(f = .05, m = 2)$?

**11.** Find the numbers to fill the blanks where $j$ and $f$ are rates corresponding to $i = \frac{6}{100}$ or $d = \frac{6}{106}$.

| $m$ | $j$ | $f$ |
|---|---|---|
| 1 | .06000000 | .05660377 |
| 2 | .05912603 | .05742828 |
| 3 | .05883847 | |
| 4 | .05869538 | .05784655 |
| 6 | .05855287 | |
| 12 | .05841061 | |
| 365 | .05827356 | |
| 1000 | .05827061 | .05826721 |
| $\infty$ | .05826891 | .05826891 |

**14. The interest and discount formulas when $m$ becomes infinite.** When the interest rate, $j$, remains constant and the number of conversion periods, $m$, increases, the value of $\left(1 + \frac{j}{m}\right)^m$ also increases. An illustration of this for $j = .06$ and $m = 1, 2, 3, 4, 6, 12, 365, 1000, \infty$ has been given in Exercise 1, Art. 13. This exercise shows further that as $m$ increases from 1 to 1000, $\left(1 + \frac{j}{m}\right)^m$ increases from 1.06 to 1.06183462 approximately, and this comparatively small increase in $\left(1 + \frac{j}{m}\right)^m$ suggests that as $m$ increases beyond limit, $\left(1 + \frac{j}{m}\right)^m$ approaches some definite number as a limiting value. It may be readily seen, in fact, that the limiting value of $\left(1 + \frac{j}{m}\right)^m$ when $m$ becomes infinite, written $L_{m \to \infty}\left(1 + \frac{j}{m}\right)^m$, is $e^j$, where $e$, the base of the Napierian logarithms, has the approximate value 2.71828183. The expression, $\left(1 + \frac{j}{m}\right)^m$, may be written in the form $\left(1 + \frac{j}{m}\right)^{\frac{m}{j} \cdot j}$ and from this it follows that

$$L_{m\to\infty}\left(1+\frac{j}{m}\right)^{m} = L_{m\to\infty}\left(1+\frac{j}{m}\right)^{\frac{m}{j}\cdot j}$$

$$= L_{x\to\infty}\left(1+\frac{1}{x}\right)^{x\cdot j}, \text{ where } x=\frac{m}{j}$$

$$= e^{j}, \text{ since, by definition, } L_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$$

In like manner it can be shown that $L_{m\to\infty}\left(1-\frac{f}{m}\right)^{m} = e^{-f}$

Hence, when $m$ becomes infinite, the compound interest and compound discount formulas can be written in the forms respectively:

$$S = Pe^{jn} \qquad (6_1)$$

$$P = Se^{-fn} \qquad (10_1)$$

When $j$ or $f$ is given and $m$ becomes infinite, the interest or discount is said to be converted continuously, at this interest or discount rate. When interest or discount is converted continuously, it follows, from Art. 13, that the relations connecting the corresponding rates $i$, $j$ and the corresponding rates $d$, $f$ are

$$1+i = e^{j} \text{ and } 1-d = e^{-f}$$

If $j$ and $f$ are given, these equations determine the values of $i$ and $d$. The value of $i$ is the largest amount of interest that can be earned on \$1 in one year at a fixed rate $j$ and a variable $m$ which increases without limit. Likewise the value of $d$ is the largest amount of discount on \$1 in one year at a fixed rate $f$, and a variable $m$ which increases without limit. If $i$ and $d$ are given, these equations determine the corresponding values of $j$ and $f$ when the number of conversions per year becomes infinite. These values of $j$ and $f$ are denoted by $\delta$ and $\delta'$ respectively and they are called the *force of interest* and the *force of discount* which correspond to the given effective rates $i$ and $d$. Using $\delta$ and $\delta'$ in place of $j$ and $f$, the above relations become

$$e^{\delta} = 1+i \text{ and } e^{-\delta'} = 1-d$$

from which $\qquad \delta = \log_e(1+i) \text{ and } \delta' = -\log_e(1-d)$

When $i$ and $d$ are corresponding effective rates, they satisfy the relation $1+i = \frac{1}{1-d}$. On substituting the values of $1+i$ and $1-d$ in terms of $\delta$ and $\delta'$, this relation becomes

$$e^{\delta} = \frac{1}{e^{-\delta'}} = e^{\delta'}$$

It follows that when $i$ and $d$ are corresponding effective rates that $\delta = \delta'$ (Exercise 11, Art. 13).

Since the force of interest, $\delta$, is the value of $j$ when $m$ becomes infinite that corresponds to a given effective interest rate $i$, it follows from the relation $j = m[(1+i)^{\frac{1}{m}} - 1]$, (formula 11′) that $\delta = L_{m\to\infty}\, m[(1+i)^{\frac{1}{m}} - 1]$. Equating this value of $\delta$ to that given above gives

$$\delta = L_{m\to\infty}\, m[(1+i)^{\frac{1}{m}} - 1] = \log_e (1+i)$$

EXAMPLE 1. Find the amount of \$1000 for 10 years at a nominal rate of 4% converted continuously.

SOLUTION. By formula $(6_1)$

$S = 1000\, e^{.4}$
$= \$1491.83$

$\log e = .434294$
$\log e^{.4} = .173718$
$\log 1000 = 3.000000$
$\log S = 3.173718$

EXAMPLE 2. Find the effective interest rate which corresponds to the force of interest of 6%.

SOLUTION. By the formula

$$e^{\delta} = 1 + i$$
$$e^{.06} = 1 + i$$

Taking logarithms, $\log_{10}(1+i) = .06 \log_{10} e = (.06)(.434294)$
$= .02605$
$1 + i = 1.0618$
$i = .0618$

EXAMPLE 3. Find the force of interest which corresponds to an effective interest rate of 6%.

SOLUTION. By the formula $e^{\delta} = 1 + i$

$$e^{\delta} = 1.06$$

Taking logarithms, $\delta \log e = \log 1.06$

$$\delta = \frac{\log 1.06}{\log e} = (.025306)(2.302585)$$
$= .0583$ (See Exercise 11, Art. 13.)

EXERCISE. Show that $L_{m\to\infty}\, m[(1 + .06)^{\frac{1}{m}} - 1] = .058269$

## EXERCISES

**1.** Find the amount of \$1000 at 6% for 1 year if the interest is converted continuously. Ans. \$1061.84.

**2.** Same as Exercise 1, except that the rate is $4\frac{3}{4}$%.

**3.** What sum will amount to \$1000 in one year at $4\frac{1}{2}$% converted continuously. Ans. \$956.00.

**4.** Find the force of interest correct to 4 decimals corresponding to an effective rate of 4%, of $4\frac{1}{2}$%, of 5%. Ans. .0392; .0440; .0487.

**5.** Find the effective rate of interest correct to 4 decimals corresponding to a force of interest of 4%, of $4\frac{1}{2}$%, of 5%. Ans. .0408; .0460; .0512.

**6.** Find the rate converted continuously which corresponds to ($j = .06$, $m = 1$), to ($j = .06$, $m = 2$), to ($j = .06$, $m = 4$). Ans. .0582; .0591; .0596.

**7.** What is the discount on \$1000 for 1 year at ($f = .07$, $m \to \infty$). Ans. \$67.60.

**8.** Find the force of discount correct to 4 decimals corresponding to an effective rate of discount of 4%, of 5%, and of 6%. Ans. .0408; .0513; .0618.

**9.** Interpret the following table:

| $n$ | $1 + .06\,n$ | $(1 + .06)^n$ | $e^{.06\,n}$ |
|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 |
| $\frac{1}{12}$ | 1.0050 | 1.0049 | 1.0050 |
| $\frac{2}{12}$ | 1.0100 | 1.0098 | 1.0101 |
| $\frac{3}{12}$ | 1.0150 | 1.0147 | 1.0152 |
| $\frac{4}{12}$ | 1.0200 | 1.0196 | 1.0202 |
| $\frac{5}{12}$ | 1.0250 | 1.0246 | 1.0353 |
| $\frac{6}{12}$ | 1.0300 | 1.0296 | 1.0305 |
| $\frac{7}{12}$ | 1.0350 | 1.0346 | 1.0356 |
| $\frac{8}{12}$ | 1.0400 | 1.0396 | 1.0408 |
| $\frac{9}{12}$ | 1.0450 | 1.0447 | 1.0460 |
| $\frac{10}{12}$ | 1.0500 | 1.0498 | 1.0513 |
| $\frac{11}{12}$ | 1.0550 | 1.0549 | 1.0565 |
| $\frac{12}{12}$ | 1.0600 | 1.0600 | 1.0618 |

Verify the entries for $n = \frac{7}{12}$.

**10.** The following inequalities hold for the amounts of $P$ for $n$ years:

$$n > 1,\ Pe^{in} > P(1+i)^n > P(1+ni)$$
$$n < 1,\ Pe^{in} > P(1+ni) > P(1+i)^n$$
$$n = 1,\ Pe^{in} > P(1+ni) = P(1+i)^n$$

Verify for $P = 1$, $i = .06$, and $n = 10$, $\frac{11}{12}$, and 1.

**11.** Show that the inequalities in Exercise 10 are special cases of the following inequalities:

$$n > \frac{1}{m},\ Pe^{in} > P\left(1+\frac{j}{m}\right)^{mn} > P(1+nj)$$
$$n < \frac{1}{m},\ Pe^{in} > P(1+nj) > P\left(1+\frac{j}{m}\right)^{mn}$$
$$n = \frac{1}{m},\ Pe^{in} > P(1+nj) = P\left(1+\frac{j}{m}\right)^{mn}$$

**15. The value of a set of sums.** By the value at a given time of a set containing just one sum is meant the accumulated or discounted value of the sum to that time. By the value at a given time of a set containing more than one sum is meant the sum of the values of the separate sums of the set at that time. In the preceding articles interest and discount formulas have been developed for finding the value at any given time of one sum due at another time. By application of these formulas to the separate sums of any set the expression for the value of the set at a given time can be easily written. The work of computing the value of an expression of this kind may be rather laborious when the number of sums in the set is large. The study of methods of computing is important in all work in the mathematics of finance and it is especially important in finding the values of sets of sums, particularly of those sets which occur frequently in finance. Some sets of frequent occurrence consist of equal sums at periodic intervals during a definite term, as the dividend on bonds. Such sets are called *annuities certain*, and methods for finding their values are given in Chapter II. Other sets of frequent occurrence consist of sums at periodic intervals during a term not definitely known, as a pension payable during the life of some person. Sets of this type are called *contingent annuities*, and methods for finding their values are given in Chapter IV.

In writing the expression for the value of a set of sums it is necessary to know the formula to be used with each sum. *The*

*compound interest formula is the one generally sanctioned by business practice and, except when otherwise stated, it will be assumed in what follows.*

In finding the values at two or more times of a given set of sums by the compound interest formula, use will frequently be made of the following

**Theorem I.** *If $V_o$ represents the value at a given time of a set of sums, and $V_t$ represents the value of the set $t$ years later than the given time, if $t$ is positive, and $t$ years earlier, if $t$ is negative,*

$$V_t = V_o\left(1 + \frac{j}{m}\right)^{mt}$$

Proof: Let $A$ denote any sum of the set when it is due and $A_{t'}$ represent its value at the given time where $t'$ is the number of years from the time $A$ is due to the given time. If $t'$ is positive, $A_{t'}$ represents the amount of $A$ for $t'$ years; if $t'$ is negative, $A_{t'}$ represents the present value of $A$ due in $t'$ years. If $A_{t'+t}$ represents the value of $A$, $t' + t$ years from the time it is due, then by formula (7)

$$\begin{aligned} A_{t'+t} &= A\left(1 + \frac{j}{m}\right)^{m(t'+t)} \\ &= A\left(1 + \frac{j}{m}\right)^{mt'} \cdot \left(1 + \frac{j}{m}\right)^{mt} \text{ by } a^{x+y} = a^x \cdot a^y \\ &= A_{t'}\left(1 + \frac{j}{m}\right)^{mt} \text{ by formula (7)} \end{aligned}$$

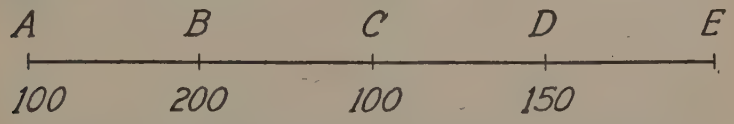
It follows that the theorem is true for each sum of the set, and hence it is true for the set as a whole. This theorem is also true when the compound discount formula at a fixed rate is used, but it is not true in general when the simple interest or simple discount formula is used.

So far in finding expressions for the value at a given time of a set of one or more sums the same rate has been assumed for the entire time during which the value of each sum is found. More general expressions for the value at a given time of a set of sums can be found by resolving the term, during which the value of each sum is found, into parts and using different rates with the separate parts. A case which is sometimes useful is that in which the

entire range over which values are to be found is divided into parts, each part having a constant interest rate for all sums whose values depend on it. For example, the amount of \$100 for 10 years at ($j = .06, m = 2$) during the first five years and at ($j = .07, m = 2$) during the last five is $100(1.03)^{10}(1.035)^{10}$. The above theorem can be extended to this case by using the appropriate interest rate for each part of the range.

### EXERCISES

The diagram below, each section of which represents one year, shows a set of sums:




1. Find the value of this set at $E$ at ($j = .06, m = 1$). Ans. \$635.81.

2. Find the value of this set at $A$ at ($j = .06, m = 1$), first by discounting \$635.81 for four years, second by summing the discounted values at $A$ of the separate sums. Ans. \$503.62.

3. Find the value of this set at $C$ at ($j = .06, m = 1$), first by discounting \$635.81 for two years, second by accumulating \$503.62 for two years, third by summing the values at $C$ of the separate sums.

4. Find the value of the set at $E$, using a simple interest rate of 6%. Ans. \$631.90.

5. Find the value of this set at $A$, using a simple interest rate of 6%. Show that this value at $A$ cannot be found by discounting \$631.00 for four years at the simple interest rate of 6%. Ans. \$505.08.

6. Find the value of the set at $A$ if ($j = .05, m = 2$) for the first two years and ($j = .06, m = 2$) for the last two years.

**16. Sets of sums having equal values. Equation of value.** The equations needed to find the unknowns in solving problems in the mathematics of finance are usually gotten by equating the values of two sets of sums at a definite time. As seen in Art. 15, the expressions for values of the sets are found by use of the formulas for accumulating or discounting a single sum, and hence the accumulating and discounting operations are of primary importance in determining equations (Art. 1). Each of the interest or discount formulas expresses equality in value between the sums

$S$ and $P$ in the sense that each has the same value at a given time. For example, $S = P\left(1 + \frac{j}{m}\right)^{mn}$ states that the value of $P$ when accumulated for $n$ years at the interest rate $j$ converted $m$ times per year is $S$. In these formulas each set consists of a single sum. More general formulas or equations arise when the number of sums in one or both of the sets is greater than one. In all of these equations there are involved sums of money, rates, and terms of years, so that an unknown may be a sum of money, a rate, or a term of years.

In writing an equation expressing equality in value between two sets of sums, use may be made of the following

**Theorem II.** — *If two sets of sums are equal in value at a known time, they are also equal at any other time t years from the known time when the compound interest formula at a nominal interest rate j converted m times per year is used in finding values.*

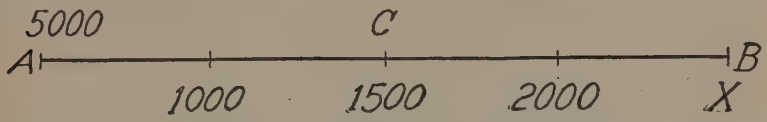
If $V_o$ is the value of either set at the known time, it follows from Theorem I that the value of either set $t$ years from this time is $V_t$ where $V_t = V_o\left(1 + \frac{j}{m}\right)^{mt}$; hence Theorem II is true. This theorem is also true when the compound discount formula is used, but it is not true for every value of $n$ when the simple interest or simple discount formula is used. By virtue of this theorem it is permissible to select that time for equating the values of two sets of sums which leads to the simplest equation in the unknown to be found. Theorem II is also true for the special case considered at the end of Art. 15.

Equations of value which arise in elementary finance usually have a single sum for one of the two sets. Some examples based on sets of sums having equal values at a known time will now be solved.

Example 1. A debt of \$5000 is to be paid in instalments, including principal and interest, as follows: \$1000 at the end of one year, \$1500 at the end of two, \$2000 at the end of three, and the balance at the end of four years. Find the last payment if unpaid principal accumulates at ($j = .06$, $m = 1$).

The diagram $AB$, each section of which represents one year, shows the two sets of sums having equal values; the set above the line represents the

debt, the set below the line the payments on the debt or the credits. The last payment is denoted by $x$.* In this diagram $A$ denotes the present time; points to the right of $A$ denote later times.




Solution. The two sets of sums shown in the diagram have equal values at $A$ and hence, by Theorem II, they have equal values at any other time. Equating values at $B$, that is, at the time the last payment is made, gives

$$x + 2000(1.06) + 1500(1.06)^2 + 1000(1.06)^3 = 5000(1.06)^4$$

Solving, using Table III, $x = \$1315.97$

Exercise 1. Find $x$ by equating values at $A$.

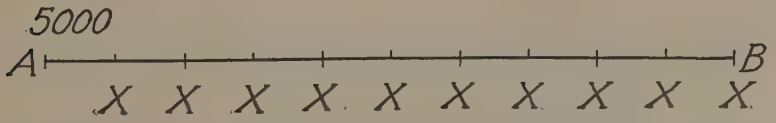
Exercise 2. Find $x$ by equating values at $C$.

Exercise 3. Find $x$ by computing, as in arithmetic, the unpaid principal of the debt just after each payment is made. This work may be arranged in a table or schedule as follows:

| Year | Principal at Beginning of Year | Interest at ($j = .06$, $m = 1$) | Payment at End of Year | Principal Repaid |
|---|---|---|---|---|
| 1 | 5000.00 | 300.00 | 1000.00 | 700.00 |
| 2 | 4300.00 | | | |
| 3 | | | | |
| 4 | | | | |

Example 2. A debt of \$5000 is to be paid in ten equal semi-annual instalments, including principal and interest, the first to be paid at the end of six months. Find the amount of each instalment if unpaid principal accumulates at ($j = .06$, $m = 2$).

The diagram $AB$, each section of which represents a half year, shows the two sets of sums having equal values. Each instalment is denoted by $x$.




* Diagrams of this kind may be helpful to the learner in specifying both known and unknown data.

SOLUTION. Equating values at $A$ gives

$$x\left(\frac{1}{1.03}+\frac{1}{(1.03)^2}+\cdots+\frac{1}{(1.03)^{10}}\right)=5000$$

Solving, using TABLE IV:

$$x = \$586.15$$

EXERCISE 1. Find $x$ by equating values at $B$.

EXERCISE 2. Check the value found for $x$ by constructing a schedule similar to that in Exercise 3, Example 1.

EXERCISE 3. Find the sum

$$\frac{1}{1.03}+\frac{1}{(1.03)^2}+\cdots+\frac{1}{(1.03)^{10}}$$

by use of the formula for summing a geometric progression.

EXAMPLE 3. Two non-interest-bearing notes, one for \$100 due in one year, the other for \$200 due in two years, are bought for \$264.06. Find the interest rate $i$ converted annually for which the purchase price has the same value as the two notes.

SOLUTION. Equating the present value of the notes to 264.06 gives

$$\frac{200}{(1+i)^2}+\frac{100}{1+i}=264.06$$

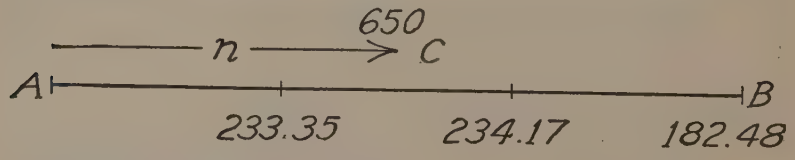
Solving, $$i = .08$$

EXERCISE 1. Check the value found for $i$ by constructing a schedule.

EXERCISE 2. Solve Example 3, if a discount rate, $d$, converted annually, is used in place of the interest rate, $i$.

EXAMPLE 4. Three non-interest-bearing notes, one for \$233.35 due in 3 months, one for \$234.17 due in 6 months, and one for \$182.48 due in 9 months, are to be retired by a single payment equal in amount to the sum of three notes. On the basis of 6% simple discount when should this payment of \$650 be made if its present value equals the sum of the present values of the three notes?

The diagram $AB$ shows the two sets of sums having equal values; $n$ is the unknown time.




SOLUTION. Equating values at $A$ gives

$$650[1-(.06)n]=233.35[1-(.06)\tfrac{1}{4}]+234.17[1-(.06)\tfrac{1}{2}]+182.48[1-(.06)\tfrac{3}{4}].$$

Solving, $$n = .48 \text{ years, approximately.}$$

EXERCISE 1. Solve Example 4 on the basis of 6% simple interest.

EXERCISE 2. Solve Example 4 on the basis of ($j = .06$, $m = 1$).

EXERCISE 3. Solve Example 4, if the two sets of sums have equal values at $C$.

It should be noted that the result found in Example 4 is independent of the discount rate used. When the same method is applied to a set of sums $S_1$ due in $n_1$ years, $S_2$ due in $n_2$ years, and so on, and to their sum $S_1 + S_2 \cdots$ there results

$$n = \frac{S_1 n_1 + S_2 n_2 + \cdots}{S_1 + S_2 + \cdots}$$

This formula gives a satisfactory means for finding the time at which the payment of a set of sums may be replaced by a single payment equal in value to their sum in case the numbers $n_1$, $n_2$, $\cdots$ denote short terms, each one year or less.

In the simple examples just solved, the unknown in each of the first two is a sum of money, in the third it is a rate, and in the fourth, a term of years. The method used to set up the equations is the same, however, in each case. Before giving other illustrations of this fundamental process, methods will be given for finding the values of sets of sums which occur at periodic intervals during a definite term of years, that is, of annuities certain. This is done in Chapter II.

## EXERCISES

**1.** A man offers to sell his home for \$8500 cash or for \$1000 cash, \$2000 at the end of the first, \$3000 at the end of the second, and \$3600 at the end of the third year. On an interest basis of ($j = .06$, $m = 1$), how much better from the standpoint of the buyer is the cash price at the time of sale, at the end of the third year? Ans. \$79.41. \$94.58.

**2.** A washing machine is bought on the instalment plan for \$160. The contract of sale calls for a cash payment of \$20, and 10 monthly payments of \$14 each, the first of which is to be made one month from date of sale. Find the cash price on the day of sale if 6% simple discount is used. Ans. \$156.15.

**3.** A farm was purchased for \$12,000, \$3000 of which was paid in cash. The balance was paid in four equal annual instalments which began one year from date of purchase. On the basis of ($j = .05$, $m = 2$), find the amount of each instalment. Check by constructing a schedule. Ans. \$2541.79.

**4.** The cost of a piano is \$450 cash or \$125 cash and \$125 at the end of 3, 6, and 9 months. On the basis of 6% simple discount, how much better from the standpoint of the buyer is the cash price on the day of sale? Ans. \$38.75.

**5.** A man rents a house for \$600 per year payable in advance. What monthly rent payable in advance is equivalent at the beginning of the year to this annual rent? Use a 6% simple interest rate. Ans. \$51.33.

**6.** The sum of \$585 paid at the end of one year will retire two non-interest-bearing notes, one for \$300 due in 2 years, and the other for \$350 due in 3 years. If $m = 1$, find the interest rate correct to four decimal places.

Ans. $j = .0712$.

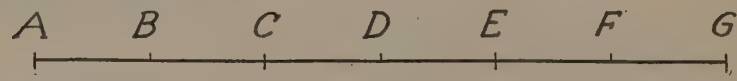
**7.** A owes B the sum of \$500. In partial payment A gives B a note drawn by C for \$300 due in one year without interest, and agrees to pay the balance in two equal annual instalments, the first of which is to be paid one year after the note matures. If an interest rate of ($j = .05$, $m = 2$) is used, find the amount of each payment. Ans. \$121.28.

**8.** A owes B \$15,000. One year from date A pays B \$3000, and at the end of the second year \$4000. A then agrees to pay the balance in four equal semi-annual instalments, the first of which is to be paid 6 months after the \$4000 payment was made. On the basis of ($j = .06$, $m = 2$) find the amount of each instalment. Check by constructing a schedule. Ans. \$2609.54.

**9.** A man owes two non-interest-bearing notes, one for \$150 due in 2 months, and one for \$300 due in 3 months. When must the sum of \$445 be paid if its present value equals that of the two sums? When must the sum of \$450 be paid? Use a simple discount rate of 7%. Which result is independent of the rate?

**10.** Four non-interest-bearing notes, one for \$125 due in 60 days, one for \$100 due in 90 days, one for \$75 due in 30 days, and one \$125 due in 10 days, are retired by a single payment. If this payment is \$425 and is the equivalent of the four notes at the present time, when must it be made? If the payment is \$430, when must it be made? Use a 5% simple discount rate.

The following exercises have reference to the diagram given below in which each section represents 6 months.




**11.** Show that:

(*a*) \$100 at $A$, \$150 at $C$, and \$200 at $D$ compose a set of sums which is equivalent to a set consisting of the single sum of \$487.22 at $G$, if ($j = .04$, $m = 2$).

(*b*) The value at $B$ of each set in 11 (*a*) is \$441.29.

(*c*) The values of the two sets in 11 (*a*) are equal at $F$.

(*d*) The values of the two sets in 11 (*a*) are not equal at $B$ on the basis of 4% simple interest, or simple discount.

**12.** Show that:

(*a*) A set of \$100 at $B$ and \$150 at $D$ is equivalent to a set consisting of the single sum \$265.50 at $E$ on the basis of 6% simple interest.

(*b*) On a basis of 6% simple interest, the two sets in 12 (*a*) are not equivalent at $F$. Find the simple interest rate at which the two sets in 12 (*a*) are equivalent at $F$.

**13.** If a set of sums consisting of \$100 at $B$ and \$400 at $E$ is equivalent to a single sum \$492.7754 at $D$, find $j$ if $m = 2$.

**14.** How long after $A$ must the sum of \$600 be paid if the set consisting of this single sum is equivalent at $A$ to a set consisting of \$300 at $C$, \$200 at $E$, and \$100 at $F$. Use a simple discount rate $d$. Interpret the cancellation of $d$.

**15.** Same as Exercise 14 except that the two sets are equivalent at $G$, and a simple interest rate of $i$ is used. Interpret the cancellation of the $i$.

**16.** Same as Exercise 14 except that the first set consists of the single sum of \$650.

**17.** If a set of sums consisting of $x$ at $B$ and $y$ at $D$ is equivalent to a set $z$ at $E$ on the basis of a simple interest rate $i$, show that they are not equivalent at $C$ unless $x$ and $y$ satisfy the equation $y = (2 + i)x$.

## CHAPTER II

### ANNUITIES CERTAIN

**17. Annuities.** An annuity is a set of sums, usually equal in value, paid at periodic intervals. The word *annuity* implies that the interval is one year; but, as used, the interval may be any fractional or integral multiple of a year. The value of each sum is called the *rent*, the interval between successive sums the *rent period*, and the time between the *beginning* of the first rent period and the *end* of the last, the *term* of the annuity. The sum of the rent payments in one year is called the *annual rent*.

Annuities are classified with respect to their terms. When the term is definite, the annuity is called an *annuity certain;* when the term becomes infinite, it is called a *perpetuity*. When the term is indefinite, depending on some contingency such as the life of a person who receives the rent, the annuity is called a *contingent annuity*. The weekly wages of workmen, the monthly rentals on property, the semi-annual taxes on real estate, the annual interest payments on a sum of money each for a fixed term are illustrations of annuities certain.

Annuities are classified also with respect to the time in the rent period when the rent is paid. When the rent is paid at the beginning of its period, the annuity is called an *annuity due;* when paid at the end of its period, the annuity is called an *annuity immediate*, or more briefly an *annuity*.

Classifications of annuities based on the times when their terms begin, and on the value of the rent, will be given in Arts. 27 and 32 respectively. In what follows in this chapter the word *annuity* will mean an annuity certain.

In this chapter the fundamental formulas for finding the values of annuities certain are developed, and methods of solving problems based on these formulas are discussed. In deriving these formulas, use is made of the formula for summing a geometric progression.

### EXERCISES

**1.** If, in a geometric progression, the first term is represented by $a$, the last term by $l$, the constant ratio by $c$, the number of terms by $n$, and the sum by $s$, show that

$$l = ac^{n-1}$$

$$s = a\frac{c^n - 1}{c - 1}$$

What does this formula become if $c < 1$ and $n \to \infty$?

**2.** Given the geometric progression

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \cdots \text{ to } n \text{ terms}$$

Find the last term and the sum of the terms when $n = 10$. Find the sum when $n \to \infty$.

**3.** If the first term is 1, the constant ratio, $1 + .06$, and the number of terms, 6, find the sum.

**4.** Same as Exercise 3, except that the number of terms is $n$.

**5.** If $a = 1, c = (1.05)^{-1}$, and $n = 4$, find $s$.

**6.** If $a = 1, c = (1 + i)^{-1}$, and $n = 10$, show that $s = \dfrac{1 - (1 + i)^{-10}}{i}$.

**7.** If, in an arithmetric progression, the first term is represented by $a$, the last term by $l$, the common difference by $d$, the number of terms by $n$, and the sum by $s$, show that

$$l = a + (n - 1)d$$

$$s = \frac{(a + l)n}{2}$$

**18. The value of an annuity at the end of its term.** In this chapter $R$ denotes the rent, $n$ the number of years in the term, and $r$ the number of years in the rent period of any annuity or of any annuity due. Unless otherwise stated, the compound interest formula at the rate $j$ converted $m$ times per year is used in finding values.

The value of any annuity at the end of its term, often called the *amount* of the annuity, will be denoted by $V_n$. In this article the formula for the amount of an annuity is derived by the use of the compound interest formula and the formula for finding the sum of a geometric progression. The derivation is given for two simple and important special cases which occur frequently in practice and then for the general case.

CASE 1. *The interest and the rent are paid annually; that is,* $m = r = 1$. In this case there are $n$ rent periods in the term of

the annuity, and one interest conversion period in each rent period, so that the successive rent payments, beginning with the first, are accumulated for $n-1$, $n-2$, $\cdots$ 1, 0 interest conversion periods. By formula (6), Chapter I, the values of the rent payments at the end of the term are $R(1+i)^{n-1}$, $R(1+i)^{n-2}$, $\cdots$, $R(1+i)$, and $R$. Hence

$$V_n = R[1+(1+i)+(1+i)^2+\cdots+(1+i)^{n-1}]$$

The right-hand member is a geometric progression, having $R$ for the first term, $(1+i)$ for the ratio, and $n$ for the number of terms. Summing this progression gives

$$V_n = R\frac{(1+i)^n-1}{i}$$

Case 2. *The interest and the rent are paid m times per year; that is,* $mr = 1$. In this case there are $mn$ rent periods in the term of the annuity and one interest conversion period in each rent period, so that the successive rent payments, beginning with the first, are accumulated for $mn-1$, $mn-2$, $\cdots$ 1, 0 interest conversion periods. By formula (6), Chapter I, the values of the rent payments at the end of the term are

$$R\left(1+\frac{j}{m}\right)^{mn-1}, \quad R\left(1+\frac{j}{m}\right)^{mn-2}, \quad \cdots, \quad R\left(1+\frac{j}{m}\right), \text{ and } R.$$

Hence

$$V_n = R\left[1+\left(1+\frac{j}{m}\right)+\left(1+\frac{j}{m}\right)^2+\cdots+\left(1+\frac{j}{m}\right)^{mn-1}\right]$$

The right-hand member is a geometric progression, having $R$ for the first term, $\left(1+\frac{j}{m}\right)$ for the ratio, and $mn$ for the number of terms. Summing this progression gives

$$V_n = R\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\frac{j}{m}}$$

General Case. *The interest and the rent may be paid at the same or different times, and* $\frac{n}{r}$ *is an integer.* In this case there are $\frac{n}{r}$ rent pe-

riods in the term of the annuity and $mr$ interest conversion periods in each rent period, so that the successive rent payments, beginning with the first, are accumulated for $mr\left(\frac{n}{r}-1\right)$, $mr\left(\frac{n}{r}-2\right)$, ···, $mr$, 0 interest conversion periods. By formula (6), Chapter I, the values of the rent payments at the end of the term are $R\left(1+\frac{j}{m}\right)^{mr\left(\frac{n}{r}-1\right)}$, $R\left(1+\frac{j}{m}\right)^{mr\left(\frac{n}{r}-2\right)}$, ···, $R\left(1+\frac{j}{m}\right)^{mr}$ and $R$. Hence

$$V_n = R\left[1+\left(1+\frac{j}{m}\right)^{mr}+\left(1+\frac{j}{m}\right)^{2mr}, \cdots +\left(1+\frac{j}{m}\right)^{mr\left(\frac{n}{r}-1\right)}\right]$$

The right-hand member is a geometric progression having $R$ for the first term, $\left(1+\frac{j}{m}\right)^{mr}$ for the ratio and $\frac{n}{r}$ for the number of terms. Summing this progression gives

$$V_n = R\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\left(1+\frac{j}{m}\right)^{mr}-1} \qquad (1)$$

In the derivation of formula (1), $\frac{n}{r}$ is an integer; the formula, however, determines a positive value for $V_n$ when $\frac{n}{r}$ is not an integer. In the applications in elementary finance $\frac{n}{r}$ is usually integral.

Formula (1) may be stated verbally in the form: *To find the value of an annuity at the end of its term divide the compound interest on R for the annuity term by the compound interest on* 1 *for one rent period.*

EXAMPLE. If \$100 are deposited at the end of each half year in a bank which pays 5% converted semi-annually, find the amount of the deposits at the end of 5 years.

SOLUTION. By formula (1)

$$V_5 = 100\,\frac{1.025^{10}-1}{.025}$$

$$= \$1120.34 \qquad \text{(Table III)}$$

## EXERCISES

**1.** Find the sum of the geometric progression

$$1, \left(1+\frac{j}{m}\right), \left(1+\frac{j}{m}\right)^2, \left(1+\frac{j}{m}\right)^3.$$

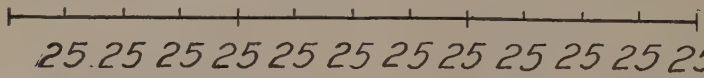
Evaluate this sum if $j = .06$ and $m = 2$.

**2.** Find the sum of the geometric progression

$$1, \left(1+\frac{.06}{4}\right), \cdots, \left(1+\frac{.06}{4}\right)^7$$

Compute the value of this sum by use of Table I.

**3.** The following diagram, each section of which denotes three months, represents the annuity having $R = 25$, $n = 3$, $r = \frac{1}{4}$:




Show that the amount of this annuity at ($j = .06$, $m = 4$) is given by

$$V_3 = 25[1 + 1.015 + (1.015)^2 + \cdots + (1.015)^{11}];$$

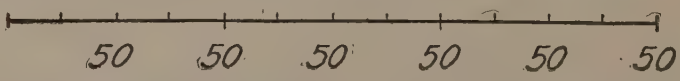
that the amount at ($j = .06$, $m = 2$) is given by

$$V_3 = 25[1 + (1.03)^{\frac{1}{2}} + 1.03 + \cdots + (1.03)^{\frac{11}{2}}]$$

Compute the value of $V_3$ in each case by use of the formula for the sum of a geometric progression, and by use of Tables III and VIII.

Ans. 326.03; 325.83

**4.** The following diagram, each section of which denotes three months, represents the annuity having $R = 50$, $n = 3$, $r = \frac{1}{2}$:




Show that the amount of this annuity at ($j = .06$, $m = 4$) is given by

$$V_3 = 50[1 + (1.015)^2 + (1.015)^4 + \cdots + (1.015)^{10}];$$

that the amount at ($j = .06$, $m = 2$) is given by

$$V_3 = 50[1 + 1.03 + (1.03)^2 + \cdots + (1.03)^5].$$

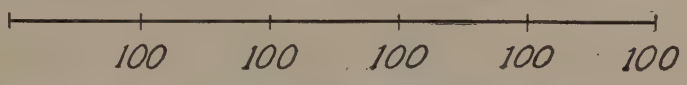
Compute the value of $V_3$ in each case by use of the formula for the sum of a geometric progression, and by use of Table III.

**5.** By formula (1) find the amount of an annuity having $R = 300$, $n = 10$, and $r = \frac{1}{4}$ at ($j = .05$, $m = 4$). Ans. 15446.87

**6.** By formula (1) find the amount of an annuity having $R = 1200$, $n = 10$, and $r = 1$ at ($j = .05$, $m = 2$). Ans. 15137.57

**7.** By formula (1) find the amount of an annuity having $R = 100$, $n = 10$, $r = \frac{1}{12}$ at ($j = .05$, $m = 4$).

**8.** The following diagram, each section of which denotes one year, represents the annuity having $R = 100$, $n = 5$, and $r = 1$:




Show that the amount of this annuity at 5% simple interest is

$$100[1.00 + 1.05 + 1.10 + 1.15 + 1.20];$$

that the amount of this annuity at ($j = .05$, $m = 1$) is

$$100[1 + 1.05 + (1.05)^2 + (1.05)^3 + (1.05)^4]$$

Compute the value of each of these amounts; use the formula for the sum of an arithmetic progression in the first instance, that for a geometric progression in the second. Ans. 550; 552.56

**9.** At the simple interest rate $i$, show that the amount of the annuity whose rent is $R$, term $n$ years, and rent period $r$ years is

$$R\left(\frac{n}{r}\right)\left(1 + \frac{n - r}{2}i\right)$$

**19. The value of an annuity at the beginning of its term.** The value of an annuity at the beginning of its term, often called the *present value* of the annuity, will be denoted by $V_o$. The formula for the present value of an annuity can be derived by the method used in Art. 18 for deriving the formula for the amount. It can be derived more briefly, however, by applying Theorem I, Art. 15, Chapter I, to the value of $V_n$. By this theorem $V_o$ is $V_n$ discounted for $n$ years; that is, $V_o = V_n\left(1 + \frac{j}{m}\right)^{-mn}$. Hence, using the value of $V_n$ given by formula (1),

$$V_o = R\frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\left(1 + \frac{j}{m}\right)^{mr} - 1} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Formula (2) may be stated verbally in the form: *To find the value of an annuity at the beginning of its term divide the compound discount on R for the annuity term by the compound interest on 1 for one rent period.*

EXAMPLE. Find the present value of an annuity whose rent is \$20, term is 3 years, and rent period is 1 month, at ($j = .06$, $m = 1$).

SOLUTION. By formula (2)

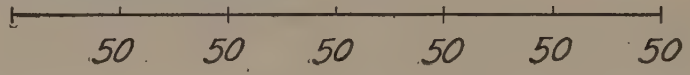
$$V_o = 20\,\frac{1 - 1.06^{-3}}{1.06^{\frac{1}{12}} - 1}$$

$$= 20\,\frac{.16038072}{.00486755} \quad \text{(Tables IV and VIII)}$$

$$= \$658.98$$

## EXERCISES

**1.** The following diagram, each section of which denotes 6 months, represents the annuity having $R = 50$, $n = 3$, and $r = \frac{1}{2}$:




Show that the present value of this annuity at ($j = .06$, $m = 2$) is given by

$$V_o = 50[(1.03)^{-1} + (1.03)^{-2} + \cdots + (1.03)^{-5} + (1.03)^{-6}];$$

that the present value of this annuity at ($j = .06$, $m = 4$) is given by

$$V_o = 50[(1.015)^{-2} + (1.015)^{-4} + \cdots + (1.015)^{-10} + (1.015)^{-12}]$$

Compute the value of $V_o$ in each case by use of the formula for the sum of a geometric progression, and by use of Table IV.

Ans. 270.86; 270.66

**2.** By formula (2) find the present value of the annuity having $R = 100$, $n = 5$, and $r = \frac{1}{4}$ at ($j = .05$, $m = 2$); at ($j = .05$, $m = 1$).

**3.** Show that the present value of the annuity of Exercise 1, at the simple discount rate $d = .06$ is

$$50[.97 + .94 + .91 + .88 + .85 + .82];$$

that the present value at the compound discount rate ($f = .06$, $m = 1$) is

$$50[(.94)^{\frac{1}{2}} + .94 + (.94)^{\frac{3}{2}} + (.94)^2 + (.94)^{\frac{5}{2}} + (94)^3]$$

Compute the present value of each. Use the formula for the sum of an arithmetic progression in the first instance, that for the sum of a geometric progression in the second. Ans. 268.50; 269.59

**20. Annuity tables. Notation.** When $R = r = 1$, $V_n$ is denoted by $s_{\overline{n}|}$ and $V_o$ is denoted by $a_{\overline{n}|}$. When $R = r = \frac{1}{p}$, $V_n$ is denoted by $s^{(p)}_{\overline{n}|}$ and $V_o$ is denoted by $a^{(p)}_{\overline{n}|}$. That is, $s_{\overline{n}|}$ denotes the amount and $a_{\overline{n}|}$ denotes the present value of an annuity whose rent is 1 payable annually for $n$ years, and $s^{(p)}_{\overline{n}|}$ denotes the amount and $a^{(p)}_{\overline{n}|}$ denotes the present value of an annuity whose rent is $\frac{1}{p}$

payable $p$ times per year for $n$ years. In each of these annuities the annual rent is 1.

When $m$ is greater than 1, the interest rate $j$ converted $m$ times per year may be indicated by writing $(m)$ to the upper left of the symbol, and $j$ to the lower right; when $m$ is 1, it is omitted and $j$ may be replaced by $i$.* For example,

${}^{(4)}s_{\overline{10}|.06}$ denotes the amount of an annuity whose rent is 1 payable annually for 10 years at $(j = .06, m = 4)$;

$s^{(4)}_{\overline{10}|.06}$ denotes the amount of an annuity whose rent is $\frac{1}{4}$ payable quarterly for 10 years at $(j = .06, m = 1)$;

$a_{\overline{10}|.06}$ denotes the present value of an annuity whose rent is 1 payable annually for 10 years at $(j = .06, m = 1)$.

By formulas (1) and (2) and the notation just given it follows that

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \qquad (1_1), \qquad a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} \qquad (2_1)$$

$$s^{(p)}_{\overline{n}|i} = \frac{1}{p}\,\frac{(1+i)^n - 1}{(1+i)^{\frac{1}{p}} - 1} \qquad (1_2), \qquad a^{(p)}_{\overline{n}|i} = \frac{1}{p}\,\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{p}} - 1} \qquad (2_2)$$

Putting $n = \frac{1}{p}$ in $(1_1)$ gives $s_{\overline{\frac{1}{p}}|i} = \dfrac{(1+i)^{\frac{1}{p}} - 1}{i}$; putting $n = 1$ in $(1_2)$ gives $s^{(p)}_{\overline{1}|i} = \dfrac{1}{p}\,\dfrac{i}{(1+i)^{\frac{1}{p}} - 1}$. Hence, using formula (11′), Art. 13, Chapter I,

$$s^{(p)}_{\overline{1}|i} = \frac{1}{p s_{\overline{\frac{1}{p}}|i}} = \frac{i}{j_{(p)}} \qquad (1_3)$$

Table V gives the values, correct to eight decimals, of $s_{\overline{n}|i}$ for values of $i$ and integral values of $n$ which occur frequently in practice; Table VI gives analogous values of $a_{\overline{n}|i}$. It may be

* The notation here used is that of Glover in *Tables of Applied Mathematics in Finance, Insurance and Statistics* (Part I, page 3), published by George Wahr, Ann Arbor, Michigan.

readily verified that $\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i$. Table VII gives the reciprocals of the numbers in Table VI, that is, the values of $\frac{1}{a_{\overline{n}|i}}$. Table X gives the values of $s^{(p)}_{\overline{1}|i} = \frac{i}{j_{(p)}}$ for certain integral values of $p$. The tables just cited will be referred to as annuity tables.

### EXERCISES

1. Interpret the meaning of the following symbols; draw a diagram in each case:

(a) $5\,s_{\overline{10}|.06}$ (d) $40\,s^{(4)}_{\overline{1}|.06}$

(b) $10\,a_{\overline{10}|.06}$ (e) $5^{(2)}s_{\overline{10}|.06}$

(c) $40\,s^{(4)}_{\overline{10}|.06}$ (f) $10^{(4)}a_{\overline{8}|.06}$

2. By use of Tables V, VI, VII, and X find the values of the following:

(a) $5\,s_{\overline{50}|.05}$ (d) $\dfrac{50}{a_{\overline{25}|.045}}$

(b) $10\,a_{\overline{100}|.025}$ (e) $40\,s^{(4)}_{\overline{1}|.06}$

(c) $\dfrac{100}{s_{\overline{10}|.03}}$ (f) $\dfrac{10}{s_{\overline{20}|.005}}$

**21. New forms of the values of $V_n$ and $V_o$.** Formulas (1) and (2) can be put into forms well suited to computations with annuity tables by dividing numerator and denominator of the right-hand members by $\frac{j}{m}$ and then using formulas $(1_1)$ and $(2_1)$. This gives

$$V_n = R\frac{\dfrac{\left(1+\frac{j}{m}\right)^{mn}-1}{\frac{j}{m}}}{\dfrac{\left(1+\frac{j}{m}\right)^{mr}-1}{\frac{j}{m}}} \qquad V_o = R\frac{\dfrac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\frac{j}{m}}}{\dfrac{\left(1+\frac{j}{m}\right)^{mr}-1}{\frac{j}{m}}}$$

$$V_n = R\frac{s_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} \quad (3), \qquad V_o = R\frac{a_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} \quad (4)$$

Formula (3) may be stated verbally in the form: *To find the amount of an annuity of rent R, term n, and rent period r, at the rate j converted m times per year, divide the amount of an annuity of rent R, term mn, and rent period 1 at the annual rate,* $\frac{j}{m}$, *by the amount of an annuity of rent 1, term mr and rent period 1 at the annual rate* $\frac{j}{m}$.

Formula (4) may be stated in an analogous form; this is left as an exercise.

When $mr = 1$, formulas (3) and (4) become, since $s_{\overline{1}|i} = 1$,

$$V_n = Rs_{\overline{mn}|\frac{j}{m}} \qquad (3_1), \qquad V_o = Ra_{\overline{mn}|\frac{j}{m}} \qquad (4_1)$$

When $mr = \frac{1}{p}$, formulas (3) and (4) become, since $\frac{1}{s_{\overline{\frac{1}{p}}|i}} = ps^{(p)}_{\overline{1}|i}$ by formula $(1_3)$

$$V_n = Rps_{\overline{mn}|\frac{j}{m}} \cdot s^{(p)}_{\overline{1}|\frac{j}{m}} \qquad (3_2)$$

$$V_o = Rpa_{\overline{mn}|\frac{j}{m}} \cdot s^{(p)}_{\overline{1}|\frac{j}{m}} \qquad (4_2)$$

Formulas (1), (2), (3), and (4) should be thoroughly mastered. Formulas (3) and (4) are adapted to computations based on annuity tables; formulas (1) and (2) are adapted to computations other than those based on annuity tables.

EXAMPLE 1. If \$100 are deposited at the end of each half year in a bank which pays 5% converted semi-annually, find the amount of the deposits at the end of 5 years. (See Example, Art. 18.)

SOLUTION. By formula $(3_1)$

$$V_5 = 100\, s_{\overline{10}|.025}$$
$$\$1120.34 \quad \text{(Table V)}$$

EXAMPLE 2. Find the present value of an annuity whose rent is \$20, term is 3 years, and rent period is 1 month, at $(j = .06, m = 1)$. (See Example, Art. 19.)

SOLUTION. By formula $(4_2)$

$$V_o = 240\, a_{\overline{3}|.06}\, s^{(12)}_{\overline{1}|.06}$$
$$= 240\,(2.67301195) \cdot (1.0272107) \quad \text{(Tables VI and X)}$$
$$= \$658.98$$

### EXERCISES

**1.** Write the expressions for $V_n$ and $V_o$ by use of formulas (1) and (2), divide numerator and denominator by $\frac{j}{m}$, and write these results in the forms given by formulas (3) and (4), for each of the following:

| $R$ | $n$ | $r$ | $j$ | $m$ |
|---|---|---|---|---|
| 100 | 10 | 1 | .06 | 4 |
| 100 | 10 | $\frac{1}{4}$ | .06 | 2 |

**2.** Find $V_n$ and $V_o$ at ($j = .06$, $m = 4$), for each of the following:

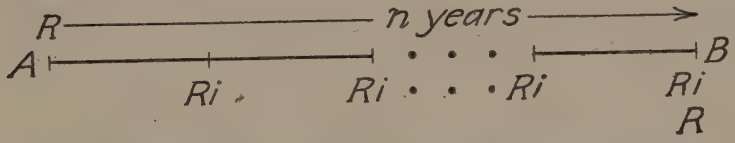
| $R$ | $n$ | $r$ | |
|---|---|---|---|
| 100 | 2 | $\frac{1}{4}$ | Tables V and VI |
| 10 | 8 | $\frac{1}{2}$ | Tables V, VI, and VII |
| 1 | 2 | $\frac{1}{12}$ | Tables V, VI, and X |

**3.** Write the expressions for the values of $V_n$ and $V_o$, in the forms best suited for computations, for each of the following:

| $R$ | $n$ | $r$ | $j$ | $m$ |
|---|---|---|---|---|
| 100 | 10 | 1 | .055 | 4 |
| 100 | 10 | 1 | .06 | 4 |
| 10 | 5 | $\frac{1}{4}$ | .05 | 2 |
| 10 | 5 | $\frac{1}{52}$ | .05 | 12 |
| 25 | $2\frac{1}{6}$ | $\frac{1}{12}$ | .05 | 4 |
| 25 | $\frac{1}{2}$ | $\frac{1}{12}$ | .04 | 1 |

**22. Another derivation of the formulas for the values of $V_n$ and $V_o$.** The interest payments on $R$ for $n$ years at the rate $i$ payable annually constitute an annuity of rent $Ri$ and term $n$. The value of this annuity at the end of its term at the annual rate $i$ is represented by $iV_n$ and the value at the beginning of its term by $iV_o$.

It follows that, at this interest rate, the amount of $R$ for $n$ years is $R + iV_n$; likewise the present value of $R$ due in $n$ years is $R - iV_o$. The line diagram, $AB$, may be used to illustrate these statements:




In this diagram the $R$ above is the original investment and the set of sums below is the return on this investment, so that the set below has the same value at any time as the $R$ above.

Equating the values of these sets at $B$ and $A$ gives respectively

$$R + iV_n = R(1+i)^n \text{ from which } V_n = R\frac{(1+i)^n - 1}{i},$$

$$R(1+i)^{-n} + iV_o = R \text{ from which } V_o = R\frac{1-(1+i)^{-n}}{i}$$

An analogous treatment of the annuity composed of the interest payments on $R$ for $n$ years at the rate $j$ payable $m$ times per year leads to the equations

$$R + \frac{j}{m}V_n = R\left(1+\frac{j}{m}\right)^{mn} \text{ from which } V_n = R\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\frac{j}{m}},$$

$$R\left(1+\frac{j}{m}\right)^{-mn} + \frac{j}{m}V_o = R \text{ from which } V_o = R\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\frac{j}{m}}$$

In general the compound interest amounts at the end of each $r$ years on $R$ for $n$ years at the rate $j$ converted $m$ times per year, as given by formula (8), Art. 9, Chapter I, constitute an annuity, when $\frac{n}{r}$ is an integer, of rent $R\left[\left(1+\frac{j}{m}\right)^{mr} - 1\right]$, term $n$, and rent period $r$. An analogous treatment of this annuity at the rate $j$ converted $m$ times per year leads to the general equations

$$R + \left[\left(1+\frac{j}{m}\right)^{mr} - 1\right]V_n = R\left(1+\frac{j}{m}\right)^{mn}$$

from which
$$V_n = R\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\left(1+\frac{j}{m}\right)^{mr}-1} \qquad (1)$$

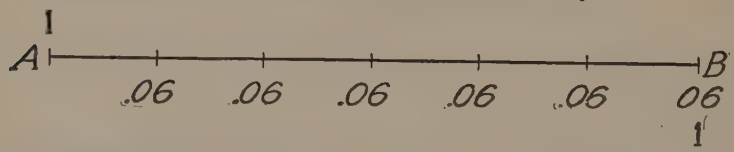
$$R\left(1+\frac{j}{m}\right)^{-mn}+\left[\left(1+\frac{j}{m}\right)^{mr}-1\right]V_o = R$$

from which
$$V_o = R\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^{mr}-1} \qquad (2)$$

It may be noted that the derivations in this article do not use the formula for summing a geometric progression.

### EXERCISES

**1.** In the diagram each section of the line represents one year:




An investment of 1 is made at $A$ at ($j = .06$, $m = 1$). The return on this investment is an annuity of .06 whose term is 6 years and in addition, 1 at $B$. Equating the values of the investment and of the return on the investment at $B$ and at $A$ gives respectively

$$(1.06)^6 = 1 + .06\, s_{\overline{6}|.06} \quad \text{or} \quad s_{\overline{6}|.06} = \frac{(1.06)^6 - 1}{.06},$$

$$1 = (1.06)^{-6} + .06\, a_{\overline{6}|.06} \text{ or } a_{\overline{6}|.06} = \frac{1-(1.06)^{-6}}{.06}$$

**2.** By the use of a diagram similar to that in Exercise 1 show that

$$(1+i)^{-n} = 1 + i\, s_{\overline{n}|i} \quad \text{or} \quad s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i},$$

$$1 = (1+i)^{-n} + i\, a_{\overline{n}|i} \text{ or } a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

**23. Relations connecting $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$.** The relations given in this article are of service in computation.

By Theorem I, $\quad s_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^n$

By formula $(1_1)$, $\quad s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$

Solving each of these equations for $(1 + i)^n$ and equating the results gives

$$\frac{s_{\overline{n}|i}}{a_{\overline{n}|i}} = 1 + i\, s_{\overline{n}|i}$$

Dividing both members of this equation by $s_{\overline{n}|i}$ gives the relation

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i \qquad (5)$$

Relation (5) shows that the reciprocal of $a_{n|i}$ is the reciprocal of $s_{n|i}$, plus $i$, and that the reciprocal of $s_{\overline{n}|i}$ is the reciprocal of $a_{\overline{n}|i}$, minus $i$. Since $i$ can be subtracted easily from $\frac{1}{a_{\overline{n}|i}}$ it follows that one table (Table VII) suffices for both $\frac{1}{a_{\overline{n}|i}}$ and $\frac{1}{s_{\overline{n}|i}}$.

To find the value of $s_{\overline{53}|.07}$ by means of Table V use may be made of the relation $s_{\overline{53}|.07} = s_{\overline{50}|.07}\,(1.07)^3 + s_{\overline{3}|.07}$. This relation may be readily verified by substituting the values of $s_{\overline{53}|.07}$, $s_{\overline{50}|.07}$, and $s_{\overline{3}|.07}$ given by formula $(1_1)$. It may be derived by direct reasoning as follows: the annuity whose amount, $s_{\overline{53}|.07}$, is to be found can be resolved into two annuities, one composed of the first 50 rent payments and the other composed of the last three. The value of the first of these annuities at the end of 50 years is represented by $s_{\overline{50}|.07}$ and hence, by Theorem I, its value at the end of 53 years is $s_{\overline{50}|.07}(1.07)^3$. The value of the second annuity at the end of its term is represented by $s_{\overline{3}|.07}$. It follows that $s_{\overline{53}|.07} = s_{\overline{50}|.07}(1.07)^3 + s_{\overline{3}|.07}$. This same method of reasoning leads directly to the general relations

$$s_{\overline{n+n_1}|i} = s_{\overline{n}|i}\,(1 + i)^{n_1} + s_{\overline{n_1}|i} \quad \ldots\ldots \quad (6)$$

$$a_{\overline{n+n_1}|i} = a_{\overline{n}|i}\,(1 + i)^{-n_1} + a_{\overline{n_1}|i} \quad \ldots\ldots \quad (7)$$

When $n$ and $n_1$ are found in the annuity tables but $n + n_1$ is not, these relations can be used to compute the amount and the present value of an annuity whose term is outside the range of the table. For example, $a_{\overline{120}|.06} = a_{\overline{60}|.06}\,(1.06)^{-60} + a_{\overline{60}|.06}$. In deriving formulas (6) and (7) the annuity whose term is $n + n_1$ years is resolved into two annuities whose terms are $n$ and $n_1$ years. Resolving a given annuity into two or more components is frequently of service in computations.

When $n_1$ is 1, formulas (6) and (7) take the special forms, since $s_{\overline{1}|i} = 1$ and $a_{\overline{1}|i} = \dfrac{1}{(1+i)}$,

$$s_{\overline{n+1}|i} = s_{\overline{n}|i}(1+i) + 1 \tag{6'}$$

$$a_{\overline{n}|i} = a_{\overline{n+1}|i}(1+i) - 1 \tag{7'}$$

These special formulas may be used in constructing tables of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$. For example, when $n$ is 1, formula (6′) determines $s_{\overline{2}|i}$, then when $n$ is 2, it determines $s_{\overline{3}|i}$, and so on. If $a_{\overline{50}|i}$ is computed from formula ($3_1$), then, when $n$ is 49, formula (7′) determines $a_{\overline{49}|i}$, then when $n$ is 48, it determines $a_{\overline{48}|i}$, and so on. In computing these tables by this process checks by direct computation should be made every 10 years or so to avoid errors.

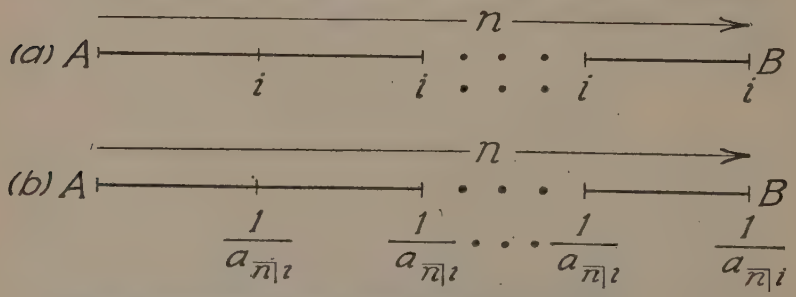
## EXERCISES

**1.** Verify by use of Tables V and VI that $s_{\overline{5}|.06}$ and $a_{\overline{5}|.06}$ satisfy the equation

$$\frac{1}{a_{\overline{5}|.06}} = \frac{1}{s_{\overline{5}|.06}} + .06$$

**2.** Verify that $\dfrac{1}{a_{\overline{n}|i}} = \dfrac{1}{s_{\overline{n}|i}} + i$ by substituting $a_{\overline{n}|i} = \dfrac{1-(1+i)^{-n}}{i}$ and $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$

**3.** Each section of the lines represents one year in the following diagrams:*




(1) Show that the sets of sums represented by these diagrams are equivalent at the interest rate $i$ by showing that each is equivalent to 1 at $A$.

*Insert "1" under last $i$ in diagram ($a$).

(2) Show that the value at $B$ of the set of sums represented by diagram ($a$) is $i\,s_{\overline{n}|i} + 1$, and by diagram ($b$) is $\dfrac{s_{\overline{n}|i}}{a_{\overline{n}|i}}$.

(3) Equate the results in (2) and show that $\dfrac{1}{a_{\overline{n}|i}} = \dfrac{1}{s_{\overline{n}|i}} + i$.

**4.** Show that:

($a$) $s_{\overline{5}|i} = s_{\overline{2}|i}(1+i)^3 + s_{\overline{3}|i} = s_{\overline{3}|i}(1+i)^2 + s_{\overline{2}|i} = s_{\overline{4}|i}(1+i) + 1$

($b$) $s_{\overline{n+1}|i} = s_{\overline{n}|i}(1+i) + 1 = (1+i)^n + s_{\overline{n}|i}$

($c$) $a_{\overline{5}|i} = a_{\overline{3}|i} + a_{\overline{2}|i}(1+i)^{-3} = a_{\overline{2}|i} + a_{\overline{3}|i}(1+i)^{-2} = (1 + a_{\overline{4}|i})(1+i)^{-1} = a_{\overline{4}|i} + (1+i)^{-5}$

($d$) $a_{\overline{n+1}|i} = (a_{\overline{n}|i} + 1)(1+i)^{-1} = a_{\overline{n}|i} + (1+i)^{-(n+1)}$

**5.** Find the value of

($a$) $s_{\overline{117}|.06}$ (Tables III and V)

($b$) $a_{\overline{153}|.06}$ (Tables IV and VI)

($c$) $\dfrac{1}{s_{\overline{117}|.06}}$

**24. Graphs of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$.** Each of the formulas for the values of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ involves three letters and when one of these letters is given, each formula expresses a relation between the other two. These relations can be represented graphically.

Figure 5 shows the graphs of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ for $i = .04$ and $n = 0, 1, 2 \cdots 14$. Tables V and VI were used in constructing them. It may be noted that the slope of the $s_{\overline{n}|i}$ graph increases as $n$ increases and that the slope of the $a_{\overline{n}|i}$ graph decreases as $n$ increases.

Figure 6 shows the graphs of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ for $n = 15$ and $i = .005, .01, 015 \cdots .07$.

It may be noted that the segments of the graphs in Figure 6 which lie between adjacent values of $i$ given in Tables V and VI are nearly straight lines.

Corresponding graphs of the more general functions $V_n$, $V_o$ may also be constructed; when $mr$ is different from one, it is necessary, however, to compute tables of values of $V_n$, $V_o$ for assigned values of $n$ or of $i$.

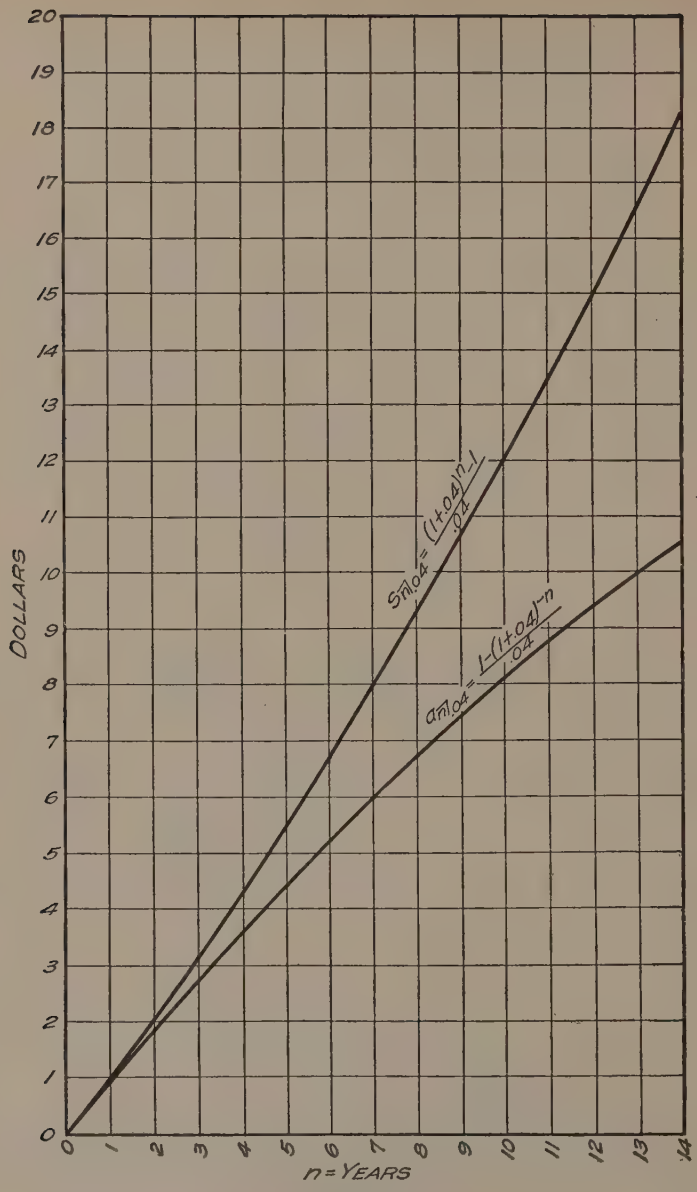



FIGURE 5

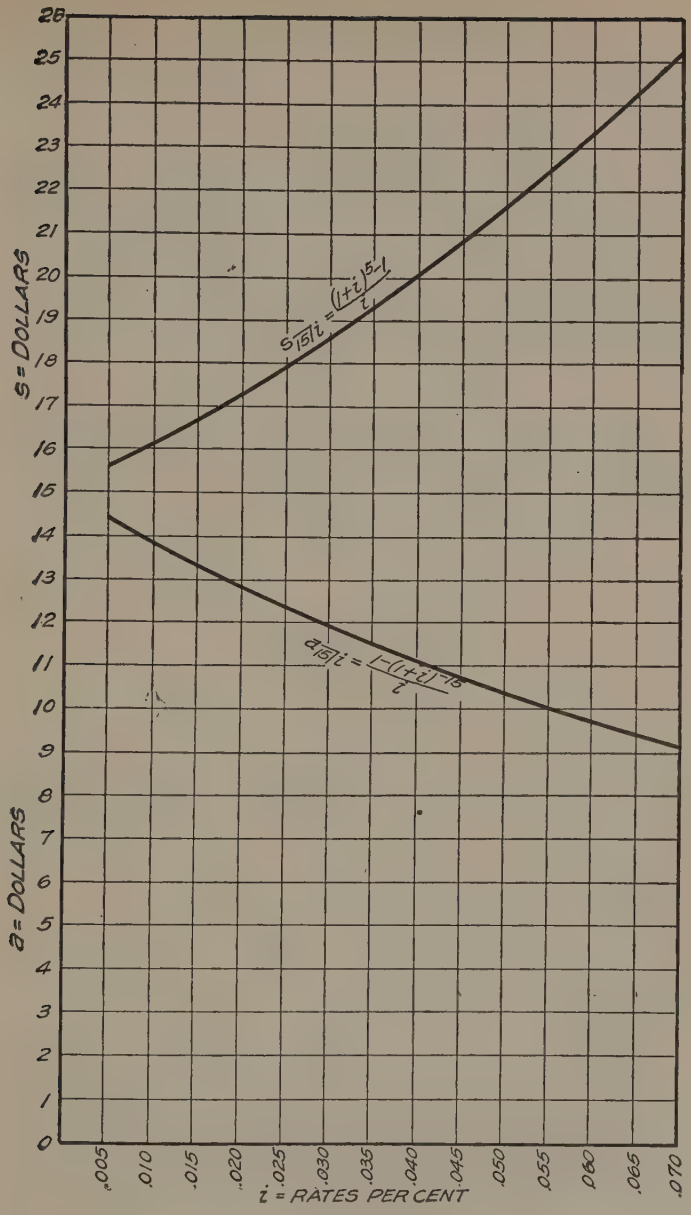

s = DOLLARS
a = DOLLARS
$s_{\overline{15}|i} = \frac{(1+i)^{15}-1}{i}$
$a_{\overline{15}|i} = \frac{1-(1+i)^{-15}}{i}$
i = RATES PER CENT
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
.005
.010
.015
.020
.025
.030
.035
.040
.045
.050
.055
.060
.065
.070

FIGURE 6

**25. Problems based on the annuity formulas.** Problems whose solutions are based directly on formulas (1), (2), (3), and (4) have $V_n$, $V_o$, $R$, $n$, or $j$ for the unknown; $m$ is given. The processes needed to find these unknowns are similar to those used in Art. 12, Chapter I, in solving problems based on the compound interest formula except when the interest rate is the unknown. In Arts 18, 19, and 21 some examples have been given in which $V_n$ or $V_o$ is the unknown. In this article some examples will be solved in which $R$ or $n$ is the unknown. The determination of the interest rate will be treated in Art. 26.

EXAMPLE 1. A man desires to accumulate \$4000 in 12 years by investing equal amounts at the end of each year. Find the annual investment if he can realize 4% converted quarterly.

SOLUTION. By formula (3)

$$4000 = R\frac{s_{\overline{48}|.01}}{s_{\overline{4}|.01}}$$

Solving,

$$R = 4000\, s_{\overline{4}|.01}\ \frac{1}{s_{\overline{48}|.01}}$$

$$= 4000(4.060401)(.01633384)$$

$$= \$265.29$$

EXAMPLE 2. An investment of \$5000, bearing 4% payable annually is to be used, principal and interest, to repair a building at the end of each 5 years for 30 years. If the same amount is spent each time repairs are made, find this amount.

SOLUTION. By formula (4)

$$5000 = R\frac{a_{\overline{30}|.04}}{s_{\overline{5}|.04}}$$

Solving,

$$R = 5000\, s_{\overline{5}|.04}\frac{1}{a_{\overline{30}|.04}}$$

$$= 5000(5.41632256)(.0578301)$$

$$= \$1566.13$$

EXERCISE. Check the value of $R$ by constructing a schedule showing the outstanding principal at the end of each 5 years.

EXAMPLE 3. What payment made at the end of each six months for 3 years will discharge a debt of \$1200 at ($j = .06$, $m = 1$).

SOLUTION. By formula (4)

$$1200 = R\frac{a_{\overline{3}|.06}}{s_{\overline{\frac{1}{2}}|.06}}$$

Solving,* $$R = 1200\frac{s_{\overline{\frac{1}{2}}|.06}}{a_{\overline{3}|.06}}$$

$$= 1200\frac{1}{a_{\overline{3}|.06}}\frac{(1+.06)^{\frac{1}{2}}-1}{.06}$$

$$= 20000(.37410981)(.02956302) \qquad \text{(Tables VII and VIII)}$$

$$= \$221.20$$

EXERCISE. Solve Example 3 by use of formula 2.

EXAMPLE 4. If instalments of \$300 including principal and interest are paid at the end of each six months on a debt of \$5000 bearing 6% payable semi-annually, find how many instalments are needed before the principal of the debt is less than \$300. Also find the payment necessary to extinguish the debt at the end of the next six months.

SOLUTION. By formula (2)

$$5000 = 300\frac{1-(1.03)^{-2n}}{.03}$$

Solving, $$1.03^{2n} = 2$$

$$2n = \frac{\log 2}{\log 1.03} = 23.44$$

This result indicates that 23 instalments of \$300 each are needed to reduce the debt to less than \$300. Let $x$ be the additional payment at the end of 12 years which extinguishes the debt. Equating the present value of the payments to 5000 gives

$$\frac{x}{(1.03)^{24}} + 300\,a_{\overline{23}|.03} = 5000$$

Solving, $$x = \$136.029$$

EXERCISE 1. Find $x$ by equating the value of the payments at the time the 23d instalment is made to the value of the debt at that time.

EXERCISE 2. Find $x$ by constructing a schedule showing the amount of the debt after each payment is made.

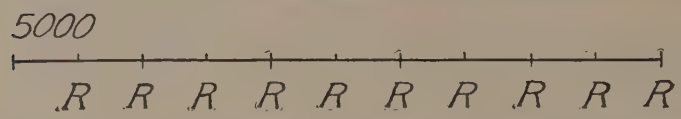
EXERCISE 3. Find $2n$ by interpolating in Table VI. (Hint. In this case $a_{\overline{2n}|.03} = 16.66666667$.)

EXERCISE 4. Solve $s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$ for $n$ in terms of $i$ and $S_{\overline{n}|i}$

*Notice that both interest and annuity tables are used in this computation.

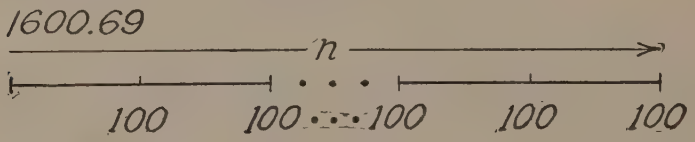
### EXERCISES

**1.** In the following diagram each section of the line represents six months:




Find $R$ if the set of sums consisting of the $R$'s is equivalent to that consisting of the single sum of \$5000. Use ($j = .06$, $m = 2$). Ans. \$586.15.

**2.** Each section of the line represents six months:




If ($j = .06$, $m = 1$), find $n$ if the two sets of sums are equivalent. Ans. 11 years.

**3.** A savings deposit of \$5 is made at the end of each week for 10 years. Find the amount of the deposits if ($j = .04$, $m = 4$). Ans. \$3192.27.

**4.** A man buys a house for \$5500. He pays \$500 cash, and agrees to pay \$60 at the end of each month thereafter until the debt is paid. If the interest rate is ($j = .06$, $m = 12$), how much will still be due at the end of four years? at the end of six years? How many payments will reduce the debt to less than \$60? If the debt is paid in full one month after it is reduced to less than \$60, find the last payment. Ans. \$3106.58; \$1975.69; 108; \$4.12.

**5.** The present value of an annuity of \$10 paid at the end of each six months for $n$ years at ($j = .05$, $m = 2$) is \$194.64. Find $n$, (*a*) by use of Table I, (*b*) by use of Table VI. Ans. 13.5 years.

**6.** The amount of an annuity of \$10 paid at the end of each six months for $n$ years at ($j = .05$, $m = 2$) is \$333.16. Find $n$, (*a*) by use of Table I, (*b*) by use of Table V. Interpret the meaning of the fractional value of $n$.
Ans. 12.27 years.

**7.** The rent of a house is \$60 a month. If the rent is to be paid at the end of each month, find the equivalent annual rent payable at the beginning of the year if ($j = .06$, $m = 1$). Ans. \$697.73.

**8.** A debt of \$5000 is extinguished in 5 years by equal monthly payments. If ($j = .06$, $m = 4$) and the payments are made at the end of each month, find the amount of each payment.

**9.** What is the cash price of a piano which is equivalent to \$50 cash and monthly payments of \$25 thereafter until \$500 in all is paid. Use ($j = .08$, $m = 4$). Ans. \$472.19.

**10.** What sum deposited in a savings bank at the end of each month for 8 years will amount to \$1000, if ($j = .04$, $m = 4$). Ans. \$8.86.

**11.** A cash payment of \$3500 was made on a farm bought for \$10000. What payment made at the end of each year for six successive years paid the balance in full, if ($j = .055$, $m = 2$)? Ans. \$1304.28.

**26. Methods for finding the interest rate.** In this problem the equation to be solved is gotten by substituting the known values of $V_n$ or $V_o$, $n$, $r$, and $m$ into formulas (1) or (2). In practice, the value of $j$ determined by one of these equations is usually an irrational number. There are two methods in common use for solving such equations. One of these is the *interpolation method;* and the other is *Newton's method.* For the accuracy needed in many problems of this type in finance, the interpolation method is preferable. A third method, well suited for solving rate equations, is the *method of iteration.**

In problems in which interpolation has been used heretofore to determine an unknown approximately, there has been just one interpolation for the unknown and the tables used for interpolating have been given; they have been logarithmic, interest and annuity tables. In problems in finance in which the rate is determined to a desired degree of accuracy by the method of interpolation more than one interpolation are often needed and the tables used for interpolating and for testing the accuracy of any approximation must be constructed. In a rate problem based on formula (1) or on formula (2), the relation between $V_n$ or $V_o$ and $j$, gotten by substituting the known values of $m$, $n$, and $r$ into the one or the other of these formulas, can be used to construct such a table of values of $V_n$ or $V_o$ corresponding to assigned values of $j$.

Newton's method consists in replacing $j$ in the equation to be solved by $j' + h$, where $j'$ is any approximation already found and $h$ is the correction or error, arranging the resulting equation in ascending powers of $h$, dropping the terms which contain powers of $h$ greater than one, and solving for $h$. Then $j' + h$ is a closer approximation than $j'$. By repeating this process the root can

* See papers in Volume XXXII, 1925, of the *American Mathematical Monthly* by C. H. Forsythe, March Number, page 126, and by L. R. Ford, June–July Number, page 272.

be found to any desired degree of accuracy. The initial value of $j'$ is usually gotten by selecting one of two numbers between which the root lies in a table constructed as in the method of interpolation, by selecting the approximation found by one interpolation in such a table, or by use of some approximation formula.

By means of a graph the geometric significance of an interpolation and also of an application of Newton's process may be seen. Let $f(x) = k$, where $k$ is a constant, be any equation having a root, $x$, between the two numbers $a$ and $b$, and let Figure 7 represent the graph of $y = f(x)$ from $a$ to $b$ inclusive, $MP$ being the ordinate of length $k$, $OD$ being the abscissa $a$, and $OE$ the abscissa $b$.

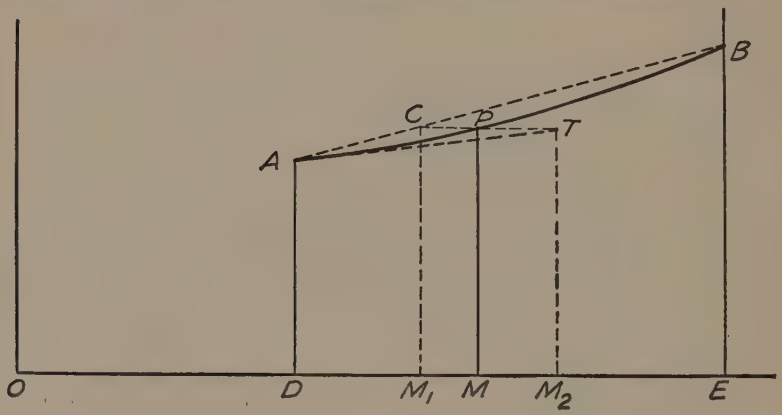



FIGURE 7.

To solve the equation $f(x) = k$ is to find the value of $OM$. Since $OM = a + DM$, this is equivalent to finding the value of $DM$. In the interpolation method the arc $APB$ is replaced by the chord $AB$, the given ordinate, $MP$, takes the position $M_1C$, and $DM_1$ is found instead of $DM$. In Newton's method, if $a$ is the approximation used, the arc $APB$ is replaced by its tangent at $A$, the given ordinate, takes the position $M_2T$, and $DM_2$ is found instead of $DM$. When the arc $APB$ is nearly straight both methods give excellent approximations.

The solution by the method of iteration of a rate equation in $j$ is based on writing the equation in a form $j = f(j)$, where $f(j)$ is a

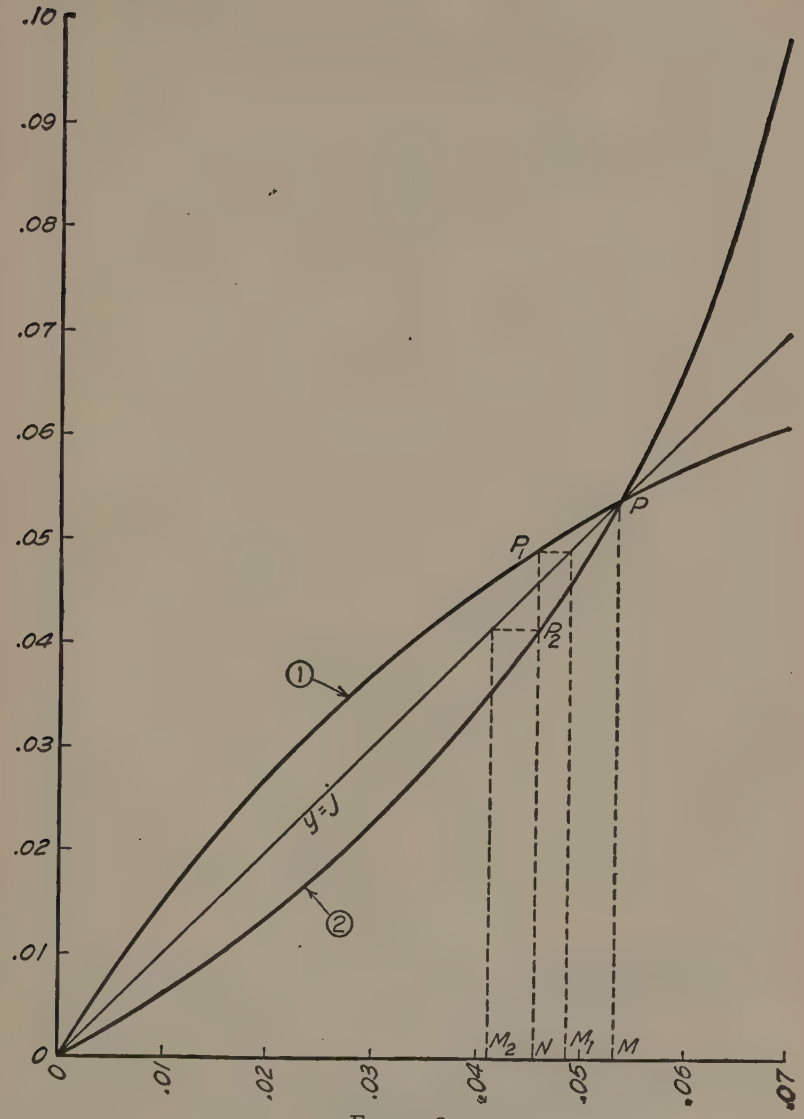

.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
0
.01
.02
.03
.04
.05
.06
.07
P
P1
P2
①
②
y=j
M2
N
M1
M

FIGURE 8.

function of $j$ whose value, $j'' = f(j')$, for a given approximate value $j'$ of $j$ is a closer approximation to the root than $j'$; likewise $f(j'')$ is a closer approximation than $j''$. By repetition or iteration of the process of computing successive values of $f(j)$ the root can be found to the accuracy desired. The rate equations which arise in elementary finance can usually be written in the form $j = f(j)$ in more than one way. For example, the equation $12102 = 1000\dfrac{1-(1+j)^{-20}}{j}$ can be solved readily for either $j$ which occurs in it, and hence it may be written in the forms: $j = \dfrac{1-(1+j)^{-20}}{12.102}$; $j = (1 - 12.102\,j)^{-\frac{1}{20}} - 1$. The first of these forms is suited to the method of iteration, but the second is not. This may be seen by observing the graphs of

$$(1)\ y = \frac{1-(1+j)^{-20}}{12.102} \text{ and } (2)\ y = (1 - 12.102\,j)^{-\frac{1}{20}} - 1.$$

To solve either of the two forms of this rate equation is to find the abscissa $OM$ of the point $P$ in which either graph (1) or graph (2), Figure 8, is cut by the line $y = j$. From graph (1) it is seen that the first form of the $j$ equation replaces any approximation $ON$ by $NP$, $= OM_1$, which is a closer approximation to $OM$ than $ON$; from graph (2) it is seen that the second form of the $j$ equation replaces the approximation $ON$ by $NP_2 = OM_2$ which is not as good an approximation to $OM$ as $ON$. Observation of the graph of a like form of any rate equation will show if the form is suited to the method of iteration.

Some examples will now be solved to illustrate these methods.

Example 1. An annuity of \$100 per year has a value of \$2065 at the end of 15 years at the interest rate $i$ converted annually. Find the interest rate $i$.

Substituting into formula (1) and dividing both members by 100 gives, for the equation to be solved,

$$\frac{(1+i)^{15}-1}{i} = 20.65 \quad \text{or} \quad s_{\overline{15}|i} = 20.65$$

Solution by interpolation. In this case $s_{\overline{15}|i} = \dfrac{(1+i)^{15}-1}{i}$ can be used to construct a table of values for interpolation. By Table IV $s_{\overline{15}|.04} = 20.02358764$ and $s_{\overline{15}|.045} = 20.78405429$. Hence $i$ lies between .04

and .045. Interpolation from the table

| $i$ | $s_{\overline{15}\rceil i}$ |
|---|---|
| .04 | 20.0236 |
| $i$ | 20.65 |
| .045 | 20.7841 |

gives $\dfrac{i - .04}{.045 - .040} = \dfrac{.6264}{.7605}$.

Solving, $i = .0441$

Figure (6) in Art. 24 shows that the graph of $s_{\overline{15}\rceil i}$ is similar to that in Figure (7). It follows that the root to be found is greater than .0441. By use of a 7-place table of logarithms it is found that $s_{\overline{15}\rceil .0442} = 20.660$. Hence the root lies between .0441 and .0442, so that .0441 is correct to four decimals. Computing $s_{\overline{15}\rceil .0441}$ and interpolating from the table

| $i$ | $s_{\overline{15}\rceil i}$ |
|---|---|
| .0441 | 20.644 |
| $i$ | 20.65 |
| .0442 | 20.660 |

gives $\dfrac{i - .0441}{.0442 - .0441} = \dfrac{6}{16}$.

Solving, $i = .04413$

It follows again that the root is greater than .04413. This is seen also from the fact that $s_{\overline{15}\rceil .04413} = 20.649$. The root is less than .04414 since $s_{\overline{15}\rceil .04414} = 20.650+$. Hence the root lies between .04413 and .04414.

The separate tables showing pairs of values of $s_{\overline{15}\rceil i}$ and $i$ which satisfy the relation $s_{\overline{15}\rceil i} = \dfrac{(1 + i)^{15} - 1}{i}$ can be combined into the single table

| $i$ | $s_{\overline{15}\rceil i}$ |
|---|---|
| .04 | 20.0236 |
| .0441 | 20.644 |
| .04413 | 20.649 |
| $i$ | 20.650 |
| .04414 | 20.650+ |
| .0442 | 20.660 |
| .045 | 20.7841 |

Finding the first five digits of the root is equivalent to constructing this table of pairs of corresponding values of $s_{\overline{15}\rceil i}$ and $i$. The numbers in the first and the last rows are the first pair, those in the second and the next to the last are the second pair, and so on. A similar table can be constructed for any rate equation.

Solution by Newton's Method. The equation to be solved may be written

$$(1 + i)^{15} - 20.65\, i - 1 = 0$$

One interpolation shows that the root is .0441 approximately. Replacing $i$ by $.0441 + h$, expanding by the binomial theorem, and dropping powers of $h$ higher than the first gives

$$(1.0441)^{15} - 1 - 20.65(.0441) + (15(1.0441)^{14} - 20.65)\, h = 0$$

On computing by seven-place logarithms, this gives

$$h = .000034 \text{ or } .000035$$
$$i = .044134 \text{ or } .044135$$

A second application of Newton's method, using $i = .04413 + h$, shows that $h = .000004$ or $.000005$. A seven-place table of logarithms is not of sufficient extent to show which of these values of $h$ is correct.

EXAMPLE 2. The present value of an annuity of \$1000 per year for twenty years is \$12102. Find the interest rate $i$ if interest is converted annually.

Substituting into formula (2) and dividing both members by 1000 gives for the equation to be solved,

$$\frac{1-(1+i)^{-20}}{i} = 12.102 \quad \text{or} \quad a_{\overline{20}|i} = 12.102.$$

SOLUTION BY INTERPOLATION. In this case $a_{\overline{20}|i} = \dfrac{1-(1+i)^{-20}}{i}$ can be used to construct a table of values for interpolation. By means of Table VI and a seven-place table of logarithms, the following table of pairs of values of $a_{\overline{20}|i}$ and $i$ can be readily computed:

| $i$ | $a_{\overline{20}\|i}$ |
|---|---|
| .05 | 12.4622 |
| .0534 | 12.1106 |
| $i$ | 12.1020 |
| .0535 | 12.1005 |
| .0550 | 11.9504 |

This table shows that the root lies between .0534 and .0535. Another interpolation gives $i = .05349$.

SOLUTION BY ITERATION.* As seen by the solution by interpolation the root $i$ lies between .05 and .055. The equation to be solved can be written in the following form, which, by the above discussion, is suited to the method of iteration:

$$i = \frac{1-(1+i)^{-20}}{12.102}$$

Employing this form and using arrows to indicate substitutions, the work can be outlined as follows:

$$.05 \longrightarrow .0515 \qquad .055 \longrightarrow .0534$$
$$\text{Average} = .0529$$
$$.053 \longrightarrow .0532 \qquad .054 \longrightarrow .0538$$
$$\text{Average} = .0535$$
$$.0535 \longrightarrow .05349$$

## EXERCISES

1. An annuity having $R = 10$, $n = 5$, $r = \frac{1}{2}$ has an amount of \$112.30. Find $\frac{j}{2}$ correct to four places of decimals if $m = 2$. Ans. .0255.

2. An annuity having $R = 100$, $n = 5$, $r = \frac{1}{4}$ has a present value of \$1700. Find $\frac{j}{4}$ correct to four decimals if $m = 4$.

* See paper by Forsythe referred to on page 69.

**3.** An electric washer was priced at \$100 cash or \$5 cash and \$8.50 at the end of each month for twelve months. If $m = 12$, find $\frac{j}{12}$ correct to five decimals, which will make the two prices equivalent. Ans. .01111.

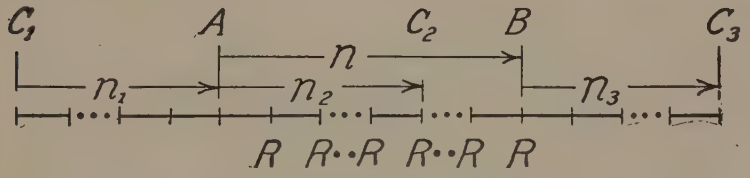
**27. The value of an annuity at any time. Deferred and forborne annuities.** An annuity whose term begins $n_1$ years after a specified time is called a *deferred annuity* with respect to this time; it is said to be deferred $n_1$ years. An annuity whose term begins $n_1$ years before a specified time is called a *forborne annuity;* it is said to be forborne $n_1$ years. Formulas (1), (3) and (2), (4) give the values of an annuity at the end and at the beginning of its term.

The expression for the value of a deferred or a forborne annuity can be written at once by use of either one of the following methods:

*Method 1. To find the value of an annuity at any time, apply Theorem I, Art. 15, Chapter I to $V_o$ or $V_n$.*

*Method 2. To find the value of an annuity at any time, resolve the annuity into the difference or sum of two annuities each of which has the beginning or end of its term at the given time, and then find the difference or sum of the values of these component annuities at this time.*

The following line diagram can be used to visualize these two methods:




$AB$ represents an annuity of rent $R$, term $n$ years, and rent period $r$ years. The arrows indicate increasing time. The value of this annuity at $C_1$ will be denoted by $V_{-n_1}$, at $A$ by $V_o$, at $C_2$ by $V_{n_2}$, at $B$ by $V_n$, and at $C_3$ by $V_{(n+n_3)}$.

By Method 1, the value at $C_1$ of the annuity $AB$ deferred $n_1$ years is given by

$$V_{-n_1} = V_o\left(1 + \frac{j}{m}\right)^{-mn_1}$$

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By Method 2, the value at $C_1$ of the annuity $AB$ deferred $n_1$ years can be found by first filling in the missing values of $R$ between $C_1$ and $A$ and then subtracting the present value of the annuity $C_1A$ from that of the annuity $C_1B$. This gives, using formula (4),

$$V_{-n_1} = R\frac{a_{\overline{m(n+n_1)}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} - R\frac{a_{\overline{mn_1}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} = R\frac{a_{\overline{m(n+n_1)}|\frac{j}{m}} - a_{\overline{mn_1}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

Analogous expressions for the values of the annuity $AB$ at $C_2$ and $C_3$ are given in Exercises 3 and 4 at the end of this article.

Some examples will now be solved to illustrate these methods.

EXAMPLE 1. Find the present value of an annuity of \$100 every six months for five years, deferred three years, at ($j = .055$, $m = 2$).

SOLUTION BY METHOD 1.

$$\begin{aligned} V_{-3} &= V_0\,(1.0275)^{-6} \\ &= 100\,a_{\overline{10}|.0275}\,(1.0275)^{-6} \\ &= \$734.22 \end{aligned}$$

SOLUTION BY METHOD 2.

$$\begin{aligned} V_{-3} &= 100(a_{\overline{16}|.0275} - a_{\overline{6}|.0275}) \\ &= \$734.22 \end{aligned}$$

EXAMPLE 2. Find the value of an annuity of \$100 every six months for five years, forborne eight years, that is, three years after the payments are discontinued, at ($j = .06$, $m = 2$).

SOLUTION BY METHOD 1.

$$\begin{aligned} V_3 &= V_5(1.03)^6 \\ &= 100\,s_{\overline{10}|.03}\,(1.03)^6 \\ &= \$1368.85 \end{aligned}$$

SOLUTION BY METHOD 2.

$$\begin{aligned} V_3 &= 100(s_{\overline{16}|.03} - s_{\overline{6}|.03}) \\ &= \$1368.85 \end{aligned}$$

EXAMPLE 3. Find the value of the annuity in Example 1 if it is forborne two years, that is, find its value at the time of the fourth payment.

SOLUTION BY METHOD 1.

$$\begin{aligned} V_2 &= V_0(1.0275)^4 \\ &= 100\,a_{\overline{10}|.0275} \cdot (1.0275)^4 \\ &= \$963.04 \end{aligned}$$

SOLUTION BY METHOD 2.

$$V_2 = 100(s_{\overline{4}|.0275} + a_{\overline{6}|.0275})$$
$$= \$963.04$$

Exercises 1–4 which follow refer to the annuity $AB$ diagrammed above.

EXERCISE 1. Show that

$$V_{-n_1} = R\frac{\left(1+\frac{j}{m}\right)^{-mn_1} - \left(1+\frac{j}{m}\right)^{-m(n+n_1)}}{\left(1+\frac{j}{m}\right)^{mr}-1}$$ By Method 1.

$$= R\frac{\frac{1-\left(1+\frac{j}{m}\right)^{-m(n+n_1)}}{\frac{j}{m}} - \frac{1-\left(1+\frac{j}{m}\right)^{-mn_1}}{\frac{j}{m}}}{\frac{\left(1+\frac{j}{m}\right)^{mr}-1}{\frac{j}{m}}}$$

$$= R\frac{a_{\overline{m(n+n_1)}|\frac{j}{m}} - a_{\overline{mn_1}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

This result shows that the two methods give equivalent results in case of a deferred annuity.

EXERCISE 2. When $R = r = 1$, $V_{-n_1}$ is denoted by $n_1 \mid a_{\overline{n}|i}$; when $R = r = \frac{1}{p}$, $V_{-n_1}$ is denoted by $n_1 \mid a^{(p)}_{\overline{n}|i}$. Show by use of the result of Exercise 1, that

$$n_1 \mid a_{\overline{n}|i} = a_{\overline{n+n_1}|i} - a_{\overline{n_1}|i}$$
$$n_1 \mid a^{(p)}_{\overline{n}|i} = a^{(p)}_{\overline{n+n_1}|i} - a^{(p)}_{\overline{n_1}|i}$$

EXERCISE 3. Show that, $(n_2 < n)$

$$V_{n_2} = R\frac{\left(1+\frac{j}{m}\right)^{mn_2} - \left(1+\frac{j}{m}\right)^{m(n_2-n)}}{\left(1+\frac{j}{m}\right)^{mr}-1}$$

$$= R\frac{s_{\overline{mn_2}|\frac{j}{m}} + a_{\overline{m(n-n_2)}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

EXERCISE 4. Show that

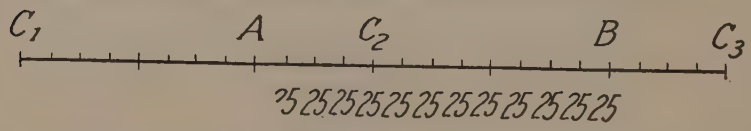
$$V_{(n+n_2)} = R\frac{\left(1+\frac{j}{m}\right)^{m(n+n_2)} - \left(1+\frac{j}{m}\right)^{mn_2}}{\left(1+\frac{j}{m}\right)^{mr} - 1}$$

$$= R\frac{s_{\overline{m(n+n_2)}|\frac{j}{m}} - s_{\overline{mn_2}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

EXERCISE 5. Use the results of Exercises 1, 3, and 4 to show that Methods 1 and 2 give the same result for the value of an annuity at any time.

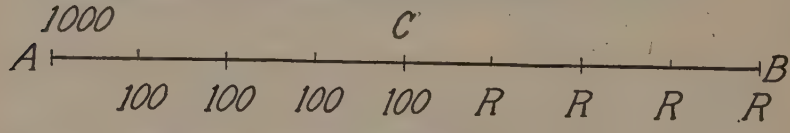
## EXERCISES

1. Each section of the following diagram represents three months:




Find the values of the annuity $AB$ at $C_1$, at $C_2$, at $C_3$, (*a*) at the rate ($j = .045$, $m = 4$); (*b*) at the rate ($j = .06$, $m = 12$); (*c*) at the rate ($j = .045$, $m = 2$). How long is the annuity $AB$ deferred with reference to $C_1$? How long is it forborne with reference to $C_2$, and how long with reference to $C_3$?

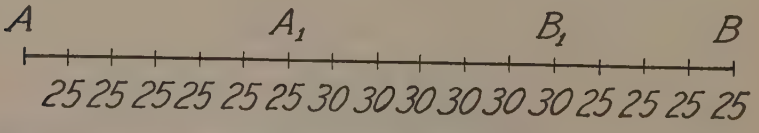
2. Each section of the following diagram represents six months:




The set of sums below the line is equivalent to that above it. Find $R$ by equating the values of these sets at $A$; at $C$; at $B$. Use ($j = .05$, $m = 2$).

Ans. \$183.03.

3. Find the values at $A$ and at $B$ of the set of sums, represented by the diagram below, where each section of the line represents one year, at ($j = .06$, $m = 2$).

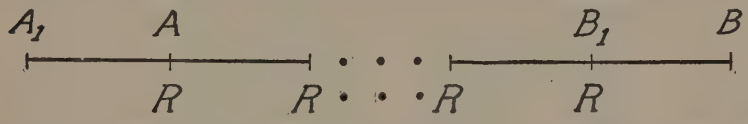



[Resolve the set of sums into the annuity $AB$ whose rent is \$25 and the annuity $A_1B_1$ whose rent is \$5]. Ans. \$268.29; \$690.87.

4. A man purchased a house for \$6800, paying \$1200 cash, and agreeing to make monthly payments on the balance. He paid \$50 for 65 months, \$40 for 15 months, and \$55 for 20 months. What was the amount of the debt after the last payment if ($j = .06$, $m = 12$)? How many additional \$55 payments will reduce the debt to less than \$55? If the debt is paid in full one month after it is reduced to less than \$55, find the last payment.

Ans. \$2821.43; 59; \$23.33.

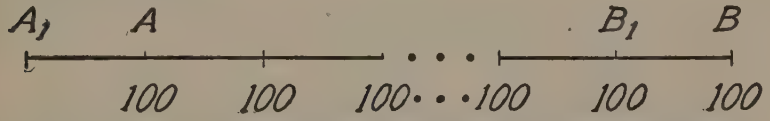
**28. The value of an annuity due at any time.** In an annuity due the rent is paid at the beginning of each rent period; in an annuity it is paid at the end of each rent period. It follows that an annuity due of given rent, rent period, and term, is equivalent to an annuity of the same rent, the same rent period, but whose term begins and ends one rent period earlier. A line diagram may be used to represent the annuity due and its equivalent annuity:




In this diagram $AB$ represents an annuity due whose term begins at $A$ and ends at $B$, and $A_1B_1$ represents the equivalent annuity whose term begins at $A_1$ and ends at $B_1$. The value at a given time of any annuity due, $AB$, is then the value at this time of the annuity $A_1B_1$. It follows that the methods in Art. 27 are directly applicable to the problem of finding the value at a given time of an annuity due.

EXAMPLE 1. Find the present value of an annuity due having $R = 100$, $n = 20$, $r = \frac{1}{2}$ at ($j = .06$, $m = 2$).

In the following diagram * let $AB$ represent this annuity due and $A_1B_1$ the equivalent annuity.




The present value of the annuity due, $AB$, is then the value at $A$ of the annuity $A_1B_1$.

SOLUTION BY METHOD 1.

$$V_{\frac{1}{2}} = V_0(1.03)$$
$$= 100\, a_{\overline{40}|.03}\,(1.03)$$
$$= \$2380.82$$

* Delete the 100 under $B$.

SOLUTION BY METHOD 2.

$$V_{\frac{1}{2}} = 100 + 100\, a_{\overline{39}|.03}$$
$$= \$2380.82$$

EXAMPLE 2. Find the value of the annuity due in Example 1, if it is deferred 10 years; if it is forborne 30 years.

SOLUTION BY METHOD 1.

$$V_{9\frac{1}{2}} = V_o\,(1.03)^{-19}$$
$$= 100\, a_{\overline{40}|.03}\,(1.03)^{-19}$$
$$= \$1318.20$$

$$V_{30\frac{1}{2}} = V_o\,(1.03)^{61}$$
$$= 100\, a_{\overline{40}|.03}\,(1.03)^{61}$$
$$= \$14026.86$$

SOLUTION BY METHOD 2.

$$V_{9\frac{1}{2}} = 100 + 100(a_{\overline{39}|.03} - a_{\overline{19}|..03})$$
$$= \$1318.20$$

$$V_{30\frac{1}{2}} = 100(s_{\overline{61}|.03} - s_{\overline{21}|.03})$$
$$= \$14026.86$$

The symbols used for the value of an annuity due are the same as those for an annuity with the exception that $a$ is replaced by $\mathbf{a}$ and $s$ by $\mathbf{s}$. For example, $\mathbf{a}_{\overline{n}|}$ represents the present value, and $\mathbf{s}_{\overline{n}|}$ the amount of an annuity due of rent 1 payable annually for $n$ years.

EXERCISE 1. Show that

$$\mathbf{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i} = a_{\overline{n}|i}\,(1+i)$$
$$\mathbf{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1 = s_{\overline{n}|i}\,(1+i)$$

[Use methods 1 and 2, Art. 27.]

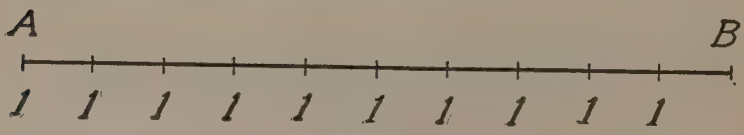
EXERCISE 2. Show that

$$\mathbf{a}^{(p)}_{\overline{n}|i} = \frac{1}{p} + a^{(p)}_{\overline{n-\frac{1}{p}}|i} = a^{(p)}_{\overline{n}|i} \cdot (1+i)^{\frac{1}{p}}$$

$$\mathbf{s}^{(p)}_{\overline{n}|i} = s^{(p)}_{\overline{n+\frac{1}{p}}|i} - \frac{1}{p} = s^{(p)}_{\overline{n}|i} \cdot (1+i)^{\frac{1}{p}}$$

## EXERCISES

1. Each section of the diagram represents one year:




Find the value of the annuity due $AB$ at $A$, and at $B$, if $(j = .06, m = 1)$.

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**2.** Show that the annuity due $AB$ of annual rent 1 in Exercise 1 is equivalent to the annuity $AB$ of annual rent 1.06 if ($j = .06$, $m = 1$). Use this fact to show that

$$s_{\overline{10}|.06} = (1.06)\, s_{\overline{10}|.06}$$
$$a_{\overline{10}|.06} = (1.06)\, a_{\overline{10}|.06}$$

**3.** Show that an annuity due of rent $R$ is equivalent to an annuity of rent $R\left(1+\frac{j}{m}\right)^{mr}$, having the same term and rent period if the interest rate $j$ is converted $m$ times a year.

**4.** What is the annual rent payable in advance which is equivalent to a monthly rental of \$60 payable at the beginning of each month if ($j = .06$, $m = 1$)? (See Exercise 7, Art. 25.) Ans. \$701.12.

**5.** Deposits of \$100 were made in a savings bank at the beginning of each quarter year for 5 years. Find the amount of these deposits at the end of 5 years; at the end of 10 years; at the time of the last deposit. Use ($j = .04$, $m = 4$). Ans. \$2223.92; \$2713.60; \$2201.90.

**6.** Same as Exercise 5, except that the deposits were made at the beginning of each month. Ans. \$6649.70; \$8213.89; \$6627.68.

**29. The value of a perpetuity.** By the definition in Art. 1, a perpetuity is an annuity whose term becomes infinite. From this definition it follows that the value of a perpetuity at any time can be found by making $n$ infinite in the expression for the value of an annuity at the given time. The result of making $n$ infinite in any annuity formula can be written at once by noting that $L_{k\to\infty}\left(1+\frac{j}{m}\right)^{-k} = 0$ and $L_{k\to\infty}\left(1+\frac{j}{m}\right)^{k} \to \infty$ and hence that $a_{\overline{\infty}|\frac{j}{m}} = \dfrac{1}{\frac{j}{m}}$ and $s_{\overline{\infty}|\frac{j}{m}} \to \infty$.

For example, when $n \to \infty$, $V_o = R\dfrac{1}{\left(1+\frac{j}{m}\right)^{mr}-1}$ $(2_3)$

$$= R\frac{1}{\frac{j}{m}\, s_{\overline{mr}|\frac{j}{m}}} \qquad (4_3)$$

The value of a perpetuity at any time other than the beginning of its term can be found by Methods 1 and 2, Art. 11. In applying Method 2, the given perpetuity is resolved into the sum or difference of a perpetuity and an annuity.

EXERCISE 1. Show that, when $n \to \infty$, the value of a perpetuity deferred $n_1$ years is given by

$$V_{-n_1} = R\frac{\left(1+\frac{j}{m}\right)^{-mn_1}}{\left(1+\frac{j}{m}\right)^{mr}-1} = R\frac{\frac{1}{j}-a_{\overline{mn_1}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

EXERCISE 2. Show that, when $n \to \infty$, the value of a perpetuity forborne $n_1$ years is given by

$$V_{n_1} = R\frac{\left(1+\frac{j}{m}\right)^{mn_1}}{\left(1+\frac{j}{m}\right)^{mr}-1} = R\frac{s_{\overline{mn_1}|\frac{j}{m}}+\frac{1}{j}}{s_{\overline{mr}|\frac{j}{m}}}$$

EXAMPLE 1. Find the present value of the perpetuity ($R = 100$, $n \to \infty$, $r = 5$) at the rate ($j = .06$, $m = 2$).

SOLUTION. By formula $(4_3)$

$$V_o = 100\frac{1}{.03\, s_{\overline{10}|.03}}$$

$$= \frac{100}{.03}(.08723051) = \$290.77$$

EXERCISE. Solve Example 1 by using $V_o = \dfrac{100}{(1.03)^{10}-1}$ [Formula $(2_3)$]

EXAMPLE 2. Find the present value of the perpetuity ($R = 100$, $n \to \infty$, $r = \frac{1}{2}$) at the rate ($j = .06$, $m = 4$).

SOLUTION. By formula $(4_3)$

$$V_o = \frac{100}{.015}\frac{1}{s_{\overline{2}|.015}}$$

$$= \frac{100}{.015}(.49627792) = \$330.85$$

## EXERCISES

**1.** Find the present value of an annuity having $R = r = 1$ at ($j = .06$, $m = 1$) if $n = 10$; if $n = 100$; if $n = 150$; if $n \to \infty$.

Ans. \$7.36; \$16.62; \$16.66; \$16.67.

**2.** Find the present value of a perpetuity whose term is deferred 10 years if $R = 10$ and $r = \frac{1}{2}$ at ($j = .06$, $m = 4$); find the value if the term is forborne 10 years. Ans. \$737.76; \$2427.73.

**3.** How much could a railroad company afford to spend to eliminate a dangerous crossing requiring the attention of two watchmen at \$150 a month each? Use ($j = .05$, $m = 2$). Ans. \$72746.32.

**4.** If ($j = .05$, $m = 2$) what is the amount of an endowment which will provide a perpetuity of \$1000 at the beginning of each year? Ans. \$20753.09.

**5.** By the terms of a will an annual perpetuity of \$1000 is equally divided between two charities. If the first charity receives the entire income until it has received its share, find the number of payments it receives and the amount of the last payment if ($j = .05$, $m = 2$). Ans. 29; \$35.48.

**6.** Derive formula ($2_3$) by use of the formula for the sum of a geometric progression when the number of terms become infinite.

**7.** Derive formula ($2_3$) by noting that $V_o$ is the principal of an investment which yields $R$ dollars of interest at the end of each $r$ years.

[In this type of investment, the principal is not returned.]

**8.** Solve Example 1 above by use of the principle stated in Exercise 7.

**30. The value of a continuous annuity.** A continuous annuity is one having an infinite number of rent payments each year and a fixed annual rent $K$; that is, in a continuous annuity $R = \frac{K}{p}$, and $p$ becomes infinite. In Art. 14, Chapter I, it was seen, by equating two expressions for the force of interest, that $L_{x\to\infty} x[(1+i)^{\frac{1}{x}} - 1] = \log_e(1+i)$. Upon replacing $x$ by $\frac{p}{m}$ and $i$ by $\frac{j}{m}$ and multiplying by $m$ it follows from this limit that

$$L_{p\to\infty} p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}} - 1\right] = m\log_e\left(1+\frac{j}{m}\right)$$

and hence that

$$L_{p\to\infty}\left(ps_{\overline{\frac{m}{p}}\,|\,\frac{j}{m}}\right) = \frac{m^2}{j}\log_e\left(1+\frac{j}{m}\right).$$

By means of these results the value of a continuous annuity at any time can be written from the formula for the value of an annuity at that time. For example,

$$\text{When } p\to\infty \text{ and } R = \frac{K}{p},\ V_o = K\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{m\log_e\left(1+\frac{j}{m}\right)} \qquad (2_4)$$

$$= K\frac{j}{m^2}\frac{a_{\overline{mn}\,|\,\frac{j}{m}}}{\log_e\left(1+\frac{j}{m}\right)} \qquad (4_4)$$

When $p \to \infty$ and $R = \frac{K}{p}$, $V_n = K \frac{\left(1 + \frac{j}{m}\right)^{mn} - 1}{m \log_e\left(1 + \frac{j}{m}\right)}$ $(1_4)$

$$= K \frac{j}{m^2} \frac{s_{\overline{mn}|\frac{j}{m}}}{\log_e\left(1 + \frac{j}{m}\right)} \qquad (3_3)$$

When $K = m = 1$ and $p \to \infty$, $V_o$ is denoted by $\bar{a}_{\overline{n}|i}$, and $V_n$ is denoted by $\bar{s}_{\overline{n}|i}$.

EXERCISE. Show that

$$\bar{a}_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{\log_e(1+i)} = \frac{i}{\delta} a_{\overline{n}|i}$$

$$\bar{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{\log_e(1+i)} = \frac{i}{\delta} s_{\overline{n}|i}$$

EXAMPLE. Find the present value of the continuous annuity ($K = 100$, $n = 10$) at the rate ($j = .06$, $m = 2$).

SOLUTION. By formula $(4_4)$

$$V_o = 100(.015) \frac{a_{\overline{20}|.03}}{\log_e(1.03)}$$

$$= \$754.99$$

## EXERCISES

**1.** At ($j = .06$, $m = 1$) find $V_n$ for each of the following annuities: $K = Rp = 1000$, $n = 10$, $\frac{1}{r} = p = 2, 12, 52, 365$ and for $\frac{1}{r} = p \to \infty$.

Ans. \$13375.62; \$13539.46; \$13564.82; \$13571.60; \$13572.38.

**2.** Compute the values of $1000\, a^{(365)}_{\overline{10}|.06}$ and $1000\, \bar{a}_{\overline{10}|.06}$

Ans. \$7578.31; \$7578.75.

**31. An extension of the compound interest formula.** The interest payments on a principal, $P$, bearing the interest rate $j$ payable $m$ times per year form an annuity whose rent is $\frac{Pj}{m}$ and whose rent period is $\frac{1}{m}$. By formula (3) the amount of this annuity at the rate $j$ converted $m$ times per year is given by $V_n = \frac{Pj}{m} s_{\overline{mn}|\frac{j}{m}}$. It follows that the compound amount of $P$ for $n$

years at the rate $j$ converted $m$ times per year can be written in the form, $S = P\left(1 + \frac{j}{m}\, s_{\overline{mn}|\frac{1}{m}}\right)$. The two formulas

$$S = P\left(1 + \frac{j}{m}\right)^{mn}$$

$$S = P\left(1 + \frac{j}{m}\, s_{\overline{mn}|\frac{1}{m}}\right)$$

give the same value for $S$. In the derivation of each of these formulas, the interest payments bear the same rate as the original principal.

If often happens in business transactions that the interest payments on $P$ are invested at rates which differ from the rate which $P$ yields. In such cases the above formulas are not applicable. The method used in getting the second formula is applicable, however, to any case. Let $P$ bear the rate $j$ payable $m$ times per year. When the interest payments bear the rate $j'$ payable $m$ times per year, the formula for $S$ becomes

$$S = P\left(1 + \frac{j}{m}\, s_{\overline{mn}|\frac{j'}{m}}\right)$$

When the interest payments bear the rate $j'$ payable $m'$ times per year the formula for $S$ becomes

$$S = P\left(1 + \frac{j}{m}\, \frac{s_{\overline{m'n}|\frac{j'}{m'}}}{s_{\overline{\frac{m'}{m}}|\frac{j'}{m'}}}\right)$$

Still more general formulas for $S$ can be gotten by separating the interest payments on $P$ into sets of one or more each, and then accumulating these sets at different rates by the use of the formulas for the amounts of a single sum, an annuity, and a forborne annuity. Many interesting formulas can be found in this way. In problems in elementary finance these formulas need not be used; it is better to make direct use of the method just stated, by means of which they can be written.

Each of these formulas, when solved for $P$, gives an analogous formula for the present value of a sum.

EXERCISE 1. If $S = P\left(1 + \frac{j}{m} s_{\overline{mn}|\frac{j}{m}}\right)$ show that

$$P = \frac{S}{\left(1 + \frac{j}{m} s_{\overline{mn}|\frac{j}{m}}\right)} = S\left(1 - \frac{j}{m} a_{\overline{mn}|\frac{j}{m}}\right).$$

EXERCISE 2. If $S = P(1 + i\, s_{\overline{n}|i'})$ show that

$$P = \frac{S}{(1 + i\, s_{\overline{n}|i'})} \neq S(1 - i\, a_{\overline{n}|i'}).$$

EXAMPLE. A \$1000 bond bearing 4% payable semi-annually and maturing in 5 years at par is bought at its face value. Find the amount of this investment at the end of the five years if the buyer can invest the interest payments (1) at $(j = .04, m = 2)$; (2) at $(j = .05, m = 2)$; (3) at $(j = .06, m = 4)$; (4) at $(j = .05, m = 2)$ during the first three years and at $(j = .06, m = 2)$ during the last two years.

SOLUTION. In this example the interest payments on the bond form the annuity $(R = 20, n = 5, r = \frac{1}{2})$. The amount of this annuity

at $(j = .04, m = 2)$ is $V_5 = 20\, s_{\overline{10}|.02} = 218.99$;

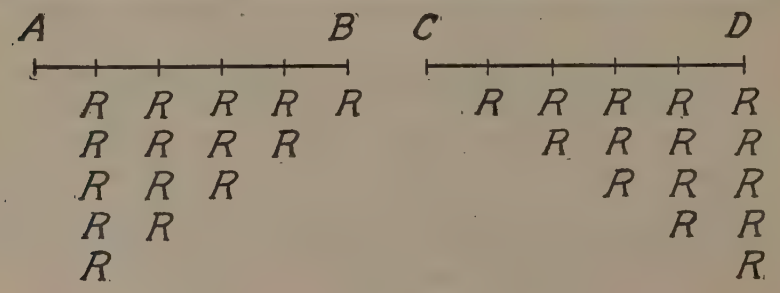
at $(j = .05, m = 2)$ is $V_5 = 20\, s_{\overline{10}|.025} = 224.07$;

at $(j = .06, m = 4)$ is $V_5 = 20\, s_{\overline{20}|.015} = 229.52$;

at $\begin{pmatrix} j = .05,\ m = 2 \text{ for 3 yrs.} \\ j = .06,\ m = 2 \text{ for 2 yrs.} \end{pmatrix}$ is $V_5 = 20(s_{\overline{10}|.025} - s_{\overline{4}|.025}) + 20\, s_{\overline{4}|.03} = 224.69.$

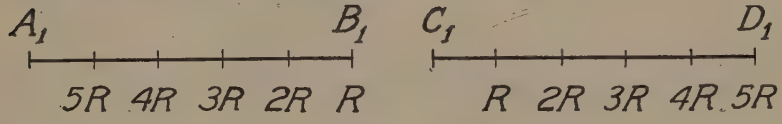
Hence the amounts of the investments are (1) \$1218.99, (2) \$1224.07, (3) \$1229.52, (4) \$1224.69.

**32. The value of a set of annuities. Increasing and decreasing annuities.** Some transactions in finance involve sets of annuities. The value at any given time of a set of annuities can be found by adding the values at that time of the separate annuities which compose the set. The values of some sets can be found more easily, however, by basing the computations on simplified forms



of the expressions for the sum of the values of the given sets. The line diagrams on page 86, each section of which denotes $r$ years, represent such sets.

The terms of the sets in $AB$ all begin at $A$; the terms of those in $CD$ all end at $D$. In each case the terms of the separate annuities are $r$, $2r$, $3r$, $4r$, $5r$. In general sets of these types, $5r$ is replaced by $n$. The separate sets represented by each of these diagrams can be combined into a single set by adding vertically. These single sets can be represented by the diagrams




When $5r$ is replaced by $n$ the set represented by $A_1B_1$ is called a *decreasing annuity* of term $n$, and that represented by $C_1D_1$ is called an *increasing annuity* of term $n$.

In this article simple formulas are found for the values of increasing and decreasing annuities of the kinds illustrated by the above diagrams. Symbols for the values of these increasing and decreasing annuities may be obtained by prefixing the letters $I$ and $D$ to the symbols used in the preceding articles. For example, $(Is_{\overline{n}|i})$ denotes the amount at the annual interest rate $i$ of an increasing annuity of term $n$ and rent period 1, whose successive rent payments are $1, 2, \cdots, n$.

The expression for the present value of a decreasing annuity, obtained by writing the sum of the present values of the annuities which compose it, as illustrated by the diagrams $A_1B_1$ and $AB$, can be changed into a simple form. The process is as follows:

$$(DV_o) = R\left\{\frac{a_{\overline{mr}|\frac{1}{m}}}{s_{\overline{mr}|\frac{1}{m}}} + \frac{a_{\overline{2mr}|\frac{1}{m}}}{s_{\overline{mr}|\frac{1}{m}}} + \cdots + \frac{a_{\overline{mn}|\frac{1}{m}}}{s_{\overline{mr}|\frac{1}{m}}}\right\} \text{ by formula (4),}$$

$$= \frac{R}{s_{\overline{mr}|\frac{1}{m}}} \cdot \frac{\frac{n}{r} - \left[\left(1+\frac{j}{m}\right)^{-mr} + \left(1+\frac{j}{m}\right)^{-2mr} + \cdots + \left(1+\frac{j}{m}\right)^{-mn}\right]}{\frac{j}{m}}$$

by formula $(2_1)$,

$$= \frac{R}{s_{\overline{mr}|\frac{1}{m}}} \frac{\frac{n}{r} - \frac{a_{\overline{mn}|\frac{1}{m}}}{s_{\overline{mr}|\frac{1}{m}}}}{\frac{j}{m}}$$

$$= R \frac{\frac{n}{r} - \frac{a_{\overline{mn}|\frac{1}{m}}}{s_{\overline{mr}|\frac{1}{m}}}}{\frac{j}{m} s_{\overline{mr}|\frac{1}{m}}} \tag{8}$$

When $mr = 1$ $\qquad (DV_o) = R \dfrac{mn - a_{\overline{mn}|\frac{1}{m}}}{\frac{j}{m}}$ $\qquad (8_1)$

When $m = 1, r = 1$ $\quad (DV_o) = R \dfrac{n - a_{\overline{n}|i}}{i}$ $\qquad (8_2)$

Exercise. Find a formula for $(DV_n)$, the amount of a decreasing annuity. [Use Theorem I and formula (8).]

The expression for the amount of an increasing annuity obtained by writing the sum of the amounts of the annuities which compose it, as illustrated by the diagrams $C_1D_1$ and $CD$, can be changed into a simple form. The process is analogous to that used in deriving formula (8).

$$(\mathrm{IV}_n) = R \left\{ \frac{s_{\overline{mr}|\frac{1}{m}}}{s_{\overline{mr}|\frac{1}{m}}} + \frac{s_{\overline{2mr}|\frac{1}{m}}}{s_{\overline{mr}|\frac{1}{m}}} + \cdots + \frac{s_{\overline{mn}|\frac{1}{m}}}{s_{\overline{mr}|\frac{1}{m}}} \right\} \text{ by formula (3)}$$

$$= \frac{R}{s_{\overline{mr}|\frac{1}{m}}} \frac{\left(1 + \frac{j}{m}\right)^{mr} + \left(1 + \frac{j}{m}\right)^{2mr} + \cdots + \left(1 + \frac{j}{m}\right)^{mn} - \frac{n}{r}}{\frac{j}{m}}$$

by formula $(1_1)$

$$= \frac{R}{s_{\overline{mr}|\frac{1}{m}}} \frac{\left(1 + \frac{j}{m}\right)^{mr} \left[1 + \left(1 + \frac{j}{m}\right)^{mr} + \cdots + \left(1 + \frac{j}{m}\right)^{m(n-r)}\right] - \frac{n}{r}}{\frac{j}{m}}$$

$$= \frac{R}{s_{\overline{mr}|\frac{j}{m}}} \frac{\left(1+\frac{j}{m}\right)^{mr} s_{\overline{mn}|\frac{j}{m}} - \frac{n}{r}}{s_{\overline{mr}|\frac{j}{m}}}$$

$$= R \frac{\left(1+\frac{j}{m}\right)^{mr} s_{\overline{mn}|\frac{j}{m}} - \frac{n}{r}}{\frac{j}{m} s_{\overline{mr}|\frac{j}{m}}} \tag{9}$$

When $mr = 1$, $$(IV_n) = R \frac{\left(1+\frac{j}{m}\right) s_{\overline{mn}|\frac{j}{m}} - mn}{\frac{j}{m}} \tag{9_1}$$

When $m = 1, r = 1$, $$(IV_n) = R \frac{(1+i) s_{\overline{n}|i} - n}{i} \tag{9_2}$$

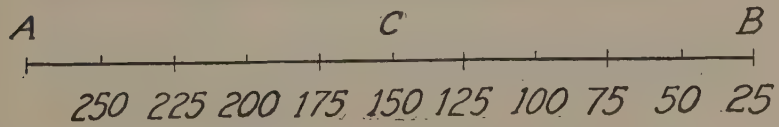
EXERCISE 1. Show that $(Is_{\overline{n}|i}) = \frac{s_{\overline{n+1}|i} - (n+1)}{i}$

EXERCISE 2. Find a formula for $(IV_0)$. [Use Theorem I and formula (9).]

The processes used for finding the values of these increasing and decreasing annuities can be used for finding the values of other sets of annuities. (See Exercise 3, below.)

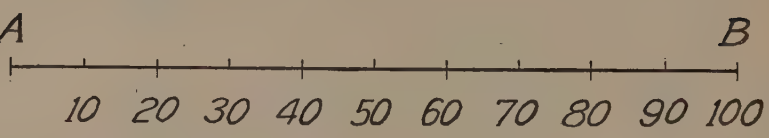
## EXERCISES

1. Each section of the following diagram represents six months:




Find the value of the decreasing annuity $AB$ at $A$; at $B$; and at $C$. Use $(j = .04, m = 2)$. Ans. \$1271.77; \$1550.28; \$1376.60.

**2.** Each section represents six months:




Find the values of the increasing annuity $AB$ at $B$ and at $A$. Use ($j = .06$, $m = 2$). Ans. \$602.60; \$448.39.

**3.** Show that the present value at ($j = .04$, $m = 2$) of the set of annuities having $R = 25$, $r = \frac{1}{2}$, and $n = 5, 7, 9, 11, 13, 15, 17, 19, 21$, and 23 is

$$\frac{25}{.02}\left(10 - \frac{v^{10} - v^{50}}{1 - v^4}\right) \text{ where } v = (1.02)^{-1}$$

Compute the value of this expression. Ans. \$5133.07.

**4.** A bank established a Christmas savings club which required that each member deposit 5 cents at the end of the first week, 10 cents at the end of the second, 15 cents at the end of the third, and so on for 50 weeks, the last payment being \$2.50. Show that the amount $A$ of this increasing annuity at 4% exact simple interest is, where $w = 49$ weeks,

$$A = .05\,[(w+1) + w + (w-1) + (w-2) + \cdots + 2 + 1] +$$
$$\frac{(.05)(.04)(7)}{(365)(2)}\,[(w+1)w + w(w-1) + (w-1)(w-2) + \cdots + (3)(2) + (2)(1)]$$
$$= \frac{(.05)(w+2)(w+1)}{2} + \frac{(.05)(.04)7}{(365)(2)}\left[\frac{w(w+1)(w+2)}{3}\right]$$
$$= \$63.75 + .80$$
$$= \$64.55$$

# CHAPTER III

## APPLICATIONS

**33. Introduction.** In Chapter I interest and discount formulas, both simple and compound, have been derived for finding values of a single sum of money, and in Chapter II compound interest formulas have been derived for finding values of certain sets of sums, namely the annuities certain. Various applications of the formulas for a single sum have been given; also a few applications to problems involving sets of sums have been given. In this chapter the applications will be treated more fully and systematically, especially those that involve sets of sums. In these applications the equations in the unknowns to be found will usually be set up by means of the method discussed in Art. 16, Chapter I, that is, by equating values of sets of sums.

In solving many of the problems given in this chapter the following procedure will be found helpful: (1) specify clearly all sums, rates, and terms involved, both known and unknown; (2) note the sets having equal values; (3) set up the equations by equating values; (4) solve the equations and perform the computations. When the data are complex a line diagram may be used to advantage in visualizing the problem. In determining the sets having equal values it is advisable to view each problem as an actual business transaction. The solutions of the equations are based on the same processes that are used in the illustrative examples in the text of Chapters I and II.

The applications are arranged under three headings as follows: (1) Debts and Sinking Funds; (2) Investments; (3) Depreciation and Capitalized Costs.

## DEBTS AND SINKING FUNDS

**34. Methods of paying debts.** Unless otherwise stated, the word "debt" will mean an interest-bearing debt. The interest payments on a debt are due at specified times, and the principal of the debt may be extinguished by a set of one or more payments. The extinction of a debt through a set of payments is called *amortization*. A schedule, like that in Example 1, Art. 16, Chapter I, which shows the payments on the principal, the interest payments, and the unpaid principal in the process of amortizing a debt is called an *amortization schedule*.

Different methods of amortizing a debt may be readily formulated by specifying different sets of payments on the principal. Those which follow occur frequently in practice and they suffice to illustrate the general processes. The unknown to be determined may be the set of payments of principal and interest, the term, or the rate. Methods for finding the first two of these types of unknowns are given in Arts. 35 to 43 inclusive; methods for finding an unknown rate are given in Arts. 56 to 60 inclusive.

**35. The debt is retired in a known time by periodic payments, including interest and principal, which are equal in amount.** In this method the equal payments, including interest and principal, constitute an annuity of unknown rent $R$ having a given term $n$ and $mr = 1$. The value of $R$ can be found by solving the equation obtained by equating the present value of this annuity to the original principal $P$; this gives,

$$Ra_{\overline{mn}|\frac{j}{m}} = P,$$

or

$$R = \frac{P}{a_{\overline{mn}|\frac{j}{m}}}$$

EXAMPLE 1. Annuity bonds to the amount of \$10,000, bearing interest at 5 per cent payable semi-annually, are to be repaid, interest and principal, in ten equal semi-annual instalments. Find the amount of each instalment and construct an amortization schedule.

SOLUTION. By the above formula,

$$R = \frac{1000}{a_{\overline{10}|.025}}$$

$$= \$1142.59$$

AMORTIZATION SCHEDULE

| Year | Interest Period | Principal at Beginning of Period | Interest Due at End of Period | Payment of Interest and Principal | Principal Repaid |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 1 | 10,000.00 | 250.00 | 1142.59 | 892.59 |
| 1 | 2 | 9,107.41 | 227.69 | 1142.59 | 914.90 |
| $1\frac{1}{2}$ | 3 | 8,192.51 | 204.81 | 1142.59 | 937.78 |
| 2 | 4 | 7,254.73 | 181.37 | 1142.59 | 961.22 |
| $2\frac{1}{2}$ | 5 | 6,293.51 | 157.34 | 1142.59 | 985.25 |
| 3 | 6 | 5,308.26 | 132.71 | 1142.59 | 1009.88 |
| $3\frac{1}{2}$ | 7 | 4,298.38 | 107.46 | 1142.59 | 1035.13 |
| 4 | 8 | 3,263.25 | 81.58 | 1142.59 | 1061.01 |
| $4\frac{1}{2}$ | 9 | 2,202.24 | 55.05 | 1142.59 | 1087.54 |
| 5 | 10 | 1,114.70 | 27.87 | 1142.59 | 1114.72 |
| Totals | | | 1425.88 | 11425.90 | 10000.02 |

Exercise. Reconstruct the amortization schedule above by carrying all the work to mills, using $R = 1142.588$.

Example 2. A \$10,000 debt is to be paid, interest and principal, in ten equal semi-annual instalments. Find the amount of each instalment if the unpaid principal bears 5 per cent payable semi-annually during the first three years, and 6 per cent semi-annually during the last two years.

Solution. Equating the value of the instalments at the end of three years to the value of the debt at the same date gives

$$R(s_{\overline{6}|.025} + a_{\overline{4}|.03}) = 10{,}000\ (1.025)^6$$

$$R = \frac{11596.934}{10.1048351}$$

$$= \$1147.66$$

Exercise. Construct an amortization schedule for Example 2. (Note that the interest rate per half year changes from .025 to .03 at the end of three years.)

### EXERCISES

1. Annuity bonds to the amount of \$25,000 bearing interest at $5\frac{1}{2}\%$ payable annually are to be repaid, interest and principal, in nine equal annual instalments. Find the amount of each instalment and construct an amortization schedule. Ans. \$3595.99.

2. A cash payment of \$1500 is made on a house sold for \$6000. The balance is to be paid in 10 years by equal semi-annual payments which include interest and principal. If unpaid principal bears interest at 6% payable semi-annually, find the semi-annual payment and construct an amortization schedule.

3. A trust fund of \$125,000 is created to provide equal quarterly payments for 10 years, at the end of which time the fund is to be exhausted. Find the quarterly payment if unexpended portions of the fund bear interest at 5% payable quarterly. Ans. \$3990.18.

4. A balance of \$1200 on the purchase price of an automobile is repaid in equal instalments, including interest and principal, at the end of each month for one year. Find the monthly instalment if the debt bears interest at 12% payable monthly. Construct a schedule. Ans. \$106.62.

**36. The debt is retired in a known time by periodic payments, including interest and principal, which are nearly equal in amount.** In some transactions in which it would be desirable to make equal periodic payments of interest and principal the conditions are such that it is impossible to do so. For example, in paying a debt in a given time represented by bonds of given denomination it is not possible to make equal payments of interest and principal since the interest must be paid in full and the payments on the principal must be integral multiples of the redemption value of the bonds. In such cases the method in Art. 35 can be used to find the amount of the periodic payment under the assumption that this is uniform, and then the difference between this amount and the interest due at the end of any period determines the number of bonds that should be retired to make the payments of interest and principal nearly equal.

EXAMPLE. A debt of \$10,000 consisting of 100 bonds of \$100 denomination, bearing interest at 5% payable semi-annually, is to be paid, interest and principal, in ten semi-annual instalments as nearly equal as possible. Construct an amortization schedule.

SOLUTION. If the instalments were equal in value, the amount of each instalment would be \$1142.59 by Example 1, Art. 35. The interest due in a half year is \$250; the difference, $1142.59 - 250 = 892.59$, shows that 9 bonds

should be retired at this time. The new principal is \$10,000 − \$900 = \$9100 and the interest due at the end of one year is \$227.50; the difference, 1142.59 − 227.50 = 915.09, shows that 9 bonds should be retired at the end of one year. Continuing this process gives the following:

AMORTIZATION SCHEDULE

| Year | Interest Period | Principal at Beginning of Period | Interest Due at End of Period | Number of Bonds Retired at End of Period | Principal Repaid at End of Period | Total Payment of Interest and Principal |
|---|---|---|---|---|---|---|
| ½ | 1 | 10000 | 250.00 | 9 | 900 | 1150.00 |
| 1 | 2 | 9100 | 227.50 | 9 | 900 | 1127.50 |
| 1½ | 3 | 8200 | 205.00 | 9 | 900 | 1105.00 |
| 2 | 4 | 7300 | 182.50 | 10 | 1000 | 1182.50 |
| 2½ | 5 | 6300 | 157.50 | 10 | 1000 | 1157.50 |
| 3 | 6 | 5300 | 132.50 | 10 | 1000 | 1132.50 |
| 3½ | 7 | 4300 | 107.50 | 10 | 1000 | 1107.50 |
| 4 | 8 | 3300 | 82.50 | 11 | 1100 | 1182.50 |
| 4½ | 9 | 2200 | 55.00 | 11 | 1100 | 1155.00 |
| 5 | 10 | 1100 | 27.50 | 11 | 1100 | 1127.50 |
| Totals | | | 1427.50 | 100 | 10000 | 11427.50 |

## EXERCISES

**1.** A city issues street improvement bonds to the amount of \$25,000 in denominations of \$500, bearing interest at 5½% payable semi-annually. Construct a schedule showing the number of bonds to be retired each half year for five years in order to make the payments of interest and principal as nearly equal as possible.

**2.** A \$10,000 issue of bonds in denominations of \$1000, bearing interest at 5% payable semi-annually, is to be retired in eight semi-annual instalments as nearly equal as possible. Construct an amortization schedule.

**3.** A \$100,000 issue of bonds, bearing interest at 6% payable annually and consisting of 250 bonds of \$100, 50 bonds of \$500, and 50 bonds of \$1000, denomination, is to be retired, interest and principal, in ten annual instalments as nearly equal as possible. Construct an amortization schedule.

**37. The debt is retired by payments, including interest and principal, all of which are equal and known except the last, which is equal to or less than the known payment.** In Arts. 35 and 36 methods of payment of debts are presented in which the number of periodic payments of interest and principal is given and the problem is to determine the payments. In the method presented in this article a given amount is paid periodically until the debt is liquidated, or, as is usually the case, until the unpaid principal becomes less than the given payment; the problem is to find the number of the given payments and the amount of the last payment. The solution of Example 4, Art. 25, Chapter II shows how these quantities can be found. Another illustration is given by the

EXAMPLE. If instalments of \$100, including interest and principal, are paid at the end of each month on a debt of \$10,000, bearing interest at 5% payable monthly, find how many instalments are needed before the unpaid principal is less than \$100; also find the amount needed to pay off the balance of the debt at the end of the following month.

SOLUTION. By formula (2), Art. 3, Chapter II,

$$10000 = 100\,\frac{1 - \dfrac{1}{\left(1 + \dfrac{.05}{12}\right)^{12n}}}{\dfrac{.05}{12}}$$

Solving,
$$\left(1 + \frac{.05}{12}\right)^{12n} = \frac{12}{7}$$

$$12\,n = \frac{\log 12 - \log 7}{\log 12.05 - \log 12} = 129.6$$

This result indicates that 129 monthly instalments of \$100 each are needed to reduce the debt to an amount less than \$100. Let $x$ be the additional amount needed at the end of 130 months to extinguish the debt. Equating the value of the payments to the value of the debt at the end of 130 months gives

$$x + 100\,s_{\overline{130}|\frac{.05}{12}} - 100 = 10000\left(1 + \frac{.05}{12}\right)^{130}$$

$$x = \$62.90 \quad \text{Tables III and V.}$$

EXERCISE 1. Find $x$ by equating the present value of the payments to \$10,000.

EXERCISE 2. What amount is needed to extinguish the debt at the end of 129 months? [Discount for one month the value of $x$ found in the above

solution; also solve by equating the value of the payments to that of the debt at the end of 129 months.]

EXERCISE 3. Find $n$ by interpolation from Table VI.

This method of paying debts is frequently used in repaying loans made by building and loan associations. The number of payments and the value of the last payment can be found by constructing an amortization schedule, and this is done by the association in keeping its accounts. The advantage of the solution given above lies in the fact that it requires much less computation than is needed to make an amortization schedule.

### EXERCISES

**1.** On a \$6000 mortgage, bearing interest at 6% payable semi-annually, payments of \$360 were made at the end of each six months. Find the number of payments which will reduce the unpaid principal to less than \$360; also find the amount needed to pay off the remainder of the debt at the end of the next six months. Ans. 23; \$163.24.

**2.** Same as Exercise 1, except that the interest rate is 7% payable semi-annually.

**3.** A trust fund of \$100,000 is invested at $(j = .05,\ m = 2)$. From the fund semi-annual payments of \$3500 are to be made as long as possible. Find the number of full payments and the amount of the last one.

Ans. 50; \$2578.54.

**4.** On a \$3000 mortgage payments of \$200 are made at the end of each six months. If the mortgage bears interest at 7% payable semi-annually during the first three years, and at 6% payable semi-annually after three years, find the number of payments and the amount of the last payment.

Ans. 20; \$184.65.

**38. The debt is retired by equal known periodic payments on the principal.** In this method the number of payments is found by dividing the principal by the amount of each payment, and the interest payment for each period can be readily computed since the principal on which interest is computed decreases by a known amount each time a payment on the principal is made. This method includes the case in which the principal is extinguished by a single payment as in an ordinary bond.

EXAMPLE. A debt of \$10,000, bearing interest at 5% payable semi-annually, is retired in 5 years by semi-annual payments of \$1000 on the principal. Construct an amortization schedule.

AMORTIZATION SCHEDULE

| Year | Interest Period | Principal at Beginning of Period | Interest Due at End of Period | Payment on Principal | Total Payment |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 1 | 10000 | 250 | 1000 | 1250 |
| 1 | 2 | 9000 | 225 | 1000 | 1225 |
| $1\frac{1}{2}$ | 3 | 8000 | 200 | 1000 | 1200 |
| 2 | 4 | 7000 | 175 | 1000 | 1175 |
| $2\frac{1}{2}$ | 5 | 6000 | 150 | 1000 | 1150 |
| 3 | 6 | 5000 | 125 | 1000 | 1125 |
| $3\frac{1}{2}$ | 7 | 4000 | 100 | 1000 | 1100 |
| 4 | 8 | 3000 | 75 | 1000 | 1075 |
| $4\frac{1}{2}$ | 9 | 2000 | 50 | 1000 | 1050 |
| 5 | 10 | 1000 | 25 | 1000 | 1025 |

EXERCISES

1. A city issues \$20,000 in bonds of \$1000 denomination, bearing interest at $5\frac{1}{2}\%$ payable semi-annually. If two bonds are retired at the end of each half year, construct a table showing the amortization of the debt.

2. A farm is purchased for \$10,000, one half of which is paid in cash. The purchaser agrees to pay \$1000 at the end of each year on the unpaid principal of the debt. Construct an amortization schedule if the debt bears interest at 7% payable annually.

3. Same as Exercise I, except that the interest rate is 6% payable semi-annually.

**39. The debt is retired by payments which may be irregular both in amount and in time.** An illustration of this method has been given in Example 1, Art. 16, Chapter I. Another illustration is afforded by the

Example. A note for \$500, issued January 1, 1915, stipulated that unpaid principal was to bear interest at 6% payable semi-annually and that unpaid interest was to bear interest at 8% payable semi-annually. Interest payments were made during the first year and a payment of \$100, including interest and principal, was made January 1, 1918. If no further payments were made, find the amount due January 1, 1923.

SOLUTION. The interest due semi-annually during the first three years was \$15 and the unpaid principal January 1, 1918, just after the \$100 payment, was

$$500 + 15\, s_{\overline{4}|.04} - 100 = 400 + 15\, s_{\overline{4}|.04}$$

The interest due semi-annually during the last five years was $.03(400+15\, s_{\overline{4}|.04})$ and the total amount due January 1, 1923 was

$$400 + 15\, s_{\overline{4}|.04} + .03(400 + 15\, s_{\overline{4}|.04})s_{\overline{10}|.04} = (400 + 15\, s_{\overline{4}|.04})(1 + .03\, s_{\overline{10}|.04})$$
$$= \$630.71$$

### EXERCISES

1. Solve the above example if the payment made January 1, 1918, was \$50.

2. A note for \$500 issued January 1, 1916 stipulated that unpaid principal was to bear 6% interest payable semi-annually and that unpaid interest was to bear 7% payable semi-annually. Payments on principal and interest were made as follows: January 1, 1917, \$100; January 1, 1918, \$50; and July 1, 1920, \$200. Find amount due January 1, 1924. Ans. \$334.84.

3. On a debt of \$5000 payments of \$200 were made at the end of each half year for 4 years, after which semi-annual payments of \$300 were made until the debt was reduced to less than \$300. For the first two years the interest rate was 7% payable semi-annually; for the remainder of the term it was 6% payable semi-annually. Find the amount of the last payment if it was made six months after the debt was reduced to less than \$300. Ans. \$52.69.

**40. The unpaid principal at any time.** The unpaid principal at any specified time of a debt paid by instalments which are known can evidently be determined by constructing an amortization schedule to this time. For example, the schedule for Example 1, Art. 35, shows that the unpaid principal just after the sixth instalment is paid is \$4298.38. Two other methods can often be used to find the unpaid principal at any time; one of these may be called the *prospective* method, and the other the *retrospective*. In the prospective method the unpaid principal just after an instalment is paid is viewed as the value at that time of the subsequent instalments of interest and principal. In the retrospective method the unpaid principal just after any instalment is paid is viewed as the difference between the value of the original principal at the time and that of the prior instalments of interest and principal. If $P_k$ denotes the unpaid principal just after the $k$th instalment, an application of these methods to the method of paying debts

treated in Art. 35 gives respectively:

$$P_k = Ra_{\overline{mn-k}|\frac{j}{m}} \qquad \text{(prospective)}$$

$$P_k = P\left(1+\frac{j}{m}\right)^k - Rs_{\overline{k}|\frac{j}{m}} \qquad \text{(retrospective)}$$

EXERCISE 1. Compute the right-hand member of each of these expressions when $k = 6$ for Example 1, Art. 35.

EXERCISE 2. Compute the unpaid principal in Example 3, Art. 35, at the end of the fourth year, by both the prospective and the retrospective methods.

### EXERCISES

1. Use the prospective and the retrospective methods to find the unpaid principal at the beginning of the third year for the annuity bonds in Example 1, solved in Art. 35. Ans. $5308.26.

2. Same as Exercise 1, for the example solved in Art. 37. Ans. $7739.39.

3. A house is bought for $7800, of which $2000 is paid in cash. On the balance which bears interest at 6%, payable semi-annually, payments of $390 were made at the end of each six months. Immediately after the semi-annual payment which reduced the debt to less than $4000 it was refunded. Find the amount of the debt when it was refunded. Ans. $3879.26.

**41. Sinking funds.** A sinking fund consists of a set of sums put aside for the purpose of meeting a future obligation. The sums put aside should be productively invested. The future obligation is often an instalment on a debt, a sum of money needed to restore capital invested, or a sum of money needed to replace an article which is subject to depreciation. In the latter case the sinking fund is sometimes called a depreciation fund. When the sums are put aside at periodic intervals, they constitute an annuity and the annuity formulas of Chapter II may be used with such sinking funds. Unless otherwise stated, sinking funds having equal periodic payments will be understood in what follows.

In sinking-fund operations the unknowns are ordinarily the sums of money put aside periodically; the interest rate and the term are usually known. When the sums are equal, the value of each can be found by solving formula (1) or (3), Chapter II, for $R$. For example, when $mr = 1$, formula (3) gives

$$R = V_n \cdot \frac{1}{s_{\overline{mn}|\frac{j}{m}}}$$

where $V_n$ is the amount to be accumulated by the sinking fund, $n$ is the term of years, and $j$ is the nominal interest rate converted $m$ times per year.

The amount $V_{n_1}$ in the sinking fund at the end of $n_1$ years ($n_1 < n$) can also be found by formula (3).

$$V_{n_1} = Rs_{\overline{mn_1}|\frac{j}{m}} = V_n \cdot \frac{1}{s_{\overline{mn}|\frac{j}{m}}} \cdot s_{\overline{mn_1}|\frac{j}{m}}$$

EXAMPLE. A sinking fund was created to pay the principal of a loan of \$100,000 in 5 years by equal semi-annual instalments. If the sinking fund accumulates at ($j = .05$, $m = 2$), find (1) the semi-annual instalment and (2) the amount in the sinking fund at the end of three years.

SOLUTION. By the above formulas

$$R = 100{,}000 \cdot \frac{1}{s_{\overline{10}|.025}} = \$8925.876$$

$$V_3 = \$8925.876 \cdot s_{\overline{06}|.025} = \$57{,}016.146$$

EXERCISE. Check the values found for $R$ and $V_3$ by completing the following

SINKING–FUND SCHEDULE

| YEAR | INTEREST PERIOD | PAYMENT TO FUND AT END OF PERIOD | INTEREST DUE ON FUND AT END OF PERIOD | AMOUNT IN FUND AT END OF PERIOD |
|---|---|---|---|---|
| $\frac{1}{2}$ | 1 | 8925.876 | | 8925.876 |
| 1 | 2 | 8925.876 | 223.147 | 18174.899 |
| $1\frac{1}{2}$ | 3 | 8925.876 | 454.372 | 27555.147 |
| 2 | 4 | 8925.876 | | |
| $2\frac{1}{2}$ | 5 | 8925.876 | | |
| 3 | 6 | 8925.876 | | |
| $3\frac{1}{2}$ | 7 | 8925.876 | | |
| 4 | 8 | 8925.876 | | |
| $4\frac{1}{2}$ | 9 | 8925.876 | | |
| 5 | 10 | 8925.876 | | |

EXERCISE

A debtor creates a sinking fund to pay the principal of a debt of \$6178.75 by depositing \$120 at the end of each three months in a savings bank which pays 5% converted quarterly. Show that the deposits must be made for 10 years. Construct a sinking fund schedule.

Sinking funds to restore invested capital and to replace articles subject to depreciation are considered in Arts. 47, 50, and 64 respectively. Sinking funds to repay loans are considered in Art. 42.

**42. The sinking-fund method of retiring a debt.** In this method a sinking fund is created to retire the principal of the debt when due; the interest payments are made separately. When the payments into a sinking fund created to meet a debt are made at the times of the interest payments on the debt and when the sinking-fund payments earn the same interest rate as that borne by the debt, the sinking-fund payment plus the interest payment on the debt is the same as the periodic payment of interest and principal necessary to pay the debt (Art. 35). If $P$ denotes the principal of the debt, the sinking-fund payment, found by replacing $V_n$ by $P$ in the above formula, is $\frac{P}{s_{\overline{mn}|\frac{j}{m}}}$, and the interest payment is $P \cdot \frac{j}{m}$. The sum of these two payments is $P\left(\frac{1}{s_{\overline{mn}|\frac{j}{m}}} + \frac{j}{m}\right) = P \cdot \frac{1}{a_{\overline{mn}|\frac{j}{m}}}$ (Formula 5, Chapter II), which by the method in Art. 35 is the periodic payment of interest and principal. When, however, the sinking-fund payments bear the rate $j'$ and the debt bears the rate $j$ each payable $m$ times per year, the sinking-fund payment plus the interest payment is $P\left(\frac{1}{s_{\overline{mn}|\frac{j'}{m}}} + \frac{j}{m}\right)$.

EXAMPLE 1. Find the semi-annual payment into a sinking fund which accumulates at 5% converted semi-annually, necessary to pay a debt of \$10,000 due in 5 years and bearing interest at 5% payable semi-annually.

SOLUTION. By Art. 41, the sinking-fund payment, $R$, is given by

$$R = 10{,}000 \frac{1}{s_{\overline{10}|.025}} = \$892.588$$

It may be noted that this sinking-fund payment plus the interest payment of \$250 is \$1142.588, as found in Example 1, Art. 35.

EXERCISE 1. Find the amount in the sinking fund just after the sixth payment is made.

EXERCISE 2. Find the actual indebtedness just after the sixth sinking-fund payment is made by deducting the amount found in Exercise 1 from the original principal, \$10,000.

EXAMPLE 2. Solve Example 1 if the sinking-fund payments are accumulated at 4% converted semi-annually.

SOLUTION. By Art. 41 the sinking-fund payment is

$$R = 10{,}000 \cdot \frac{1}{s_{\overline{10}|.02}} = \$913.265$$

In this case the sinking-fund payment plus the interest payment is \$1163.265.

EXERCISE 1. Find the amount in the sinking fund just after the sixth payment is made.

EXERCISE 2. Find the actual indebtedness just after the sixth sinking-fund payment is made by deducting the amount found in Exercise 1 from the original principal \$10,000.

EXAMPLE 3. The principal of a \$10,000 debt bearing interest at 5% payable semi-annually is to be paid in two \$5000 instalments, the first in 5 years and the second in 8 years. Deposits are made semi-annually into a sinking fund which pays 4% converted semi-annually, for the purpose of meeting these instalments when due. If the deposits are such that the sum of the deposit and the interest due at any time is constant, find the amount of each deposit.

SOLUTION. If $R$ denotes the semi-annual deposit during the first 5 years, then $R + 125$ is the semi-annual deposit during the last 3 years, since interest payments decrease by \$125 at the end of 5 years. Equating the amount of the deposits to the amount of the instalments gives

$$R s_{\overline{16}|.02} + 125\, s_{\overline{6}|.02} = 5000 + 5000\ (1.02)^6$$

$$R = \$528.04$$

$$R + 125 = \$653.04$$

## EXERCISES

**1.** Find the annual payment into a sinking fund which accumulates at ($j = .055$, $m = 1$) necessary to pay a debt of \$25,000 due in 9 years, and bearing interest at $5\frac{1}{2}$% payable annually. Find also the sum of the annual interest on the debt and the sinking-fund payment. (See Ex. 1, Art. 35.)

Ans. \$2220.99; \$3595.99.

**2.** Same as Exercise 1 except that the sinking fund accumulates at ($j = .045$, $m = 1$).

**3.** Find the quarterly payment into a sinking fund which accumulates at ($j = .04$, $m = 4$) necessary to pay a debt of \$100,000 due in 10 years, and bearing interest at 6% payable quarterly. Find also the sum of the interest and sinking-fund payments. Ans. \$2045.56; \$3545.56.

**4.** Find the semi-annual payment into a sinking fund which accumulates at ($j = .045$, $m = 2$) necessary to pay the principal of the following debts: (1) \$1000 due in 1 year and bearing interest at 4% payable semi-annually; (2) \$2000 due in 2 years and bearing interest at 5% payable semi-annually; and (3) \$3000 due in three years and bearing interest at 6% payable semi-annually. Find also the sums of the interest and sinking-fund payments at the end of each half year.

Ans. \$974.21; \$1134.21; \$1134.21; \$1114.21; \$1114.21; \$1064.21; \$1064.21.

**5.** Same as Exercise 4, except that the sinking-fund payments are such that the sum of the sinking-fund payments and the interest due at the end of each six months is constant. Construct a sinking-fund schedule.

Ans. \$945.25; \$945.25; \$965.25; \$965.25; \$1015.25; \$1015.25; \$1105.25.

**43. Book Values.** In keeping accounts connected with the payments of debts the unpaid principal of a debt at any entry date is often called the *Book Value* of the debt at that date. When a sinking fund is created to meet the debt when due, the book value of the debt at each entry date is its original value less the amount in the sinking fund. "Book values" are also used in connection with other items whose values change from entry to entry. For example, in Art. 46 the book values of an investment are found at different entry dates, and in Art. 62 the book values of an article subject to depreciation are determined. The book values of a debt whose principal is paid by means of a sinking fund are illustrated by the

EXAMPLE. The principal of a \$10,000 debt bearing interest at 5% payable semi-annually is to be paid at the end of 5 years by means of equal semi-annual deposits into a sinking fund which pays 4% converted semi-annually. Construct a schedule showing the book value of the indebtedness at the end of each half year.

SOLUTION. By Example 2, Art. 42, the sinking fund payment is \$913.265.

SCHEDULE

| Year | Interest Period | Book Value of Debt at End of Period | Payment to Sinking Fund at End of Period | Interest Due on Fund at End of Period | Amount in Fund at End of Period |
|---|---|---|---|---|---|
| 0 | 0 | 10000 | | | |
| $\frac{1}{2}$ | 1 | 9086.735 | 913.265 | | 913.265 |
| 1 | 2 | 8155.205 | 913.265 | 18.265 | 1844.795 |
| $1\frac{1}{2}$ | 3 | 7205.044 | 913.265 | 36.896 | 2794.956 |
| 2 | 4 | 6235.880 | 913.265 | 55.899 | 3764.120 |
| $2\frac{1}{2}$ | 5 | 5247.333 | 913.265 | 75.282 | 4752.667 |
| 3 | 6 | 4329.015 | 913.265 | 95.053 | 5760.985 |
| $3\frac{1}{2}$ | 7 | 3210.530 | 913.265 | 115.220 | 6789.470 |
| 4 | 8 | 2161.476 | 913.265 | 135.789 | 7838.524 |
| $4\frac{1}{2}$ | 9 | 1091.441 | 913.265 | 156.770 | 8908.559 |
| 5 | 10 | .005 | 913.265 | 178.171 | 9999.995 |

### EXERCISES

**1.** Construct a schedule showing the book value of the debt at the end of each year for Exercise 1, Art. 42.

**2.** Construct a schedule showing the book value of the debt at the end of each year for Exercise 2, Art. 42.

**3.** Construct a schedule showing the book value of the total indebtedness for Exercise 5, Art. 42.

## INVESTMENTS (BONDS, STOCKS, NOTES, SAVINGS)

**44. The problems of investment.** A problem in investment, like one in debts, may have a sum of money, a term of years, or a rate for an unknown. When a sum of money is unknown, it is usually the purchase price or value of an investment, the investment rate and the term being known. The value of an investment under a given rate depends upon the set of sums which the investment returns to the investor. The method for finding the value

of an investment depends, then, upon the nature of the return. In Arts. 45 to 54 inclusive various types of return are considered. When a term of years is unknown, it is frequently the time required to pay for an investment bought on the instalment plan. Building and Loan Stocks afford a good illustration. This type of problem is treated in Art. 55. The rate, or yield, is frequently unknown in problems in investments, debts, and other fields of finance. Methods for finding rates are given in Arts. 56 to 60 inclusive.

A general principle in investments is that the capital invested should not be impaired. That is, the amount invested should be returned to the investor for reinvestment during, or at the end of, the term of the investment.

**45. The value of an investment whose return is a sum of money.** This simple problem has been treated quite fully in Chapter I. It is the problem of finding the present value of a sum of money due at some future time. The discounting of notes by banks affords a good illustration. Any one of the four formulas for finding the present value of a sum may be used to solve problems of this type.

**46. The value of an investment whose return is an annuity of fixed rent.** Annuity bonds furnish good illustrations of this type of return. They are bonds which are repaid in instalments such that the interest due plus the amount paid on the principal or *face value* is constant or as nearly constant as the denomination of the bonds permit. In this article the case is treated in which the instalments are equal; in article 54, the case is treated in which the instalments are nearly equal. When the purchase price of a bond exceeds its face value by an amount $P$, the bond is said to be bought at a *premium* $P$. When the face value of the bond exceeds its purchase price by an amount $D$, the bond is said to be bought at a *discount* $D$.

The return in this type of investment being an annuity, the purchase price is the present value of the annuity at the interest rate desired by the investor. For example, if an annuity of rent $R$, term $n$ years, and rent period $\frac{1}{m}$ years, is purchased to yield the

investor $j$ per cent payable $m$ times per year, the purchase price $V$ is given by

$$V = Ra_{\overline{mn}|\frac{j}{m}}$$

EXAMPLE. Annuity bonds to the amount of \$10,000 bearing 5% payable semi-annually are to be repaid, interest and principal, in 10 equal semi-annual instalments. Find the purchase price to yield the investor 4% converted semi-annually.

SOLUTION. By the example in Art. 35, the debtor makes semi-annual payments of \$1142.588 for a term of 5 years. These payments constitute the return to the investor; their present value at 4% semi-annually gives for the purchase price, $V = 1142.588\, a_{\overline{10}|.02} = \$10{,}263.394$. In this case the bond is purchased at a premium of \$263.39.

EXERCISE 1. Solve this example if the bonds are bought to yield 6% payable semi-annually. What is the discount?

EXERCISE 2. Solve this example if the bonds are bought to yield 4% payable annually. What is the premium?

EXERCISE 3. What is the purchase price of the bonds if bought 3 years before they mature, to yield 4% payable semi-annually. What is the purchase price if bought 2 years before they mature?

If the return is composed of two or more annuities, each of fixed rent, the value of the investment is the sum of the values of the component annuities.

In investments of this type each payment received from the return includes both interest on the investment and repayment of a portion of the capital invested. The payments on the capital may be put into a sinking fund, to restore the capital invested at the end of the term; this plan is discussed in Art. 47. Usually, however, in keeping accounts these payments are deducted in turn from the capital invested so that the book value of the investment changes each time a payment is received, to correspond to the purchase price at the time of entry. This change in purchase price is illustrated by the above example and by Exercise 3. In this method of keeping accounts the payments on the capital invested are available for reinvestment as soon as they are received. For the above example the following is an

INVESTMENT SCHEDULE

| Year | Interest Period | Book Value at Beginning of Period | Interest Due on Book Value at End of Period | Payment Received at End of Period | Amount Repaid on Capital Invested |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 1 | 10263.394 | 205.268 | 1142.588 | 937.320 |
| 1 | 2 | 9326.074 | 186.521 | " | 956.067 |
| $1\frac{1}{2}$ | 3 | 8370.007 | 167.400 | " | 975.188 |
| 2 | 4 | 7394.819 | 147.896 | " | 994.692 |
| $2\frac{1}{2}$ | 5 | 6400.127 | 128.003 | " | 1014.585 |
| 3 | 6 | 5385.542 | 107.711 | " | 1034.877 |
| $3\frac{1}{2}$ | 7 | 4350.665 | 87.013 | " | 1055.575 |
| 4 | 8 | 3295.090 | 65.902 | " | 1076.686 |
| $4\frac{1}{2}$ | 9 | 2218.404 | 44.368 | " | 1098.220 |
| 5 | 10 | 1120.184 | 22.404 | " | 1120.184 |
| Totals | | | 1162.486 | 11425.880 | 10263.394 |

Exercise 4. Construct an investment schedule for Exercise 1.

When the term of the annuity which constitutes the return becomes infinite, that is, when the return is a perpetuity, the purchase price is the present value of a perpetuity. When for example the rent is $R$ payable $m$ times per year, the purchase price of the perpetuity, to yield $j$ payable $m$ times per year is (Art. 29, Chapter II)

$$V = \frac{R}{\frac{j}{m}}$$

## EXERCISES

1. Annuity bonds to the amount of \$100,000 bearing interest at $5\frac{1}{2}\%$ payable quarterly are to repaid, interest and principal, in 40 equal quarterly instalments. Find the purchase price to yield the investor 6% converted quarterly. Construct an investment schedule. Ans. \$97733.00.

**2.** In Exercise 1, find the purchase price to yield the investor 6% converted semi-annually.

**3.** The return on a lease is \$1000 at the end of each three months. The lease was sold 5 years before the end of its term. Find the purchase price to yield the investor 5% converted semi-annually. Ans. \$17612.10.

**4.** The return on a 99 year lease is \$5500 at the end of each six months. It is purchased 5 years after it was executed. Find the purchase price to yield the investor 6% converted semi-annually. Ans. \$182806.69.

**5.** Same as Exercise 4, except that the return on the lease is \$5500 semi-annually for the first 10 years, \$6500 semi-annually for the next 10 years, and \$7000 semi-annually for the remainder (79 years) of its term.

**6.** Solve Exercise 4 if the semi-annual return of \$5500 is a perpetuity.
Ans. \$183333.33.

**7.** The return on a lease on a storeroom is \$150 at the end of each month for the first 5 years, and \$200 a month for the next 5 years. The lease was sold one year after it was executed to yield its owner 8% converted annually. Find the sale price. Ans. \$13475.49.

**47. The value of an investment whose return is an annuity when a sinking fund is created to restore the capital invested.** If the payments on the capital invested are used to create a sinking fund to restore the capital at the end of the investment term instead of being used to reduce the book value of the investment, the amount of the investment may be found by equating the sinking-fund payment plus the interest on the investment to the rent of the annuity constituting the return. When the sinking-fund rate equals the investment rate the interest on the investment for the case treated in Art. 46 is $V \cdot \frac{j}{m}$ payable $m$ times per year, and the sinking-fund payment needed to restore the purchase price $V$ in $n$ years is $V \cdot \frac{1}{s_{\overline{mn}|\frac{j}{m}}}$ (Art. 41). Hence in this case

$$V \cdot \frac{1}{s_{\overline{mn}|\frac{j}{m}}} + V \cdot \frac{j}{m} = R$$

or
$$V \frac{1}{a_{\overline{mn}|\frac{j}{m}}} = R \qquad \left(\text{since } \frac{1}{a_{\overline{mn}|\frac{j}{m}}} = \frac{1}{s_{\overline{mn}|\frac{j}{m}}} + \frac{j}{m}\right)$$

$$V = Ra_{\overline{mn}|\frac{j}{m}}$$

When the sinking-fund rate is $j'$ converted $m$ times per year, the purchase price is determined by the equation

$$V \cdot \frac{1}{s_{\overline{mn}|\frac{j'}{m}}} + V \cdot \frac{j}{m} = R$$

$$V = \frac{R}{\frac{1}{s_{\overline{mn}|\frac{j'}{m}}} + \frac{j}{m}}$$

EXAMPLE. It is estimated that a mine will yield a net return of \$10000 at the end of each year for 20 years, when it will be exhausted. Find the purchase price to yield 10% converted annually on the investment if a sinking fund which accumulates at 4% converted annually is created to restore the capital invested in 20 years?

SOLUTION. In this example the annual interest payment is $V(.1)$ and the annual sinking-fund payment is $V \cdot \frac{1}{s_{\overline{20}|.04}}$, so that

$$V \frac{1}{s_{\overline{20}|.04}} + V(.1) = 10000$$

Solving,
$$V = \frac{10000}{\frac{1}{s_{\overline{20}|.04}} + .1} = \frac{10000}{.1335818} = \$74860.50$$

## EXERCISES

**1.** It is estimated that a mine will yield a net return of \$1000 at the end of each year for 35 years, when it will become exhausted. Find the purchase price to yield 8% payable annually on the investment if a sinking fund which accumulates at 5% converted semi-annually is created to restore the capital invested in 35 years. Ans. \$10997.57.

**2.** It is estimated that a taxicab will yield a net return of \$500 at the end of each year for 4 years, after which it will be worthless. Find the purchase price to yield 10% payable annually on the investment if a sinking fund which accumulates at 4% converted annually is created to restore the capital invested in 4 years. Ans. \$1490.36.

**3.** It is estimated that the return from an oil well will average \$10000 annually for 3 years, after which it will be worthless. Find its value at the time of its development if the investment is to pay 8% payable annually and a sinking fund which accumulates at 5% converted annually is created to restore the capital invested in three years.

**48. The value at date of issue or at an interest payment date of an ordinary bond. The return is a single sum and an annuity.** Ordinary bonds furnish the best illustration of this type of return. In an ordinary bond a promise is made to pay a specified amount, called the *redemption value*, at the end of a stated term, and to pay interest at periodic intervals during this term on a specified value called the *face value* of the bond. The redemption value is usually the same as the face value; sometimes, however, to make the bond more attractive to buyers the redemption value is greater than the face value. In the former case the bond is said to be redeemed at *par;* in the latter, it is said to be redeemed *above par.* The words "premium" and "discount" for an ordinary bond are defined as in the preceding article for an annuity bond; that is, the premium is the purchase price less the face value and the discount is the face value less the purchase price.

The purchase price or value of an investment, whose return is an annuity and a single sum, is the present value of the annuity plus the present value of the single sum under the interest rate desired by the investor. For example, in the case of an ordinary bond whose

face value is $C$
redemption value is $F$
bond rate is $g$ payable $m$ times per year
investment rate is $j$ payable $m$ times per year
term is $n$ years

the value, $V$, is $V = F\left(1 + \frac{j}{m}\right)^{-mn} + \frac{Cg}{m} a_{\overline{mn}|\frac{j}{m}}$

This formula can be written also in the two forms

$$V = F + \left(\frac{Cg}{m} - \frac{Fj}{m}\right) a_{\overline{mn}|\frac{j}{m}}$$

$$V = \frac{Cg}{j} + \left(F - \frac{Cg}{j}\right)\left(1 + \frac{j}{m}\right)^{-mn}$$

by use of the relation $a_{\overline{mn}|\frac{j}{m}} = \dfrac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\frac{j}{m}}$ (Formula $2_1$, Chapter II)

To get the second formula from the first, replace $\left(1+\frac{j}{m}\right)^{-mn}$ by $1-\frac{j}{m}a_{\overline{mn}|\frac{j}{m}}$; to get the third from the first, replace $a_{\overline{mn}|\frac{j}{m}}$ by $\dfrac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\frac{j}{m}}$. It may be noted that both interest and annuity tables are needed for the first formula, annuity tables only for the second, and interest tables only for the third. Excellent bond tables which give the values of $V$ for various values of $\frac{j}{m}$ and $n$ have been constructed.

When $F = C$, the second form shows that the bond is bought at a premium when $g > j$, and at a discount when $g < j$. If $P$ and $D$ denote the premium and discount respectively, then

$$P = C\frac{g-j}{m}a_{\overline{mn}|\frac{j}{m}}, \quad D = C\frac{j-g}{m}a_{\overline{mn}|\frac{j}{m}}$$

The student is advised against substituting into any one of these formulas in finding the value of a bond; it is better to make direct use of the principles needed to write them. These principles are seen at once by expressing the formulas verbally. For example, by the first formula, the value of a bond at a given investment rate equals the present value of the redemption value of the bond plus the present value of the annuity composed of the interest payments on the bond. These three principles will now be applied in turn to solve the

EXAMPLE. A \$10000 bond issued July 1, 1923, bearing interest at 5% payable semi-annually, is to be redeemed at par in 5 years. Find its purchase price to yield 4 per cent converted semi-annually. The return on this investment consists of \$250 interest every half year for 5 years and \$10000 at the end of 5 years.

SOLUTION 1. The purchase price of the bond is the present value at 4% converted semi-annually of \$10000 due in 5 years plus the present value at 4% converted semi-annually of the interest payments on the bond. That is,

$$V = 10000(1.02)^{-10} + 250\,a_{\overline{10}|.02}$$
$$= 8203.483 + 2245.646 = \$10449.129$$

SOLUTION 2. A \$10000 investment for 5 years at 4% payable semi-annually would yield for its return \$200 every half year for 5 years and \$10000 at the end of 5 years. The return on the bond exceeds this return by an annuity whose rent is \$50, whose term is 5 years and whose rent period is ½ year. Hence to find the purchase price of the bond the \$10000 investment must be increased by the present value of this annuity whose rent is \$50. That is,

$$V = 10000 + 50\, a_{\overline{10}|02}$$
$$= 10000 + 449.129 = \$10449.129$$

SOLUTION 3. A \$12500 investment for 5 years at 4% payable semi-annually would yield for its return \$250 every half year and \$12500 at the end of 5 years. This return exceeds the return on the bond by \$2500 due in 5 years. Hence to find the purchase price of the bond the \$12500 investment must be decreased by the present value of \$2500 due in 5 years. That is,

$$V = 12500 - 2500(1.02)^{-10}$$
$$= 12500 - 2050.871 = \$10449.129$$

## EXERCISES

**1.** Find the value of the bond in the example above by each of the three methods, if the investment rate is ($j = .06$, $m = 2$). Ans. \$9573.49.

**2.** Find the value of the bond in the example above 4 years, 3 years, 2 years, and 1 year before it matures.

**3.** Find the value of the bond in the example above if the redemption value is \$11000, the face value remaining at \$10000. Ans. \$11269.48.

**4.** Find the purchase price of a \$5000 bond bearing interest at 5½% payable semi-annually, if it is redeemable at par in 10 years, and if the investment yields ($j = .05$, $m = 2$); if it yields ($j = .06$, $m = 2$). Find the premium in the first instance and the discount in the second. Ans. \$194.86; \$185.97.

**5.** Same as Exercise 1, except that the term of the bond is 20 years.
Ans. \$8844.261.

**6.** A \$20000 bond bears interest at 5½% payable semi-annually and matures at 102 in 5 years. Find the purchase price to yield ($j = .05$, $m = 2$); to yield ($j = .06$, $m = 2$). Ans. \$20750.08; \$19871.13.

**7.** A \$100000 issue of bonds bearing interest at 5% payable semi-annually, and whose term is 10 years, was sold for \$106554.19 to yield the purchaser ($j = .045$, $m = 2$). Find the redemption value of the issue.

**8.** If a sinking fund at the rate ($j = .04$, $m = 2$) is used to restore the premium for the example solved in the text, find the semi-annual deposit.
Ans. \$41.02.

**9.** It is estimated that a taxicab will yield a net return of \$500 at the end of each year for 4 years, at which time it will have a scrap value of \$100. Find the purchase price to yield 8% annually.

10. It is estimated that the income of a lumber company in developing a tract of wooded land will be \$10000 annually for 15 years, at which time production will cease and the land will be worth \$2500. Find the purchase price correct to dollars to yield 10% annually. Ans. \$76659.

**49. Investment schedules for ordinary bonds.** The usual method for keeping accounts for an ordinary bond investment is like that used for an annuity bond; each book value entry is the actual value of the bond, *at the investment rate*, at the time the entry is made. These values may be found by application of the principles used in solving the example and exercises in Art. 48. When a complete schedule is to be constructed, each new book value can be found more easily, however, by changing the value last entered by an amount which is the difference between the interest payment on the bond and the interest payment on the book value. This difference is subtracted each time when the purchase price is greater than the redemption value of the bond, and it is added when the purchase price is less than the redemption price. In the former instance the book value is said to be *written off* to the redemption value; in the latter it is said to be *written on*. The total amount to be written off, to make the book value agree with the redemption value, is, of course, the purchase price less the redemption value; the total amount to be written on, is the redemption value less the purchase price. In a schedule, the separate amounts written off are entered under the heading "For Amortization"; the amounts written on are entered under the heading "For Accumulation." When $F = C$, the amount amortized is the premium, and the amount accumulated is the discount.

| Date | Book Value | Interest on Book Value at 2% | Dividend Received | For Amortization of Premium |
|---|---|---|---|---|
| July 1, 1923 | 10449.13 | | | |
| Jan. 1, 1924 | 10408.11 | 208.98 | 250.00 | 41.02 |
| July 1, 1924 | 10366.27 | 208.16 | 250.00 | 41.84 |
| Jan. 1, 1925 | 10323.60 | 207.33 | 250.00 | 42.67 |
| July 1, 1925 | 10280.07 | 206.47 | 250.00 | 43.53 |
| Jan. 1, 1926 | 10235.67 | 205.60 | 250.00 | 44.40 |
| July 1, 1926 | 10190.38 | 204.71 | 250.00 | 45.29 |
| Jan. 1, 1927 | 10144.19 | 203.81 | 250.00 | 46.20 |
| July 1, 1927 | 10097.07 | 202.88 | 250.00 | 47.12 |
| Jan. 1, 1928 | 10049.01 | 201.94 | 250.00 | 48.06 |
| July 1, 1928 | 9999.99 | 200.98 | 250.00 | 49.02 |

INVESTMENT SCHEDULE (For Exercise 1, Art. 48)

| Date | Book Value | Interest on Book Value at 3% | Dividend Received | For Accumulation of Discount |
|---|---|---|---|---|
| July 1, 1923 | 9573.49 | | | |
| Jan. 1, 1924 | 9610.69 | 287.20 | 250.00 | 37.20 |
| July 1, 1924 | 9649.01 | 288.32 | 250.00 | 38.32 |
| Jan. 1, 1925 | 9688.48 | 289.47 | 250.00 | 39.47 |
| July 1, 1925 | 9729.13 | 290.65 | 250.00 | 40.65 |
| Jan. 1, 1926 | 9771.00 | 291.87 | 250.00 | 41.87 |
| July 1, 1926 | 9814.13 | 293.13 | 250.00 | 43.13 |
| Jan. 1, 1927 | 9858.55 | 294.42 | 250.00 | 44.42 |
| July 1, 1927 | 9904.31 | 295.76 | 250.00 | 45.76 |
| Jan. 1, 1928 | 9951.44 | 297.13 | 250.00 | 47.13 |
| July 1, 1928 | 9999.98 | 298.54 | 250.00 | 48.54 |

EXERCISES

1. Make investment schedules for Exercise 4, Art. 48, showing the amortization of the premium in the first instance, and the accumulation of the discount in the second.

2. Make an investment schedule for Exercise 6, Art. 48.

3. Make a schedule for Exercise 10, Art. 48, showing the book value of the investment at the end of each year.

**50. The value of an ordinary bond when a sinking fund is created to restore the difference between the redemption value and the purchase or selling price.** When a bond is purchased at a price in excess of the redemption value, the purchaser can create a sinking fund to restore the excess at the time the bond matures. When the purchase price is less than the redemption value, the seller can create a like sinking fund. When the sinking-fund rate equals the investment rate, the sinking-fund payment for the

case treated in Art. 48 is $\left(\frac{Cg}{m} - \frac{Vj}{m}\right)$ if $F < V$. Hence in this case

$$\left(\frac{Cg}{m} - \frac{Vj}{m}\right) s_{\overline{mn}|\frac{j}{m}} = V - F$$

Solving, $$V = \frac{F + \frac{Cg}{m} s_{\overline{mn}|\frac{j}{m}}}{1 + \frac{j}{m} s_{\overline{mn}|\frac{j}{m}}} = \frac{\frac{F}{s_{\overline{mn}|\frac{j}{m}}} + \frac{Cg}{m}}{\frac{1}{a_{\overline{mn}|\frac{j}{m}}}} = F\left(1 + \frac{j}{m}\right)^{-mn} + \frac{Cg}{m} a_{\overline{mn}|\frac{j}{m}}$$

In this case, as would be expected, the purchase price agrees with that in Art. 48.

When the sinking fund rate is $j'$ converted $m$ times per year, the sinking-fund equation becomes

$$\left(\frac{Cg}{m} - \frac{Vj}{m}\right) s_{\overline{mn}|\frac{j'}{m}} = V - F$$

$$V = \frac{F + \frac{Cg}{m} s_{\overline{mn}|\frac{j'}{m}}}{1 + \frac{j}{m} s_{\overline{mn}|\frac{j'}{m}}}$$

The same expression for $V$ would evidently be found if $F > V$.

The discussion in this article for the value of an ordinary bond is analogous to that given in Art. 47 for an investment whose return is an annuity. A similar discussion can be given to restore the capital for each of the more general investments treated in Arts. 52 to 54 inclusive. Other illustrations of this type of investment are given in the exercises below.

EXAMPLE. A \$10000 bond issued July 1, 1923, bearing interest at 5% payable semi-annually, is to be redeemed at par in 5 years. Find its purchase price to yield 4% converted semi-annually if a sinking fund which accumulates at ($j = .035$, $m = 2$) is created to restore the premium.

SOLUTION. In this case the equation in $V$ is

$$(250 - .02\,V) s_{\overline{10}|.0175} = V - 10000$$

Solving,
$$V = \frac{10000 + 250\, s_{\overline{10}|.0175}}{1 + .02\, s_{\overline{10}|.0175}}$$

$$= \$10444.94 \quad \text{(See Example, Art. 48.)}$$

### EXERCISES

**1.** Solve the above example if the sinking fund accumulates at ($j = .04$, $m = 2$). Ans. \$10449.13.

**2.** A \$5000 bond bearing interest at $5\frac{1}{2}\%$ payable semi-annually, and redeemable at par in 10 years, is bought to yield ($j = .05$, $m = 2$). If a sinking fund which accumulates at ($j = .05$, $m = 2$) is established to restore the premium, construct the sinking-fund schedule. (See Exercise 4, Art. 48.)

**3.** Same as Exercise 2, except that the sinking fund accumulates at ($j = .045$, $m = 2$). Ans. \$5191.89 (purchase price).

**4.** If the bond in Exercise 2 is bought to yield ($j = .06$, $m = 2$) and a sinking fund which accumulates at ($j = .06$, $m = 2$) is established to accumulate the discount, construct the sinking-fund schedule. (See Exercise 4, Art. 48.) Ans. \$4814.03.

**5.** Same as Exercise 4, except that the sinking fund accumulates at ($j = .055$, $m = 2$).

**6.** It is estimated that a taxicab will yield a net return of \$500 at the end of each year for 4 years, at the end of which time the taxicab will have a scrap value of \$100. If a sinking fund which accumulates at ($j = .04$, $m = 1$) is created to restore in 4 years the capital invested less the scrap value, find the purchase price to yield 10% annually. (See Exercise 2, Art. 47, and Exercise 9, Art. 48.) Ans. \$1560.55.

**7.** It is estimated that the income of a lumber company in developing a tract of wooded land will be \$10000 annually for 15 years, at which time production will cease and the land will be worth \$2500. If a sinking fund which accumulates at ($j = .04$, $m = 1$) is established to restore the capital invested less the value of the land, find the purchase price correct to dollars to yield 10% annually. (See Exercise 10, Art. 48.) Ans. \$67526.

**51. The value of a bond purchased between interest payment dates.** The expression for the value of an ordinary bond or of an annuity bond purchased, at date of issue or at an interest-bearing date, to yield a given interest rate can be used to find the value of the bond purchased at any date to yield this interest rate. If $V'$ denotes the value of a bond $n_1$ years $\left(n_1 < \frac{1}{m}\right)$ after the date at which $V$ was found in Art. 48, then, by Theorem I, Art. 15, Chapter I, it follows that $V' = V\left(1 + \frac{j}{m}\right)^{mn_1}$. In practice, however, the simple interest formula is ordinarily used for the fractional term, $n_1$, so that

$$V' = V(1 + jn_1)$$

Unless otherwise stated, this formula will be used in what follows to find the value of a bond purchased between interest payment dates *to yield a given interest rate.*

Bonds are often listed or quoted at a given price. The amount quoted is usually what must be paid by the purchaser for each \$100 of face value. For example, a \$10000 bond quoted at 105 can be bought for \$10500. In such cases the interest rate earned by the purchaser is not ordinarily known, so that the preceding formula cannot be applied. When a bond quoted at a certain price is sold between interest payment dates, the seller is entitled to part of the interest due on the bond at the next interest payment date, since he has held the bond during a part of this period. In practice, the part he gets is usually the simple interest on the face of the bond at the rate named in the bond for the part of the period ($n_1$ years) during which he holds it. The value by this method is said to be the *value at a certain price and accrued interest.* In the case of an ordinary bond, as treated in Art. 48, sold at a price $P$ and accrued interest this value is given by

$$V' = P + Cgn_1$$

EXAMPLE 1. Find the value of the bond in the example in Art. 48 to yield 4% semi-annually, if purchased October 1, 1925.

SOLUTION. By Art. 48, or by the first schedule in Art. 49, the value of this bond July 1, 1925, is \$10280.07. Hence the purchase price $V'$ to yield ($j = .04$, $m = 2$) is given by

$$\begin{aligned} V' &= 10280.07\ (1.01) \\ &= \$10382.87 \end{aligned}$$

EXAMPLE 2. If the bond in the example in Art. 48 were quoted at 102 on October 1, 1925, find its purchase price with accrued interest.

SOLUTION. The price at which it is quoted is \$10200, and the accrued interest on the bond is \$125. Hence the purchase price with accrued interest, $V'$, is given by

$$\begin{aligned} V' &= 10200 + 125 \\ &= \$10325 \end{aligned}$$

## EXERCISES

**1.** A \$100 bond dated October 15, 1920, bearing interest at 6% payable annually, and maturing in 10 years, was purchased January 1, 1924 to yield ($j = .05$, $m = 2$). Find the purchase price. Ans. \$106.52.

**2.** Same as Exercise 1, except that the bond is redeemable at 102.

**3.** A \$500 bond bearing interest at 5% semi-annually, payable June 15 and December 15, was purchased on August 1 at $98\frac{7}{8}$ and accrued interest. Find the purchase price. Ans. \$497.57.

**4.** A \$500 Liberty Bond was purchased at 99−26. If the bond bears interest at $4\frac{1}{4}$% payable semi-annually, find the purchase price 23 days before an interest date; 23 days after an interest date. Ans. \$508.33; \$500.42.

**5.** A \$1000 bond bearing interest at $5\frac{1}{2}$% payable semi-annually was purchased at $103\frac{1}{4}$, 95 days after an interest date. Find the purchase price.

**52. The value of serial bonds or bonds to be redeemed in equal periodic instalments. The return is that of a set of ordinary bonds having equal face values.** In an ordinary bond the face value is paid in a single instalment. In annuity bonds (Art. 46) the face value is paid in instalments at the interest payment dates such that the instalment plus the interest is constant or nearly constant. In *serial bonds* the face value is paid in instalments which are equal or nearly equal. The values of all bonds whose face is paid in known instalments may evidently be found by summing the values of the ordinary bonds composing them. In some important cases, however, simpler methods may be employed for computing the values of such bonds. In this article methods for finding the values of serial bonds whose face value is paid in equal periodic instalments are treated. The treatment is given first for a special case and then for the general case.

Special Case: *The value of serial bonds whose face value is paid in equal instalments at the interest payment dates.* The return in this case consists of the annuity composed of the instalments on the bonds and the decreasing annuity composed of the interest payments on the instalments. If $C_1$ denoted the amount of each instalment the present value of the annuity composed of them is $C_1 a_{\overline{mn}|\frac{j}{m}}$, and the present value of the decreasing annuity composed of the interest payments is, by Formula (8), Art. 32, Chapter II, $\frac{C_1 g}{m}\left(\frac{mn - a_{\overline{mn}|\frac{j}{m}}}{\frac{j}{m}}\right)$. Hence

$$V = C_1 a_{\overline{mn}|\frac{j}{m}} + \frac{C_1 g}{m}\left(\frac{mn - a_{\overline{mn}|\frac{j}{m}}}{\frac{j}{m}}\right)$$

This formula can be written in two other forms, analogous to those for an ordinary bond (Art. 48). These are

$$V = mnC_1 + C_1\left(\frac{g}{m} - \frac{j}{m}\right)\left(\frac{mn - a_{\overline{mn}|\frac{j}{m}}}{\frac{j}{m}}\right)$$

$$V = mn\frac{C_1 g}{j} + \left(C_1 - \frac{C_1 g}{j}\right) a_{\overline{mn}|\frac{j}{m}}$$

The right member of the first involves the values of an annuity, and of a decreasing annuity; that of the second involves the value of a decreasing annuity; and that of the third, the value of an annuity. In finding the values of serial bonds of this type the principles underlying the formulas should be used rather than the formulas. These principles are seen at once by expressing the formulas verbally. For example, by the last formula the value of the bonds equals the amount of $mn$ investments of $\frac{C_1 g}{j}$ each plus or minus the present value of an annuity whose rent is the difference between an instalment on the bond and the amount of one of these investments. These three principles will now be applied in turn to solve

EXAMPLE 1. Serial bonds to the amount of \$10000 bearing interest at 5% payable semi-annually are to be redeemed in 5 years in ten equal semi-annual instalments of \$1000 each. Find the value of these bonds to yield the purchaser 4% converted semi-annually.

SOLUTION 1. The present value of the instalments is $1000\, a_{\overline{10}|.02}$, and that of the decreasing annuity composed of the interest payments is $25\,\frac{10 - a_{\overline{10}|.02}}{.02}$. Hence

$$V = 1000\, a_{\overline{10}|.02} + 25\,\frac{10 - a_{\overline{10}|.02}}{.02}$$
$$= \$10254.354$$

SOLUTION 2. If ten \$1000 investments are made at 4% payable semi-annually for terms ranging by half years, from $\frac{1}{2}$ to 5 years, the return on these investments differs from the return on the bonds by the decreasing annuity which is the difference between the interest payments on the bonds and those on the investments. The rent payments of this decreasing annuity begin with \$50 and decrease \$5 each period. The present value of the decreasing

annuity is then $5\dfrac{10 - a_{\overline{10}|.02}}{.02}$. Hence

$$V = 10000 + 5\frac{10 - a_{\overline{10}|.02}}{.02}$$
$$= \$10254.354$$

SOLUTION 3. If ten \$1250 investments are made at 4% payable semi-annually for terms ranging by half years, from $\frac{1}{2}$ to 5 years, the return on these investments differs from the return on the bonds by an annuity which is the difference between the instalments on the bonds and those of the investments. The rent of this annuity is 250 and its present value is $250\, a_{\overline{10}|.02}$. Hence

$$V = 12500 - 250\, a_{\overline{10}|.02}$$
$$= \$10254.354$$

EXERCISE. Find the value of the bonds in this example to yield 6% converted semi-annually. Ans. \$9755.034.

GENERAL CASE: *The value of serial bonds whose face is paid in equal periodic instalments at periods of r years, the first instalment being paid in $n_1$ years.* The number of instalments is $\dfrac{n - (n_1 - r)}{r}$. When $n_1 = r = \dfrac{1}{m}$ this case reduces to the special case treated above. Each of the principles used in the special case may be applied to this general case. If the first two are used, the decreasing annuities must be replaced by sets of annuities of the type treated in Exercise 3, Art. 32, Chapter II. The third principle may be applied, however, without change. In this method, corresponding to each instalment on the bond an investment of $\dfrac{C_1 g}{j}$ is made for a term equal to that of the instalment. These investments yield the interest payments which the bonds yield, so that the difference between the return on the bonds and that on the investments is an annuity whose rent is $C_1 - \dfrac{C_1 g}{j}$, of rent period $r$, having the first payment in $n_1$ years and the last in $n$ years. The present value of this annuity, whose term begins in $n_1 - r$ years, is $\left(C_1 - \dfrac{C_1 g}{j}\right)\dfrac{a_{\overline{mn}|\frac{j}{m}} - a_{\overline{m(n_1 - r)}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$. Hence

$$V = \frac{n - (n_1 - r)}{r}\frac{C_1 g}{j} + \left(C_1 - \frac{C_1 g}{j}\right)\frac{a_{\overline{mn}|\frac{j}{m}} - a_{\overline{m(n_1 - r)}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

This method is illustrated by the solution of

EXAMPLE 2. A \$10000 loan bearing 5% payable semi-annually is to be repaid in ten \$1000 instalments, the first in 5 years and the others at periodic intervals of 2 years. Find the purchase price to yield 4% converted semi-annually.

SOLUTION. If ten investments of 1250 each are made at 4% payable semi-annually for terms of 5, 7, 9, . . , 23 years respectively, the interest payments on them are the same as those on the loan. The return on these ten investments exceeds that on the loan by an annuity whose rent is \$250, rent period 2 years, first payment in 5 years and last in 23 years. The value of this deferred annuity (deferred 3 years) is $250\,\frac{a_{\overline{46}|.02} - a_{\overline{6}|.02}}{s_{\overline{4}|.02}}$. Hence

$$V = 12500 - 250\,\frac{a_{\overline{46}|.02} - a_{\overline{6}|.02}}{s_{\overline{4}|.02}}$$

$$= \$11026.61$$

EXERCISE. Find the purchase price of the loan to yield 6% converted semi-annually.

The same method applies to the relation of serial bonds which are redeemed above par.

## EXERCISES

**1.** Solve Example 2 above by use of the result found in Exercise 3, Art. 32.

**2.** Solve Example 2 above by summing the values of the annuity consisting of the ten instalments and of the ten annuities consisting of the interest payments on these instalments.

**3.** Serial bonds to the amount of \$10000 are redeemable in 10 equal annual instalments. If the bonds bear interest at 6% payable annually, find the purchase price at date of issue to yield ($j = .065$, $m = 1$). Ans. \$9783.76.

**4.** Serial bonds to the amount of \$25000 are redeemable at 102 in ten equal semi-annual instalments. If the bonds bear interest at 5½% payable semi-annually, find the purchase price at date of issue to yield ($j = .06$, $m = 2$). Ans. \$25120.30.

**5.** Serial bonds to the amount of \$25000 are redeemable at 105 in five equal annual instalments, the first redemption to take place 3 years from date of issue. If the bonds bear interest at 6% payable annually, find the purchase price at date of issue to yield ($j = .055$, $m = 1$). Ans. \$26487.96.

**6.** Same as Exercise 3, except that the bonds are redeemable in 5 equal biennial instalments, the first redemption to take place 3 years from date of issue.

**53. Investment schedules for instalment bonds.** An investment schedule for instalment bonds differs from a schedule for an ordinary bond only in that it provides a column for the redemption payments.

EXAMPLE. $10000 of serial bonds issued January 1, 1919, bearing interest at 5% payable semi-annually, are redeemed in ten equal semi-annual instalments of $1000 each. If these bonds are bought to yield the purchaser 6% converted semi-annually, construct an investment schedule.

SOLUTION. By the first exercise in Art. 52, the purchase price of these bonds is $9755.034.

INVESTMENT SCHEDULE

| DATE | BOOK VALUE | INTEREST ON BOOK VALUE AT 3% | DIVIDEND RECEIVED | FOR ACCUMULATION OF DISCOUNT | REDEMPTION PAYMENTS |
|---|---|---|---|---|---|
| Jan. 1, 1919 | 9755.034 | | | | |
| July 1, 1919 | 8797.685 | 292.651 | 250 | 42.657 | 1000 |
| Jan. 1, 1920 | 7836.616 | 263.931 | 225 | 38.931 | 1000 |
| July 1, 1920 | 6871.714 | 235.098 | 200 | 35.098 | 1000 |
| Jan. 1, 1921 | 5902.865 | 206.151 | 175 | 31.151 | 1000 |
| July 1, 1921 | 4929.951 | 177.086 | 150 | 27.086 | 1000 |
| Jan. 1, 1922 | 3952.850 | 147.899 | 125 | 22.899 | 1000 |
| July 1, 1922 | 2971.436 | 118.586 | 100 | 18.586 | 1000 |
| Jan. 1, 1923 | 1985.579 | 89.143 | 75 | 14.143 | 1000 |
| July 1, 1923 | 995.146 | 59.567 | 50 | 9.567 | 1000 |
| Jan. 1, 1924 | .000 | 29.854 | 25 | 4.854 | 1000 |

EXERCISES

1. Construct an investment schedule for Exercise 3, Art. 52.
2. Construct an investment schedule for Exercise 4, Art. 52.
3. Construct an investment schedule for Exercise 6, Art. 52.

**54. The value of any investment whose return is known.** When the return on an investment is different from the types considered in the preceding articles, it can always be resolved into sets which belong to these types. It is evident, for example, that any return is a set of single sums. Again the return on any instalment bond is composed of sets of returns on ordinary bonds. Often a given return can be resolved into sets in two or more ways whose values can be computed with about equal ease. In finding the value of any investment whose return is known the aim should be to resolve the return into sets such that the computations needed in finding their values can be performed with minimum effort. Some illustrations are afforded by the solution of the

EXAMPLE. A debt of \$10000 consisting of 100 bonds of \$100 denomination bearing interest at 5% payable semi-annually is to be paid, principal and interest, in ten semi-annual instalments as nearly equal as possible. Find the purchase price of these bonds to yield the investor 4% converted semi-annually.

The retirement of the bonds in accordance with the solution of the example in Art. 36 will be used as a basis for finding the purchase price.

SOLUTION 1. This solution is based on resolving the return into three sets of the type treated in Art. 53. The first set consists of the first three instalments of \$900 each and their interest payments; the second consists of the next four instalments of \$1000 each and their interest payments; the third consists of the last three instalments of \$1100 each and their interest payments. By Art. 52 the respective values of these sets are:

$$3(1125) - 225\, a_{\overline{3}|.02}$$

$$4(1250) - 250\, (a_{\overline{7}|.02} - a_{\overline{3}|.02})$$

$$3(1375) - 275\, (a_{\overline{10}|.02} - a_{\overline{7}|.02})$$

Adding these gives

$$V = 12500 - 275\, a_{\overline{10}|.02} + 25\, a_{\overline{7}|.02} + 25\, a_{\overline{3}|.02}$$

$$= \$10263.69$$

SOLUTION 2. This solution is based on resolving the return into two sets; one consists of the ten instalments on the face of the bonds and the other of the interest payments. The value of the first set can be easily found by resolving it into an annuity having ten rent payments of \$1100 each less an annuity having seven rent payments of \$100 each, less again an annuity having three rent payments of \$100 each. The value of the second set can be found by resolving it into three decreasing annuities analogous to the three annuities into which the first set is resolved. Hence

$$V = 1100\, a_{\overline{10}|.02} - 100\, a_{\overline{7}|.02} - 100\, a_{\overline{3}|.02} + 27.50\left(\frac{10 - a_{\overline{10}|.02}}{.02}\right)$$
$$- 2.50\left(\frac{7 - a_{\overline{7}|.02}}{.02}\right) - 2.50\left(\frac{3 - a_{\overline{3}|.02}}{.02}\right)$$
$$= 12500 - 275\, a_{\overline{10}|.02} + 25\, a_{\overline{7}|.02} + 25\, a_{\overline{3}|.02}$$
$$= \$10263.69$$

Solution 3. This solution is based on resolving the return into two sets; one consists of the ten instalments and their interest payments at 4% payable semi-annually, the other consists of the difference between the interest payments on the bond and the interest payments in the first set. The value of the first set is 10000; that of the second can be found by resolving it into three decreasing annuities as in solution 2. Hence

$$V = 10000 + 5.50\left(\frac{10 - a_{\overline{10}|.02}}{.02}\right) - .50\left(\frac{7 - a_{\overline{7}|.02}}{.02}\right) - .50\left(\frac{3 - a_{\overline{3}|.02}}{.02}\right)$$
$$= 12500 - 275\, a_{\overline{10}|.02} + 25\, a_{\overline{7}|.02} + 25\, a_{\overline{3}|.02}$$
$$= \$10263.69.$$

### EXERCISES

1. Construct an investment schedule for the above example.

2. Solve this example if the investment rate is 6% converted semi-annually. Ans. \$9746.22.

3. In Exercise 1, Art. 36, find the purchase price of the bonds at date of issue to yield ($j = .06$, $m = 2$).

4. In Exercise 2, Art. 36, find the purchase price of the bonds at date of issue to yield ($j = .045$, $m = 2$).

5. A corporation issues bonds to the amount of \$600000 bearing interest at 5% payable semi-annually. The bonds are redeemable as follows: \$100000 in 5 years at 100; \$200000 in 8 years at 102; \$300000 in 11 years at 105. Find the purchase price at date of issue to yield ($j = .055$, $m = 2$). Construct an investment schedule. Ans. \$590029.87.

**55. The number of equal periodic payments needed to purchase an investment of known value when the interest rate is given. Building and Loan Association Stocks.** In the investments treated in the preceding articles, the purchase price is determined in a lump sum at the time the investment is made. Some investments, however, are paid for on the instalment plan. A good illustration is afforded by stocks issued by some building and loan associations. In such cases the payments are usually equal and periodic and

they are accumulated at a specified rate; the problem is to find the number of payments. When the payments accumulate to the value of the stock, the stock is said to *mature*. Stock of given value purchased by a set of equal periodic payments will usually mature at a time within the period just after the last full payment is made. The subscriber to the stock is ordinarily notified, however, of its maturity at the end of the next period at which time settlement is made.

EXAMPLE. A member of a building and loan association pays \$1.00 at the beginning of each month on a \$100 share of stock. Find the number of payments and time of maturity if the association pays 6% converted monthly.

SOLUTION. The payments constitute an annuity due of term $n$ years. Equating the value of this annuity due just after the last payment is made to 100 gives:

$$\frac{1.005^{12n} - 1}{.005} = s_{\overline{12n}|.005} = 100$$

By Table V, $s_{\overline{81}|.005} = 99.56$, and $s_{\overline{82}|.005} = 101.06$.

Hence 81 payments are needed and the stock matures in 6 years, 8 months, and $d$ days. To find $d$ the ordinary simple interest formula may be used; $100 - 99.56 = 0.44$ is the interest on 99.56 for $d$ days. This gives

$$.44 = 99.56 \frac{d}{360} (.06)$$

Solving, $d = 27$

Hence 81 payments mature the stock in 6 years, 8 months, and 27 days. In this case settlement would usually be made at the end of 6 years and 9 months, at which time the 81 payments made have a value of $99.56(1 + .005) = 101.06$. Settlement at this time requires the delivery of the stock and the cash payment of 6 cents to the subscriber.

### EXERCISES

1. A member of a building and loan association pays \$1 at the beginning of each month on a \$100 share of stock. If the association pays 5% converted monthly show that the stock will mature in 6 years and 11 months, the final payment being 67 cents.

2. Find the book values of the payments on the stock in Exercise 1 at the end of 4 years; at the end of 5 years and 6 months. Ans. \$54.24; \$77.10.

**56. To find the rate in investment, debt, and other problems in finance.** Interest and discount rates are frequently the unknowns in investment, debt, and other problems in finance. A bond purchased at a quoted value, a debt amortized by a known

set of payments, a depreciation estimated by the use of the simple discount formula, are illustrations. The equation determining the rate in any rate problem is found, in the usual manner, by equating the values of equivalent sets of sums. Methods for determining rates based on the compound interest formula and on annuity formulas have been presented in Art. 12, Chapter I, and in Art. 26, Chapter II. By use of one or more of these methods any rate equation arising in elementary finance can be solved. In the next four articles solutions of some important types of rate equations are given; the types treated are introduced in the order of their difficulty.

**57. Rate equations based on the value of one sum.** Various problems involving the determination of rates based on the interest and discount formulas for the value of one sum have been given in Chapter I. Two problems which relate to depreciation are solved in this article.

EXAMPLE 1. An article costing \$1000 has a value of \$100 at the end of 10 years. If the simple discount formula is used in estimating depreciation, find the discount rate.

SOLUTION. By the simple discount formula, $D = Snd$,

$$900 = 1000(10)d$$

Solving, $$d = .09$$

EXAMPLE 2. An article costing \$1000 has a value of \$100 at the end of 10 years. If the compound discount formula, with annual conversions, is used in estimating depreciation, find the discount rate.

SOLUTION. By the compound discount formula, $P = S\left(1 - \frac{f}{m}\right)^{mn}$

$$100 = 1000(1 - f)^{10}$$

Solving, $$1 - f = \sqrt[10]{.1}$$

$$f = 1 - \sqrt[10]{.1} = .206$$

**58. Rate equations based on the value of an annuity.** Two examples of rate equations based on the value of an annuity are given in Art. 26, Chapter II. In this article two additional examples are solved.

EXAMPLE 1. A debt of \$10000 is to be amortized by payments of \$1150 at the end of each half year for 5 years. If $m = 2$, find $\frac{j}{2}$.

Equating the present value of the payments to 10000 gives

$$1150\,\frac{1-\left(1+\frac{j}{2}\right)^{-10}}{\frac{j}{2}} = 10000$$

or

$$\frac{1-\left(1+\frac{j}{2}\right)^{-10}}{\frac{j}{2}} = 8.69565217$$

SOLUTION BY INTERPOLATION. In this case,

$$a_{\overline{10}|\frac{j}{2}} = \frac{1-\left(1+\frac{j}{2}\right)^{-10}}{\frac{j}{2}}$$

can be used to construct a table of values for interpolation. By Table VI $a_{\overline{10}|.025} = 8.75206393$, and $a_{\overline{10}|.0275} = 8.64007616$. Hence $\frac{j}{2}$ lies between .025 and .0275. Interpolating from the table

| $\frac{j}{2}$ | $a_{\overline{10}\|\frac{j}{2}}$ |
|---|---|
| .025 | 8.7521 |
| $\frac{j}{2}$ | 8.6957 |
| .0275 | 8.6401 |

gives $\dfrac{\frac{j}{2}-.025}{.0025} = \dfrac{.0564}{.1120}$

Solving, $\frac{j}{2} = .02626$

By the use of a seven-place table of logarithms it is found that $a_{\overline{10}|.02625} = 8.6958$, and $a_{\overline{10}|.02626} = 8.6953$. Hence $\frac{j}{2}$ lies between .02625 and .02626.

EXERCISE 1. Solve this example by using Table VII for interpolation. $\left(\text{In this example, } \frac{1}{a_{\overline{10}|\frac{j}{2}}} = .11500000.\right)$

EXERCISE 2. Solve this example by Newton's method, using $\frac{j}{2} = .0262 + h$.

EXERCISE 3. Solve this example by the method of iteration.

EXAMPLE 2. A debt of \$10000 is to be amortized by payments of \$1150 at the end of each half year for 5 years. Find the effective interest rate $i$.

Equating the present value of the payments to \$10000 gives

$$1150\,\frac{1-(1+i)^{-5}}{(1+i)^{\frac{1}{2}}-1} = 10000$$

or

$$\frac{1-(1+i)^{-5}}{(1+i)^{\frac{1}{2}}-1} = 8.69565217$$

SOLUTION 1. In this solution $V = \dfrac{1-(1+i)^{-5}}{(1+i)^{\frac{1}{2}}-1}$ is used to construct a table of values for interpolation.

| $i$ | $V$ |
|---|---|
| .05 | 8.7659 |
| $i$ | 8.6957 |
| .055 | 8.6564 |

$$\frac{i-.05}{.005} = \frac{.0702}{.1095}$$

Solving, $i = .0532$

SOLUTION 2. In this solution, use is made of the formula $1 + i = \left(1 + \frac{j}{2}\right)^2$ which connects the corresponding rates $i$ and $j_{(2)}$. Replacing $1 + i$ by $\left(1 + \frac{j}{2}\right)^2$, the equation to be solved becomes

$$\frac{1-\left(1+\frac{j}{2}\right)^{-10}}{\frac{j}{2}} = 8.69565217$$

By solution of Example 1, $\frac{j}{2} = .02625$ or $.02626$. Substituting either of these values for $\frac{j}{2}$ into the relation $1 + i = \left(1 + \frac{j}{2}\right)^2$ gives $i = .0532$.

## EXERCISES

**1.** A lease whose return is \$2500 at the end of each six months is sold for \$70000, 30 years before it expires. If $m = 2$, find the rate of interest on the investment. Ans. .0589.

**2.** A house valued at \$6800 is purchased for \$1000 in cash and \$1000 at the end of each year for 8 years. What interest rate converted annually will make the payments equivalent to the value of the house. Ans. .0775.

**3.** A piano valued at \$500 is purchased for \$100 in cash and \$35 at the end of each month for 12 months. Find the interest rate, converted monthly, which will make the present value of the payments equal to the value of the piano; find also the simple discount rate. Ans. $\frac{j}{12} = .0076$; $d = .0879$.

**4.** At the time of maturity of a certain life insurance policy the beneficiary was offered \$10000 in cash, or \$1161.75 in cash, and a like amount at the end of each year for 9 years. If $m = 1$, find the interest rate used.

**5.** A deposit of \$10 at the end of each month for 10 years was made in a saving's bank. At the end of this time there was a credit of \$1548.57 to this account. If $m = 2$, find the interest rate used. Ans. .0500.

## 59. Rate equations based on the value of an ordinary bond.

EXAMPLE. A \$10000 bond bearing interest at 5% payable semi-annually is to be redeemed at par in 5 years. This bond is purchased for \$10200. If $m = 2$, find $\frac{j}{2}$.

Equating the value of the bond at the rate $j$ payable semi-annually to 10200 gives

$$10200 = 10000\left(1 + \frac{j}{2}\right)^{-10} + 250\,\frac{1 - \left(1 + \frac{j}{2}\right)^{-10}}{\frac{j}{2}}$$

*Solution by interpolation.* In this case

$$V = 10000\left(1 + \frac{j}{2}\right)^{-10} + 250\,\frac{1 - \left(1 + \frac{j}{2}\right)^{-10}}{\frac{j}{2}}$$

can be used to construct a table of values of $V$ and $\frac{j}{2}$ for interpolation. The upper set of values in the following table are readily found by the use of Tables IV and VI; the lower set is evident:

| $\frac{j}{2}$ | $V$ |
|---|---|
| .0225 | 10221.6554 |
| .02274 | 10200.12 |
| $\frac{j}{2}$ | 10200.00 |
| .02275 | 10199.23 |
| .025 | 10000.00 |

Interpolating between the upper and the lower sets gives

$$\frac{\frac{j}{2} - .0225}{.0025} = \frac{21.6554}{221.6554}$$

$$\frac{j}{2} = .02274$$

The two inner sets can be computed by the use of a seven-place table of logarithms. This table shows that $\frac{j}{2}$ lies between .02274 and .02275.

### EXERCISES

**1.** A \$100 bond bearing interest at $5\frac{1}{2}\%$ payable semi-annually and redeemable at par in 10 years is purchased at date of issue for \$98.50. What rate converted semi-annually does the investment yield? Ans. .0570.

**2.** Same as Exercise 1 except that the bond is redeemable at 102.

Ans. .0585.

**3.** A \$1000 bond issued by a corporation bears interest at $5\frac{1}{4}\%$ payable semi-annually and is redeemable at par in 10 years. A \$1000 bond issued by a second corporation bears interest at 5% payable semi-annually and is

redeemable at 102 in 10 years. Which of the two bonds is the better investment if the first is quoted at $100\frac{1}{2}$ and the second at $99\frac{1}{2}$?

4. A \$100 liberty bond bearing interest at $4\frac{1}{4}\%$ payable semi-annually and redeemable at par in 5 years was purchased for 96−21. What rate converted semi-annually does the investment yield? Ans. .0501.

**60. Rate equations based on the value of any set of sums.** The rate equations which occur in elementary finance are usually of the types treated in the three preceding articles. A rate equation based on the value of any set of sums can be solved, however, by the use of the methods employed in these articles.

EXAMPLE. A share of common stock bought at par (\$100) yielded dividends at 6% payable quarterly, and a bonus of 2% at the end of each year, for 10 years, when it was sold at \$105. If $m = 4$, find $\frac{j}{4}$.

Equating the present value of the return on the stock to \$100 gives

$$100 = \frac{3}{2}\frac{1-\left(1+\frac{j}{4}\right)^{-40}}{\frac{j}{4}} + 2\frac{1-\left(1+\frac{j}{4}\right)^{-40}}{\left(1+\frac{j}{4}\right)^{4}-1} + 105\left(1+\frac{j}{4}\right)^{-40}$$

SOLUTION. In this case

$$V = \frac{3}{2}\frac{1-\left(1+\frac{j}{4}\right)^{-40}}{\frac{j}{4}} + 2\frac{1-\left(1+\frac{j}{4}\right)^{-40}}{\left(1+\frac{j}{4}\right)^{4}-1} + 105\left(1+\frac{j}{4}\right)^{-40}$$

can be used to construct a table of values for $V$ and $\frac{j}{4}$ for interpolation,

| $\frac{j}{4}$ | $V$ |
|---|---|
| .02 | 101.8609 |
| $\frac{j}{4}$ | 100.0000 |
| .0225 | 95.0710 |

$$\frac{\frac{j}{4}-.02}{.0025} = \frac{1.8609}{6.7899}$$

$$\frac{j}{4} = .0207$$

## EXERCISES

1. A debt of \$5000 was paid by payments of \$1000 at the end of each year for 3 years, and \$1600 at the end of each of the two following years. What rate of interest converted annually was used? Ans. .0690.

2. Common stock of a corporation bought at par paid dividends at 6% semi-annually for 3 years, 8% semi-annually for 2 years, and 7% semi-annually

for 2 years. At the end of the 7 years the stock was sold at 102. What rate of interest converted semi-annually was received on the investment?

Ans. .0701.

**3.** Five shares of common stock in a mercantile company were purchased for \$500. At the end of each six months for two years the dividend on the stock was \$20.00. For the following two years no dividends were paid. For the following six years, a dividend of \$15.00 was paid at the end of each six months. At the end of the 10 years the stock was sold for \$490. What rate of interest converted semi-annually was received on the investment?

Ans. .0501.

**4.** A \$100 share of a building and loan association stock bought on monthly payments matures at the time of the 80th payment. If the 80th payment is 18 cents and each of the others is \$1.00, find $\frac{j}{2}$. Ans. .0344.

**5.** The serial bonds in Example 1, Art. 52, were purchased for \$10125. If $m = 2$, find $\frac{j}{2}$. Ans. .0450.

## DEPRECIATION AND CAPITALIZED COST

**61. Methods of estimating depreciation charges. Definitions.** Buildings, machinery, and other property used in business enterprises deteriorate in value. Part of such losses in value can be provided for by current repairs. The loss in value that cannot be provided for by current repairs is called *depreciation*. When an article has depreciated to an extent that makes replacement necessary the total loss in value due to depreciation is the difference between its cost and its *salvage* or *scrap* value at the time of replacement. Here, as elsewhere in business, capital invested should not be impaired. To provide for depreciation losses a common practice is to set aside sums, called *depreciation charges*, at periodic intervals. The *book value* of an article at any time is its cost less the value at the time, of the depreciation charges. The *wearing value* of an article at any time is its book value at the time less the salvage value. The *total wearing value* is its cost less its salvage value. The *condition per cent* of an article at any time is its wearing value divided by its cost.

All articles do not depreciate in the same manner, so that different methods of estimating depreciation charges are necessary. Practically all methods of treating depreciation are alike in that they aim to restore the total wearing value of an article at the end

of its depreciation term; that is, at the time replacement is necessary. In Arts. 62 to 65 inclusive, four different methods of estimating depreciation charges are presented. In the examples and exercises in these articles $n$ denotes the number of years in the depreciation term, and $t$ denotes the number of years from the beginning to any date within the term.

**62. The straight-line or simple-discount method.** By this method the simple discount formulas $D = Snd$, $P = S(1 - nd)$, are used in estimating the depreciation and the book value at any time. It is called the straight-line method because the graphs of these formulas for given values of $S$ and $d$ are straight lines (Art. 7, Chapter I). The simple discount rate in a given problem can be found from $D = Snd$ by replacing $S$ by the cost of the article, $D$ by the total depreciation, and $n$ by the depreciation term. The annual depreciation is the cost times the simple discount rate.

EXAMPLE. An article costing \$1000 has a salvage value of \$100 at the end of 14 years. Use the straight-line method to find formulas for the depreciation $D$ and the book value $B$ at the end of $t$ years, and construct a depreciation schedule.

SOLUTION. Substituting $D = 900$, $S = 1000$, and $n = 14$ into $D = Snd$, gives $d = \frac{9}{140} = .06429$. Replacing $S$ by 1000, $P$ by $B$, $d$ by $\frac{9}{140}$, and $n$ by $t$ in the simple discount formulas gives

$$D = \frac{450\,t}{7}$$

$$B = 1000 - \frac{450\,t}{7}$$

By use of these formulas $B$ and $D$ can be computed for assigned values of $t$. The annual depreciation charge is $\frac{450}{7}$; it is also the difference between consecutive values of $B$ or consecutive values of $D$ when $t$ is assigned integral values. These results are shown in the depreciation schedule on the next page.

DEPRECIATION SCHEDULE

| Year | Book Value at End of Year | Depreciation Charge at End of Year | Total Depreciation Charge at End of Year |
|---|---|---|---|
| 0 | 1000.00 | | |
| 1 | 935.71 | 64.29 | 64.29 |
| 2 | 871.43 | 64.28 | 128.57 |
| 3 | 807.14 | 64.29 | 192.86 |
| 4 | 742.86 | 64.28 | 257.14 |
| 5 | 678.57 | 64.29 | 321.43 |
| 6 | 614.28 | 64.29 | 385.72 |
| 7 | 550.00 | 64.28 | 450.00 |
| 8 | 485.71 | 64.29 | 514.29 |
| 9 | 421.43 | 64.28 | 578.57 |
| 10 | 357.14 | 64.29 | 642.86 |
| 11 | 292.85 | 64.29 | 707.15 |
| 12 | 228.57 | 64.28 | 771.43 |
| 13 | 164.28 | 64.29 | 835.72 |
| 14 | 100.00 | 64.28 | 900.00 |

Exercise 1. If $C$ denotes the cost, $S$ the scrap value, and $n$ the depreciation term of an article, show that the straight-line method gives the formulas

$$D = \frac{C - S}{n} t, \qquad B = C - \frac{C - S}{n} t$$

Exercise 2. Apply the formulas in Exercise 1 to the above example.

## EXERCISES

**1.** A machine costing \$2500 is depreciated 2% monthly by the straight-line method. If the machine has a scrap value of \$200, construct the depreciation schedule for the life of the machine.

**2.** The owner of an automobile in Ohio is required to list his automobile for taxation in accordance with the following schedule: first year, at 70% of list

price; second year, 60%; third year, 50%; fourth year, 40%; fifth year, 30%; sixth year, 20%; after the sixth year, at its actual value in money. Construct a depreciation schedule for the first six years for an automobile bought in 1916 at a cost of \$3000, assuming that the list price remains constant.

3. If the rate of depreciation, $d$, is given, show that the straight-line method is not applicable if $n > \frac{1}{d}$.

**63. The constant percentage or compound discount method.** By this method the compound discount formulas, $P = S\left(1 - \frac{f}{m}\right)^{mn}$, $D = S - P$, are used in estimating the book value and the depreciation at any time. It is called the constant percentage method since the annual depreciation charge for each period is a constant percentage of the book value at the beginning of the period. The constant percentage or compound discount rate in a given problem can be found from $P = S\left(1 - \frac{f}{m}\right)^{mn}$ by replacing $S$ by the cost of the article, $P$ by the salvage value, $n$ by the depreciation term, and $m$ by the number of discount conversions per year.

EXAMPLE. An article costing \$1000 has a salvage value of \$100 at the end of 14 years. Use the constant percentage method to find formulas for the book value $B$ and the total depreciation $D$ at the end of $t$ years, and construct a depreciation schedule.

SOLUTION. Substituting $P = 100$, $S = 1000$, $n = 14$, and $m = 1$ into $P = S\left(1 - \frac{f}{m}\right)^{mn}$ gives $1 - f = \sqrt[14]{.1} = .84834$, or $f = .15166$. Replacing $S$ by 1000, $P$ by $B$, $1 - f$ by $(.1)^{\frac{1}{14}}$, $n$ by $t$, and $m$ by 1 in the compound discount formulas gives

$$B = 1000(.1)^{\frac{t}{14}}$$

$$D = 1000 - 1000(.1)^{\frac{t}{14}}$$

By use of these formulas and a table of logarithms, $B$ and $D$ can be computed readily for assigned values of $t$. The annual depreciation charges can then be computed by finding the differences between consecutive book values corresponding to integral values of $t$; they can also be found by multiplying each book value by $f$. The results are shown in the depreciation schedule on the next page.

DEPRECIATION SCHEDULE

| Year | Logarithm of Book Value | Book Values at End of Year | Depreciation Charge at End of Year | Total Depreciation Charge at End of Year |
|---|---|---|---|---|
| 0 | 3.0000000 | 1000.00 | | |
| 1 | 2.9285714 | 848.34 | 151.66 | 151.66 |
| 2 | 2.8571428 | 719.68 | 128.66 | 280.32 |
| 3 | 2.7857142 | 610.54 | 109.14 | 389.46 |
| 4 | 2.7142856 | 517.95 | 92.59 | 482.05 |
| 5 | 2.6428570 | 439.40 | 78.55 | 560.60 |
| 6 | 2.5714284 | 372.76 | 66.64 | 627.24 |
| 7 | 2.4999998 | 316.23 | 56.53 | 683.77 |
| 8 | 2.4285712 | 268.27 | 47.96 | 731.73 |
| 9 | 2.3571426 | 227.58 | 40.69 | 772.42 |
| 10 | 2.2857140 | 193.07 | 34.51 | 806.93 |
| 11 | 2.2142854 | 163.79 | 29.28 | 836.21 |
| 12 | 2.1428568 | 138.95 | 24.84 | 861.05 |
| 13 | 2.0714282 | 117.87 | 21.08 | 882.13 |
| 14 | 1.9999996 | 100.00 | 17.87 | 900.00 |

The first logarithm in the second column is $\log [1000\,(.1)^{\frac{1}{14}}]$; each of the other logarithms is obtained by adding $\log (.1)^{\frac{1}{14}} = \bar{1}.9285714$ to the one just preceding it.

Exercise 1. Compute the book values in the above schedule arithmetically by multiplying in turn by the value of $(1 - f)^{\frac{1}{14}}$.

Exercise 2. If $C$ denotes the cost, $S$ the scrap value, $n$ the depreciation term of an article and $m$ equals unity, show that the constant percentage method gives the formulas

$$B = C\left(\frac{S}{C}\right)^{\frac{1}{n}}, \qquad D = C - C\left(\frac{S}{C}\right)^{\frac{1}{n}}$$

Exercise 3. Apply the formulas in Exercise 2 to the above example.

### EXERCISES

1. A home costing \$6000 is depreciated at 5% annually by the constant percentage method. Find its book value at the end of the eighth year.

Ans. \$3980.52.

2. If a house costing \$10000 depreciates to \$7000 in eight years, find the constant annual rate of depreciation to four places of decimals. Ans. .0436.

3. A machine costing \$2500 is depreciated 2% monthly by the constant percentage method. If the machine has a scrap value of \$200 at the end of 46 months, construct the depreciation schedule for the life of the machine.

4. By the constant percentage method an article costing \$1000 depreciates to \$598.75 in $n$ years at 5% converted annually. Find $n$. Ans. 10.

5. Show that the constant percentage method is not applicable when $S = 0$, $S$ being the salvage value.

**64. The sinking-fund method.** By this method the annual depreciation charge equals the rent of an annuity whose amount at the end of the depreciation term, at a given interest rate, equals the cost of the article less its scrap value.

EXAMPLE. An article costing \$1000 has a scrap value of \$100 at the end of 14 years. Use the sinking-fund method, at 5% interest converted annually, to find formulas for the book value and the depreciation charge at the end of $t$ years, and construct a depreciation schedule.

SOLUTION. The annual sinking-fund payment is $900 \dfrac{1}{s_{\overline{14}|.05}} = 45.9216$. The formulas for $D$ and $B$ at the end of $t$ years are then

$$D = 45.9216\, s_{\overline{t}|.05}$$

$$B = 1000 - 45.9216\, s_{\overline{t}|.05}$$

By use of these formulas, $B$ and $D$ can be computed for assigned values of $t$. $B$ and $D$ can also be computed by constructing a schedule analogous to that in Art. 43. The results are shown in the depreciation schedule on the next page.

DEPRECIATION SCHEDULE

| Year | Book Value at End of Year | Payment to Sinking Fund at End of Year | Interest Due on Fund at End of Year | Total in Fund at End of Year |
|---|---|---|---|---|
| 0 | 1000.00 | | | |
| 1 | 954.08 | 45.92 | | 45.92 |
| 2 | 905.86 | 45.92 | 2.30 | 94.14 |
| 3 | 855.23 | 45.92 | 4.71 | 144.77 |
| 4 | 802.07 | 45.92 | 7.24 | 197.93 |
| 5 | 746.25 | 45.92 | 9.90 | 253.75 |
| 6 | 687.64 | 45.92 | 12.69 | 312.36 |
| 7 | 626.10 | 45.92 | 15.62 | 373.90 |
| 8 | 561.49 | 45.92 | 18.69 | 438.51 |
| 9 | 493.64 | 45.92 | 21.93 | 506.36 |
| 10 | 422.40 | 45.92 | 25.32 | 577.60 |
| 11 | 347.60 | 45.92 | 28.88 | 652.40 |
| 12 | 269.06 | 45.92 | 32.62 | 730.94 |
| 13 | 186.59 | 45.92 | 36.55 | 813.41 |
| 14 | 100.00 | 45.92 | 40.67 | 900.00 |

Exercise 1. Use the above formulas to compute the values of $B$ and $D$ at the end of 6 years.

Exercise 2. If $C$ denotes the cost, $S$ the scrap value, and $n$ the depreciation term of an article, show that the sinking-fund method at the rate $i$ gives the formulas

$$D = \frac{C - S}{s_{\overline{n}|i}} s_{\overline{t}|i}, \qquad B = C - \frac{C - S}{s_{\overline{n}|i}} s_{\overline{t}|i}$$

## EXERCISES

**1.** A sinking fund with annual deposits is created at ($j = .055$, $m = 1$) to replace in 5 years a machine which costs \$1850, and which has a scrap value of \$100. Construct the depreciation schedule.

**2.** Same as Exercise 1, except that ($j = .06$, $m = 2$).

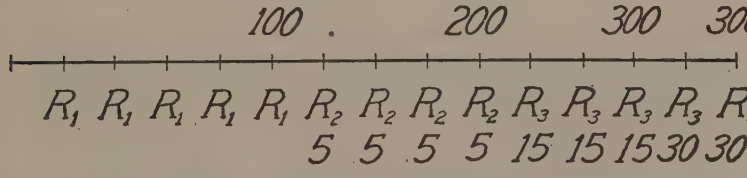
3. Same as Exercise 2, except that the deposits into the sinking fund are made semi-annually.

4. By the sinking-fund method, find the book value, at the end of the fifth year, of a building which costs \$100000 and which depreciates to \$70000 in 10 years. (Use $j = .05$, $m = 1$.) Ans. \$86820.61.

**65. The appraisal method.*** By the straight-line, compound-discount, and sinking-fund methods, an estimate is made of the depreciation charge of an article during its estimated term. In the *appraisal* method the depreciation term is divided into periods and an estimate is made of the depreciation charge for each period. The amount of depreciation for each period is often fairly well known from data at hand, and by properly utilizing this information it is possible to make a set of depreciation charges which are closer to the actual amounts of depreciation than it would be if no account whatever were taken of it. The straight-line, the constant percentage, or the sinking-fund method could be used to estimate the charges during any period; in this article, however, the sinking-fund method will be used. For each period after the first the interest payments on the fund already accumulated are used to help pay the depreciation charges for this period.

EXAMPLE. An article costing \$1000 has a scrap value of \$100 at the end of 14 years. It is estimated that this article will depreciate \$100 during the first five years, \$200 during the next four years, \$300 during the next three years, and \$300 during the last two years of the term. If the appraisal method is used with interest at 5% converted annually, find the depreciation charge at the end of each year and construct a depreciation schedule.

SOLUTION. In the following diagram, each section of which represents one year, the annual depreciation charge and the interest payments on the fund accumulated are shown below the line and the depreciation amounts for the periods into which the term is divided are shown above the line. In the diagram replace the last $R_3$ by $R_4$.




Equating the value at the end of each period of the depreciation and the interest payments during the period to the depreciation amount at the end of the period gives the following equations for determining $R_1$, $R_2$, $R_3$, and $R_4$.

*Communicated to the authors by Professor C. H. Forsythe.

$R_1 s_{\overline{5}|.05} = 100$; $(R_2+5)s_{\overline{4}|.05} = 200$; $(R_3+15)s_{\overline{3}|.05} = 300$; $(R_4+30)s_{\overline{2}|.05} = 300$

Solving, $R_1 = 18.10$; $R_2 = 41.40$; $R_3 = 80.16$; $R_4 = 116.34$

By use of these annual depreciation charges one can readily construct the following

DEPRECIATION SCHEDULE

| Year | Book Value at End of Year | Payment to Fund at End of Year | Interest Due on Fund at End of Year | Total in Fund at End of Year |
|---|---|---|---|---|
| 0 | 1000.00 | | | |
| 1 | 981.90 | 18.10 | | 18.10 |
| 2 | 962.90 | 18.10 | .90 | 37.10 |
| 3 | 942.95 | 18.10 | 1.85 | 57.05 |
| 4 | 922.00 | 18.10 | 2.85 | 78.00 |
| 5 | 900.00 | 18.10 | 3.90 | *100.00* |
| 6 | 853.60 | 41.40 | 5.00 | 146.40 |
| 7 | 804.88 | 41.40 | 7.32 | 195.12 |
| 8 | 753.72 | 41.40 | 9.76 | 246.28 |
| 9 | 700.01 | 41.40 | 12.31 | *299.99* |
| 10 | 604.85 | 80.16 | 15.00 | 395.15 |
| 11 | 504.93 | 80.16 | 19.76 | 495.07 |
| 12 | 400.02 | 80.16 | 24.75 | *599.98* |
| 13 | 253.68 | 116.34 | 30.00 | 746.32 |
| 14 | 100.03 | 116.34 | 37.31 | *899.97* |

## EXERCISES

**1.** Solve the above example if the depreciation charges are made semi-annually and the sinking fund accumulates at ($j = .05$, $m = 2$).

**2.** Solve the above example if the depreciation charges are made annually but the sinking fund accumulates at ($j = .05$, $m = 2$). [In this case,

$$R_1 \frac{s_{\overline{10}|.05}}{s_{\overline{2}|.05}} = 100,\ R_2 \frac{s_{\overline{8}|.025}}{s_{\overline{2}|.025}} + 2.50\, s_{\overline{8}|.025} = 200, \text{ etc.}]$$

**3.** In the above example, show that the depreciation fund, $D_3$, and the book value, $B_3$, for the third period, are given by the formulas

$$D_3 = 300(1.05)^{t-9} + R_3 s_{\overline{t-9}|.05}$$
$$B_3 = 1000 - D_3$$

**4.** An article costing \$3000 has a value of \$400 at the end of eight years. It is estimated that this article will depreciate \$1000 during the first year, \$1000 during the next two years, and \$600 during the next five years. If the appraisal method is used with interest at 5% converted annually, find the depreciation charge at the end of each year and construct a depreciation schedule.

**5.** Solve Exercise 4 if the straight line method is used to estimate the annual depreciation charges for each period.

**6.** Solve Exercise 4 if the constant percentage method is used to estimate the annual depreciation charges for each period.

**66. Graphs of book values and depreciation funds.** Figure 9 shows the book values and the amounts in the depreciation fund for the examples solved in Arts. 62–65 inclusive.

Graph (1) shows these values by the straight-line method, graph (2) by the constant percentage method, graph (3) by the sinking-fund method, and graph (4) by the appraisal method. For example, by the sinking fund method, $MP_3$ represents the book value of the article and $M_1P_3$ the amount in the depreciation fund at the end of five years. The slope at a point which corresponds to any time determines the rate at which the book value or the amount in the depreciation fund is changing at the time; a slope whose numerical value is large shows a rapid change while one whose numerical value is small shows a slow change.

EXERCISES

**1.** Construct the graph of book values of the article in Exercise 4, Art. 65.

**2.** Same as Exercise 1, for Exercise 5, Art. 65.

**67. Other methods of estimating depreciation.** *By the compound interest, or interest on investment method* charges are made at the end of each year to cover depreciation for the year and interest for the year on the book value of the article at the beginning of the year. The book values and the total depreciation charges are the same by this method as by the sinking-fund method. When the same interest rates are used on the book values and on

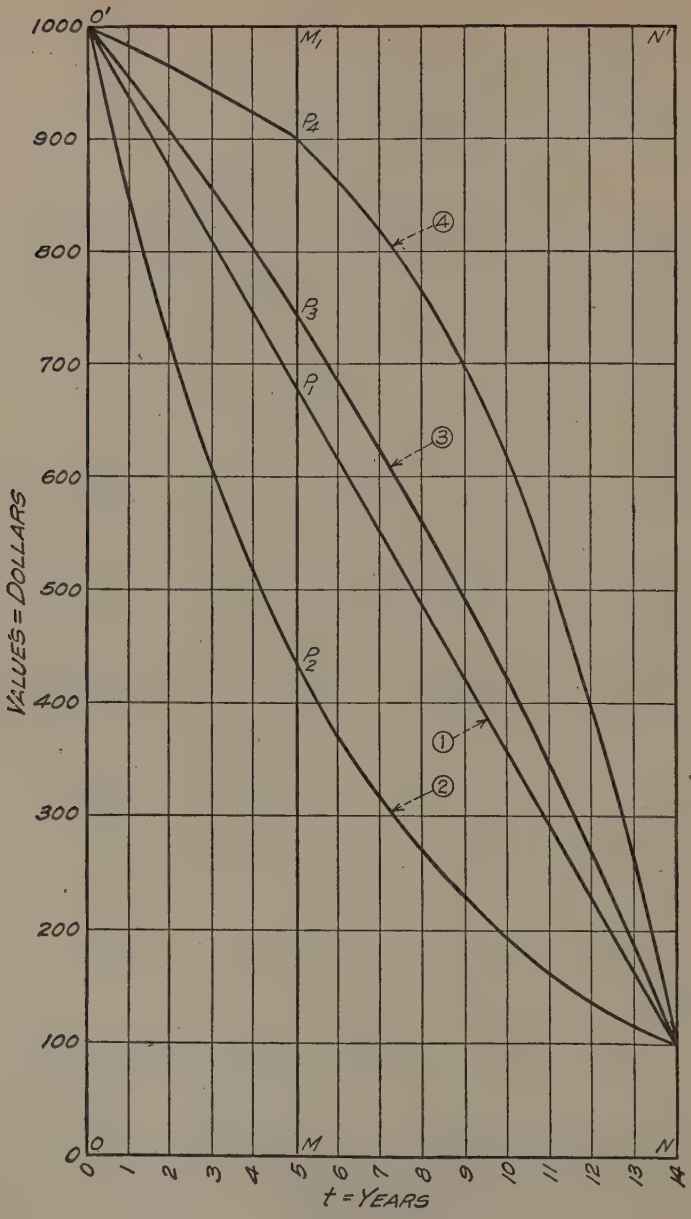

1000
900
800
700
600
500
400
300
200
100
0
VALUES = DOLLARS
O'
M,
N'
P4
P3
P1
P2
④
③
①
②
O
M
N
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
t = YEARS

FIGURE 9

the annual depreciation charges, the book value plus the depreciation charge is constant; when different rates are used, the book value plus the depreciation charge is not constant. Exercises 1 and 2 below illustrate depreciation schedules for the two cases.

By the *unit cost method* the value of a machine at any time is found by assuming that the net cost of a unit of its output equals that of a new machine which could replace it. The net cost of a unit for each machine is found by dividing the number of units it produces in a year into the sum of the annual charges for operating expenses, repairs, depreciation, and interest. Exercise 3 below illustrates the method by which the equation determining the value of the old machine is found.

It should be noted that each of the methods of estimating depreciation described in this and the preceding articles rests upon certain assumptions. In each method an assumption is made as to the amount of depreciation during the service life of an article or during an interval within the service life; the service life is also an assumed number of years. A further assumption is made as to the type of depreciation during the whole or a part of the service life. One type is given by the straight-line formula, one by the constant percentage, and another by the sinking fund. In practice that method should be used which in the judgment of the business accountant or evaluation engineer seems to be in best agreement with the data at hand and then the results determined should be modified to make them correspond to any additional data obtained later. The methods of mathematical statistics when applied to data which show with a fair amount of accuracy the depreciation which has taken place in an article for a term of years would lead to a more accurate determination of the depreciation charges of a like article than could ordinarily be obtained by the use of a method based upon assumptions which are in whole or in part *a priori*.

## EXERCISES

**1.** An article costing \$1000 has a scrap value of \$100 at the end of 14 years. Complete the following compound-interest-method schedule in which ($j = .05$, $m = 1$), for book values and for annual depreciation charges:

DEPRECIATION SCHEDULE

| Year | Book Value at End of Year | Depreciation Charges at End of Year | Total Depreciation at End of Year | Interest on Book Value at 5% at End of Year | Sum of Depreciation Charges at End of Year and Interest on Book Value |
|---|---|---|---|---|---|
| 0 | 1000.00 | | | | |
| 1 | 954.08 | 45.92 | 45.92 | 50.00 | 95.92 |
| 2 | 905.86 | 48.22 | 94.14 | 47.70 | 95.92 |
| 3 | 855.23 | 50.63 | 144.97 | 45.29 | 95.92 |
| 4 | | | | | |
| 5 | | | | | |
| 6 | | | | | |
| 7 | | | | | |
| 8 | | | | | |
| 9 | | | | | |
| 10 | | | | | |
| 11 | | | | | |
| 12 | | | | | |
| 13 | | | | | |
| 14 | | | | | |

**2.** An article costing \$1000 has a scrap value of \$100 at the end of 14 years. Complete the following compound-interest-method schedule in which the interest on book values is at ($j = .08$, $m = 1$) and the depreciation charge is at ($j = .05$, $m = 1$).

DEPRECIATION SCHEDULE

| Year | Book Value at End of Year | Depreciation Charges at End of Year at 5% | Total Depreciation at End of Year | Interest on Book Value at 8% | Sum of Depreciation Charges at End of Year on Book Values |
|---|---|---|---|---|---|
| 0 | 1000.00 | | | | |
| 1 | 954.08 | 45.92 | 45.92 | 80.00 | 125.92 |
| 2 | 905.80 | 48.22 | 94.14 | 76.33 | 124.55 |
| 3 | 852.70 | 50.63 | 147.30 | 72.46 | 123.09 |
| 4 | | | | | |
| 5 | | | | | |
| 6 | | | | | |
| 7 | | | | | |
| 8 | | | | | |
| 9 | | | | | |
| 10 | | | | | |
| 11 | | | | | |
| 12 | | | | | |
| 13 | | | | | |
| 14 | | | | | |

**3.** A machine whose cost of operation is \$500 per year and whose cost for annual repairs is \$200 has an estimated life of 5 years. A new machine whose cost of operation is \$450 per year and whose cost for annual repairs is \$100 may be purchased for \$1500 and has an estimated life of 8 years. If the scrap value of each machine is zero and if they produce the same number of units of annual output, find the value of the old machine at ($j = .06$, $m = 1$). If $x$ is the value of the old machine, then

$$\frac{x}{s_{\overline{5}|.06}} + .06\,x + 500 + 200 = \frac{1500}{s_{\overline{8}|.06}} + (.06)1500 + 450 + 100$$

Solving, $x = \$385.66$

**4.** Same as Exercise 3 except that the annual output of the new machine is 10% greater than that of the old; 13% greater. Ans. \$82.54; \$2.06.

**68. Composite life of a plant.** The various parts of a plant do not ordinarily have the same depreciation term or lifetime. The composite life of such a plant may be defined as the number of years $n$ for which the annual depreciation charge for all the parts equals the sum of the annual depreciation charge for the component parts. By means of this definition the equation determining the composite life of a plant can be readily formed.

EXAMPLE 1. Find the composite life under the sinking-fund method at 5% converted annually of a plant having the following parts:

| PART | COST | SCRAP VALUE | LIFE |
|---|---|---|---|
| 1 | 16000 | 400 | 10 |
| 2 | 23500 | 500 | 15 |
| 3 | 8600 | 200 | 8 |
| 4 | 12300 | 300 | 17 |

SOLUTION. The total wearing values of the parts are \$15600, \$23000, \$8400, and \$12000; the sum of these is \$59000. By definition the composite life, $n$ years, by the sinking-fund plan, is determined by the equation

$$\frac{59000}{s_{\overline{n}|.05}} = \frac{15600}{s_{\overline{10}|.05}} + \frac{23000}{s_{\overline{15}|.05}} + \frac{8400}{s_{\overline{8}|.05}} + \frac{12000}{s_{\overline{17}|.05}}$$

$$= \$3650.19698$$

$$s_{\overline{n}|.05} = 16.1635113$$

Interpolation in Table V gives $n = 12.1$

EXERCISE. Solve the above equation for $\frac{1}{s_{\overline{n}|.05}}$ and find $n$ by interpolation in Table VII.

EXAMPLE 2. Find the composite life of the plant in Example 1 by the straight-line method.

SOLUTION. In this case the equation which determines $n$ is

$$\frac{59000}{n} = \frac{15600}{10} + \frac{23000}{15} + \frac{8400}{8} + \frac{12000}{17}$$

$$= \$4849.2157$$

$$n = 12.2$$

## EXERCISE

Find the composite life of a manufacturing plant composed of the following units:

(*a*) building, costing \$100000 with a scrap value of \$10000 at the end of 20 years;

(*b*) heavy machinery, costing \$75000 with a scrap value of \$15000 at the end of 10 years;

(*c*) light machinery, costing \$25000 with a scrap value of \$1000 at the end of 5 years.

Use ($j = .06$, $m = 1$). Ans. 11.3.

**69. Capitalized cost.** The capitalized cost of an article is defined as the first cost plus the present value of perpetual renewals. When the renewals are made at periodic intervals and at equal cost they form a perpetuity. (Art. 29, Chapter II.)

EXAMPLE 1. A cab costing \$1600 must be replaced each 5 years. Find the capitalized cost if the scrap value of each cab is \$100. Use an interest rate of 5% converted annually.

SOLUTION. The renewals form a perpetuity whose rent is \$1500 and whose rent period is 5 years. The present value of this perpetuity is $\frac{1500}{.05}\frac{1}{s_{\overline{5}|.05}}$. Hence the capitalized cost, $V$, is given by

$$V = 1600 + \frac{1500}{.05}\frac{1}{s_{\overline{5}|.05}}$$
$$= \$7029.24$$

EXERCISE 1. Show that $V$, in Example 1, may be written in the form $V = 100 + \frac{1500}{.05}\frac{1}{a_{\overline{5}|.05}}$; compute its value from this form.

EXERCISE 2. If $C$ is the first cost and $D$ is the renewal cost of an article whose life is $r$ years, show that the capitalized cost, $V$, at the rate $i$ converted annually is given by

$$V = C - D + \frac{D}{i}\frac{1}{a_{\overline{r}|i}}$$

A definition of capitalized cost more general than the usual one given above may be stated as follows: The capitalized cost of an article is the first cost plus the present value of $n'$ renewals. When $n'$ becomes infinite, this definition reduces to the usual form. When the $n'$ renewals are made at periodic intervals and at equal costs they form an annuity.

EXAMPLE 2. A cab costing \$1600 must be replaced every 5 years. If the scrap value is \$100, find the sum, $V$, of the first cost and the present value of 8 renewals. Use an interest rate of 5% converted annually.

SOLUTION. In this case

$$V = 1600 + 1500\frac{a_{\overline{40}|.05}}{s_{\overline{5}|.05}}$$
$$= \$6258.04$$

Two articles which will serve the same purpose may have equal or different capitalized costs. When they are different the one with the smaller cost should be used.

### EXERCISES

**1.** Would it be more economical to use asbestos shingles which cost \$12.00 per thousand and last 25 years or asphalt shingles which cost \$7.00 per thousand and last 15 years? Assume that the cost of laying the shingles is the same in each case. Use ($j = .06$, $m = 1$).

**2.** What is the capitalized cost of a bridge which must be replaced every 50 years at a cost of \$350000? Use ($j = .055$, $m = 1$). Ans. \$375845.59.

**3.** What is the capitalized cost of an automobile which costs \$1800, has a scrap value of \$100, and a service life of 6 years. Use ($j = .05$, $m = 1$). Ans. \$6798.59.

**4.** A trust fund for the perpetual maintenance of a hospital building is created. How much must be deposited with the trustees of the fund to provide \$30000 for immediate use, \$5000 at the end of each year for minor repairs, and \$25000 at the end of each three years for major repairs. Use ($j = .05$, $m = 1$). Ans. \$288604.28.

**5.** A wood post which costs 45 cents and lasts 9 years can be set in soil for 10 cents. The post can be set in concrete for 25 cents, and it will then last 12 years. Find the capitalized cost of a post set in soil if 3 renewals are made, and that of a post set in concrete if 2 renewals are made. If a farmer wishes to maintain a fence for 36 years, which method of setting the posts is the more economical? Use ($j = .055$, $m = 1$). Ans. \$1.23; \$1.26.

**70. Capitalized cost equations.** An equation which expresses equality in value between the capitalized costs of articles can often be used to find an unknown in capitalized cost problems. (See Art. 15, Chapter I.) In this article two types of capitalized cost equations are presented. In one type two articles have equal capitalized costs; in the other the capitalized cost of one article equals $h$ times that of the other.

Example 1. A transfer company is using a truck having \$3000 for its cost, 3 years for its service life, and zero for its scrap value. What can it afford to pay for another truck having 5 years for its service life and zero for its scrap value. Use an interest rate of 6% converted annually.

Solution. Let $x$ denote the amount that can be paid for another truck to make its capitalized cost the same as that of the one the company is using. Equating the capitalized costs gives [Exercise 2, Example 1, Art. 69]

$$\frac{x}{.06}\,\frac{1}{a_{\overline{5}|.06}} = \frac{3000}{.06}\,\frac{1}{a_{\overline{3}|.06}}$$

Solving, $$x = 3000\, a_{\overline{5}|.06} \cdot \frac{1}{a_{\overline{3}|.06}}$$
$$= \$5727.66$$

It follows that the company can pay an amount not exceeding \$5727.66 for another truck.

EXERCISE 1. An article having $C + x$ for its first and its renewal cost and $r + k$ for its service life will serve the same purpose as an article having $C$ for its first and its renewal cost and $r$ for its service life. By solving the capitalized cost equation $\frac{C+x}{i}\frac{1}{a_{\overline{r+k}|i}} = \frac{C}{i}\frac{1}{a_{\overline{r}|i}}$ for $x$, show that

$$x = C\frac{a_{\overline{r+k}|i} - a_{\overline{r}|i}}{a_{\overline{r}|i}}$$
$$= C\frac{a_{\overline{k}|i}\,(1+i)^{-r}}{a_{\overline{r}|i}} \qquad (k > 0)$$
$$= C\frac{a_{\overline{k}|i}}{s_{\overline{r}|i}}$$

This type of equation is useful in finding the amount that one is justified in expending in order to extend the life of an article.

EXAMPLE 2. A machine costing \$1000 has a service life of 10 years and a scrap value of \$100. How much would one be justified in expending on the machine to double its output and decrease its service life by 2 years, the scrap value remaining the same. Use an interest rate of 5% converted annually.

SOLUTION. Let $x$ denote the amount that can be expended on the machine to make its capitalized cost twice that of the original machine. Then

$$100 + \frac{900+x}{.05}\frac{1}{a_{\overline{8}|.05}} = 2\left(100 + \frac{900}{.05}\frac{1}{a_{\overline{10}|.05}}\right)$$

Solving, $$x = \$638.94$$

It follows that one can afford to expend an amount not exceeding \$638.94.

EXERCISE 2. An article having $C + x$ for its first and its renewal cost, and $r + k$ for its service life will serve the same purpose as $h$ articles each of which has $C$ for its first and its renewal cost and $r$ for its service life. By solving the capitalized cost equation $\frac{C+x}{i}\frac{1}{a_{\overline{r+k}|i}} = h\left(\frac{C}{i}\frac{1}{a_{\overline{r}|i}}\right)$ for $x$, show that

$$x = C(h-1) + hC \cdot \frac{a_{\overline{r+k}|i} - a_{\overline{r}|i}}{a_{\overline{r}|i}}$$
$$= C(h-1) + hC \cdot \frac{a_{\overline{k}|i}}{s_{\overline{r}|i}} \qquad (k > 0)$$
$$= C(h-1) - hC \cdot \frac{s_{\overline{-k}|i}}{s_{\overline{r}|i}} \qquad (k < 0)$$

This type of equation is useful in finding the amount one is justified in spending to change the service life and multiply the productivity of an article.

## EXERCISES

**1.** Wood posts which cost 45 cents and last 9 years will, if treated with creosote, last 15 years. How much could a farmer afford to spend treating each post with creosote? Use ($j = .06$, $m = 1$). Ans. .19.

**2.** A machine costing \$1000 has a service life of 10 years and a scrap value of \$100. How much would one be justified in expending on the machine to increase its life to 12 years? Use ($j = .05$, $m = 1$). Ans. \$133.05.

**3.** Same as Exercise 2 except that in addition to increasing the life of the machine, its productivity is increased by 20%. Ans. \$348.52.

**4.** An automobile has a value of \$300, a life of 2 years, and a scrap value of \$25. It is estimated that replacements which would cost \$100 would increase its life to 4 years. Would it be economical to have the replacements made? Use ($j = .055$, $m = 1$).

**5.** A wood post which costs 45 cents and lasts 9 years can be set in soil for 10 cents. If the post set in concrete will last 12 years, how much can a farmer afford to spend setting the post in concrete if he wishes to maintain a fence for 36 years? Use ($j = .055$, $m = 1$).

## MISCELLANEOUS EXERCISES

**1.** On March 3, A gave B two demand notes, one for \$127.50, the other for \$1325.60. Each note bore ordinary simple interest from date, the first at $6\frac{1}{2}\%$, the second at 6%. On April 1, B offered A \$25.00 if he would refund the notes on that day. A accepted the offer and gave to a bank a note for 90 days. If the bank charged 7% ordinary simple discount and A used the \$25.00 to reduce the amount necessary to refund the debt, find the face of the note given to the bank. On the assumption that A could not have paid the two notes before June 30 how much did he gain or lose by the transaction?

**2.** A bank paid \$676.85 for a note for \$735.00 due in one year. What rate of interest did the investment yield the bank? Find the corresponding rate of discount.

**3.** Given the equation $\frac{1}{1-d} = 1 + i$. Plot the graph after completing the table of values given below:

| $i$ | 0 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 | .10 | 1 | 2 | −1 | −2 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | 0 | | | | | | | | | | | | | | | |

Is there a value of $d$ corresponding to every arbitrarily chosen positive value for $i$? What portion of the graph is of importance in the solution of problems in interest and discount? At what point does the line $i = d$ cut the hyperbola? Interpret $i = d = 0$.

**4.** A man pays \$7800 for a house and rents it at \$65.00 per month payable in advance. Taxes and upkeep cost \$273.00 annually and are charged against the investment at the end of the year. What ordinary simple interest rate does the owner make if he is able to invest each monthly rental payment as made at 5% ordinary simple interest?

**5.** A merchant buys a bill of goods for \$1275.00. He has the option of paying cash with a 2% discount, or net in 60 days. What ordinary simple discount rate could he afford to pay a bank in order to make cash payment, and how much would it be necessary for him to borrow if he did so?

**6.** Same as Exercise 5 except that $1\frac{1}{2}$% discount is allowed for cash in 10 days.

**7.** The list price of a bill of goods is \$565.85 with 10% and 5% off. If an additional 2% is given for cash, would the merchant be justified in borrowing from a bank at 8% ordinary simple discount to pay the bills rather than to pay net at end of 60 days. State the difference between the two methods of payment at the end of the 60 days.

**8.** The price of a bill of goods is \$125.00 cash or \$130.00 in 60 days. On the basis of 6% ordinary simple discount, find how much better from the purchaser's standpoint is the cash price on the date of sale; how much better at the end of the 60 days.

**9.** Two equal sums of money were placed at the same time on interest, the first at $(j = .04, m = 4)$, the second at $(j = .05, m = 2)$. When will the second sum amount to $\frac{3}{2}$ of the first?

**10.** A owes B \$1800.00 due in 3 years with interest at 5% payable semi-annually. Two years before it is due B sells the note to C at such a price that C makes $(j = .065, m = 2)$ on his investment. How much did C pay for the note?

**11.** A house is offered for \$2000 cash, \$1000 at the end of the first year, \$800 at the end of the second, and \$500 at the end of the third year, or \$1000 cash, \$1500 at the end of the first year, \$1200 at the end of the second, and \$1000 at the end of the third. What is the difference between the two options on the day of sale if an interest basis of $(j = .06, m = 2)$ is used?

**12.** A debt of \$3000 was extinguished by three annual payments, \$800 at the end of the first year, \$1000 at the end of the second, and $x$ at the end of the third. Find $x$ if the interest rate for the first two years was $(j = .06, m = 1)$ and $(j = .055, m = 1)$ for the third year.

**13.** By the will of his father, a boy aged 12 was left an annuity of \$25 payable at the end of each month until he becomes of age, and an annuity of \$500 payable at the end of each year thereafter until he becomes 30 years of age. On the assumption that the boy will certainly live until he is 30 years old, find the value of future payments on his twelfth, his eighteenth, and his twenty-fifth birthdays. Use $(j = .05, m = 2)$.

**14.** The annual premium on a certain non-participating life insurance policy for \$1000 was \$45.10 payable at the beginning of each year. Immediately after the nineteenth premium was paid, the holder of the policy died. If, instead of taking out the policy, the money had been deposited in a savings bank which pays 4% converted quarterly, how much more would the estate have received from the bank than it did receive from the insurance company? If the policyholder had died immediately after the payment of the fifth premium, how much more would the estate have received from the insurance company than it would have received from the savings bank? When would the deposits in the savings bank have had the same value as the face value of the policy?

**15.** In order to provide for the education of his daughter, a father wishes to provide an annuity of \$1500 at the beginning of each of her four years at college. How much must be deposited each month in a savings bank which pays ($j = .04$, $m = 4$) if the first deposit is made at the time of the child's birth, and the last on her eighteenth birthday when the first \$1500 is to be available?

**16.** Semi-annual instalments of \$270.00, including interest and principal, must be made on a note for \$4500 which bears interest at 6% payable semi-annually. Find the number of instalments and the amount of the last one.

**17.** Had the note in Exercise 16 been sold for \$4100 immediately after the second instalment had been paid, what rate of interest would the purchaser of the note have made if $m = 2$?

**18.** Find the selling price of the note in Exercise 16 immediately after the payment of the second instalment to yield ($j = .065$, $m = 2$).

**19.** A man wishes to borrow \$5000 for 5 years. He has the option of borrowing it from an insurance company at 5½% payable semi-annually or from a building and loan company at 6% payable semi-annually. If he chooses to borrow from the insurance company, he must pay a 2% commission to the agent who makes the loan. The loan company does not charge a commission. How much would he save by borrowing from the insurance company if he can use money at ($j = .07$, $m = 2$) in his business?

**20.** A company purchased 2250 acres of coal land at \$150 per acre. For the ten years before development took place the tax was \$3000 at the end of each year. It is estimated that (1) the life of the mine will be 45 years, (2) the recovery per acre will be 6000 tons, (3) the plant, including buildings and equipment, which cost \$400,000, must be replaced every 15 years, (4) the shafts and tunnels, which cost \$125,000, will last during the life of the mine, and (5) the value of the land after the coal is exhausted will be \$50 per acre. If the annual charge for wages is \$357,000, that for administration, repairs, and taxes is \$153,000, and the average selling price per ton of the coal at the mine is \$2.15, find how much the company had invested in the mine 5 years after development took place, and its value at that time. Use ($j = .05$, $m = 1$) and

proceed on the hypthesis that the expenditures for each year are made at the end of the year, that the yearly output of coal is sold at that time, and that the annual production is constant during the life of the mine.

**21.** What profit would the coal company of Exercise 20 have made had the mine been leased immediately after the development was completed on a basis of 25 cents a ton royalty and an annual production of 300,000 tons for 45 years? What would have been the value at that time of the future profits of the lessee? Use ($j = .05$, $m = 1$).

**22.** Find the amount the coal company in Exercise 20 had invested in the mine at the end of the tenth year of its development. What was the value of the mine at that time if the price of coal at the mine advanced to \$2.20 per ton for the remaining 35 years of the life of the mine? Use ($j = .05$, $m = 1$).

**23.** It is estimated that a slope opening of a certain mine, which would cost \$500,000, could produce coal at \$1.90 per ton; that a shaft opening, which would cost \$600,000, could produce coal at \$1.86 per ton. It is also estimated that the mine will produce 200,000 tons of coal per year for 40 years. Which method of developing the mine is the better if an interest rate of ($j = .05$, $m = 1$) is used?

**24.** The cost of a mine tie is 20 cents and its service life 3 years. If the tie is dipped in creosote, its life is 12 years. How much can a mine company afford to spend treating each tie with creosote? Use ($j = .05$, $m = 1$) and assume that the mine will be operated for 36 years.

# CHAPTER IV

## LIFE ANNUITIES AND LIFE INSURANCES

**71. Probability.** If two coins be tossed they can fall in any one of four ways: both heads; both tails; the first, head, the second, tail; the first, tail, the second, head. It will be assumed that these ways are equally likely. The event "one head and one tail" can happen in two ways. The probability that the event "one head and one tail" will happen is $\frac{2}{4}$, the numerator being the number of ways in which the event can happen and the denominator being the number of ways in which it can happen plus the number in which it can fail. In like manner the probability that both will be heads is $\frac{1}{4}$. These simple examples illustrate the following *definition of probability:*

*If an event can happen in h ways and fail in f ways, and if the $h + f$ ways are equally likely, the probability that the event will happen is $\frac{h}{h+f}$ and the probability that it will fail is $\frac{f}{h+f}$.*

Since $\frac{h}{h+f} + \frac{f}{h+f} = 1$, it follows that the probability that an event will happen plus the probability that it will fail is unity.

If two coins be tossed, each can fall in two ways. Since to each of the ways in which the first can fall there correspond two ways in which the second can fall, it follows that the two can fall in $2 \cdot 2 = 4$ ways. If a coin and a die be tossed, they can fall in $2 \cdot 6 = 12$ ways, since to each of the two ways in which the coin can fall there correspond six ways in which the die can fall. If two coins and a die be tossed, they can fall in $2 \cdot 2 \cdot 6 = 24$ ways, since to each of the $2 \cdot 2 = 4$ ways in which the coins can fall there correspond six ways in which the die can fall. Similar reasoning leads to the

**Fundamental Principle.** *If one thing can be done in $m_1$ ways, and if, after it is done, a second thing can be done in $m_2$ ways, the two*

*things taken together can be done in the order stated in $m_1m_2$ ways; more generally, if one thing can be done in $m_1$ ways, a second in $m_2$ ways, a third in $m_3$ ways, and so on, the number of ways in which they can be done when taken all together in the order stated is $m_1m_2m_3 \cdots$.*

By the number of permutations or arrangements of $n$ things taken $r$ at a time is meant the number of arrangements consisting of $r$ things which can be formed from $n$ different things. For example, twelve permutations of two letters can be formed from the four letters *a*, *b*, *c*, *d*; these twelve permutations are *ab*, *ba*, *ac*, *ca*, *ad*, *da*, *bc*, *cb*, *bd*, *db*, *cd*, and *dc*. That there are twelve such arrangements follows at once from the above fundamental principle, since the first letter can be chosen in four ways and after it is selected the second can be chosen in three ways. If ${}_nP_r$ denotes the number of permutations of $n$ different things taken $r$ at a time it follows from the same principle that

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1). $$

By the number of combinations of $n$ things taken $r$ at a time is meant the number of different sets of $r$ things which can be formed from $n$ different things. For example, six combinations of two letters can be formed from the four letters, *a*, *b*, *c*, *d*; these six combinations are *ab*, *ac*, *ad*, *bc*, *bd*, and *cd*. That there are six such combinations may be seen as follows: To each combination there correspond two and just two permutations; for example, to the combination *ab* correspond the permutations *ab* and *ba*. Since there are twelve permutations of four letters taken two at a time, there must be $\frac{1}{2}^2 = 6$ combinations of these letters taken two at a time. In general to a combination of $r$ different things there correspond ${}_rP_r = r(r-1)\cdots 2\cdot 1 = r!$ permutations of them. If ${}_nC_r$ denotes the number of combinations of $n$ different things taken $r$ at a time, it follows that ${}_nC_r \cdot r! = {}_nP_r$ or

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 2\cdot 1} = \frac{n!}{r!(n-r)!} $$

The fundamental principle and the formula for ${}_nC_r$ are often useful in determining a probability. Other useful results are presented in Art. 73.

EXAMPLE. From a bag containing 6 white and 4 red balls 5 are drawn at random. Find the probability that 3 are white and 2 are red.

SOLUTION. By the formula for ${}_nC_r$, 5 balls can be selected from 10 in $\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252$ ways; 3 white balls can be selected from 6 white in $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$ ways; and 2 red balls can be selected from 4 red in $\frac{4 \cdot 3}{1 \cdot 2} = 6$ ways. Hence, by the fundamental principle, 3 white and 2 red balls can be selected from 6 white and 4 red in $20 \cdot 6 = 120$ ways. It follows that the probability that 3 are white and 2 are red is $\frac{120}{252} = \frac{10}{21}$.

## EXERCISES

**1.** There are 4 paths up a mountain. In how many ways can one ascend the mountain and then descend by a different path? Ans. 12.

**2.** If there are 4 railroad lines from Columbus to Chicago and 5 from Chicago to St. Louis, in how many ways can one go from Columbus to St. Louis, through Chicago? Ans. 20.

**3.** How many committees can be appointed from 5 men and 4 women if each committee is made up of 2 men and 2 women? Ans. 60.

**4.** How many numbers, each of two digits, can be made if the first digit is chosen from the digits 1, 2, 3, and 4 and the second from 5, 6, 7, 8, and 9? Ans. 20.

**5.** How many numbers, each of four digits, can be made if the first two are chosen from 1, 2, 3, and 4, and the last two from 5, 6, 7, 8, and 9, and if there is no repetition of the digits? How many numbers can be made if there is no restriction as to the repetition of the digits? Ans. 240; 400.

**6.** If a bag contains 7 white and 5 red balls, and 3 balls are drawn at random, what is the probability that 2 are white and 1 is red? Ans. $\frac{105}{220}$.

**7.** If two dice are thrown, what is the probability of two threes; two fours; a three paired with a four? Ans. $\frac{1}{36}$; $\frac{1}{36}$; $\frac{1}{18}$.

**8.** If 3 coins are thrown, what is the probability of 3 heads; 2 heads and 1 tail; 1 head and 2 tails; 3 tails? Ans. $\frac{1}{8}$; $\frac{3}{8}$; $\frac{3}{8}$; $\frac{1}{8}$.

**9.** If a bag contains 3 red and 4 white balls, and another bag 4 red and 3 white balls, what is the probability of getting one red and one white ball in two drawings, one from each bag? Ans. $\frac{25}{49}$.

**72. Probability derived from observation.** In the example and exercises in Art. 71 the probabilities are derived in each case by an *a priori* determination of all the equally likely ways in which an event can happen or fail. This method is well suited to find probabilities pertaining to games of chance, in which field many

contributions to the theory of probability have had their origin. In life insurance, life annuities, mathematics of statistics, and other fields in which the notion of probability is important, it is impossible to make an *a priori* determination of the equally likely ways. In such cases the probability of an event is determined empirically by observing in what proportion of cases the event happens on a large number of occasions.

*If $h$ denotes the number of times an event happens on $n$ occasions chosen at random ($n$, large), then, in the absence of further information, $p = \frac{h}{n}$ is taken as the best estimate of the probability that the event will happen on a given occasion.*

It follows from this definition that *if $p$ denotes the probability that an event will happen, $1 - p$ denotes the probability that it will fail.* For, out of $n$ occasions, the event will happen $pn(= h)$ times and will fail $n - pn = (1 - p)n$ times. Hence, by the definition, the probability that the event will fail on a given occasion is $\frac{(1 - p)n}{n} = 1 - p$.

In the table * on page 158 are given, for the years 1890 to 1916 inclusive, the number of revenue passenger miles for railroads of all classes, the number of revenue passengers killed and injured, together with the probability of death and injury per thousand passenger miles.

### EXERCISES

1. Verify, for the year ending June 30, 1901, the probability of death and injury per thousand passenger miles as given in the above table.

2. Find, for the entire period covered by the above table, the probability of death and injury per thousand passenger miles.

**73. Independent, dependent, and mutually exclusive events.** Two or more events are said to be dependent or independent according as the occurrence of any one of them does or does not affect the occurrence of the others. Two or more events are said to be mutually exclusive when the occurrence of any one of them

* Accident Bulletin, Number 92, Interstate Commerce Commission.

| Year Ending | Revenue Passenger Miles, Roads of All Classes | Passengers Killed | Probability of Death per 1000 Passenger Miles | Passengers Injured | Probability of Injury per 1000 Passenger Miles |
|---|---|---|---|---|---|
| June 30, 1890 | 11,847,785,617 | 286 | .000024 | 2425 | .00020 |
| June 30, 1891 | 12,844,243,881 | 293 | .000023 | 2972 | .00023 |
| June 30, 1892 | 13,362,898,299 | 376 | .000028 | 3227 | .00024 |
| June 30, 1893 | 14,229,101,084 | 299 | .000021 | 3229 | .00023 |
| June 30, 1894 | 14,289,445,893 | 324 | .000023 | 3034 | .00021 |
| June 30, 1895 | 12,188,446,271 | 170 | .000014 | 2375 | .00019 |
| June 30, 1896 | 13,049,007,233 | 181 | .000014 | 2873 | .00026 |
| June 30, 1897 | 12,256,939,647 | 222 | .000018 | 2795 | .00023 |
| June 30, 1898 | 13,379,930,004 | 221 | .000017 | 2945 | .00023 |
| June 30, 1899 | 14,591,327,613 | 239 | .000016 | 3442 | .00024 |
| June 30, 1900 | 16,038,076,200 | 249 | .000016 | 4128 | .00026 |
| June 30, 1901 | 17,353,588,444 | 282 | .000016 | 4988 | .00029 |
| June 30, 1902 | 19,689,937,620 | 345 | .000012 | 6683 | .00034 |
| June 30, 1903 | 20,915,763,881 | 355 | .000017 | 8231 | .00039 |
| June 30, 1904 | 21,923,213,536 | 441 | .000020 | 9111 | .00042 |
| June 30, 1905 | 23,800,149,436 | 537 | .000023 | 10457 | .00044 |
| June 30, 1906 | 25,167,240,831 | 359 | .000014 | 10764 | .00043 |
| June 30, 1907 | 27,718,554,030 | 610 | .000022 | 13041 | .00047 |
| June 30, 1908 | 29,082,836,944 | 381 | .000013 | 11556 | .00040 |
| June 30, 1909 | 29,109,322,589 | 253 | .000009 | 10311 | .00035 |
| June 30, 1910 | 32,338,496,329 | 324 | .000010 | 12451 | .00039 |
| June 30, 1911 | 33,201,694,699 | 299 | .000009 | 12042 | .00036 |
| June 30, 1912 | 33,132,354,783 | 283 | .000009 | 14938 | .00045 |
| June 30, 1913 | 34,672,685,424 | 350 | .000010 | 15130 | .00044 |
| June 30, 1914 | 35,357,221,302 | 232 | .000007 | 13887 | .00039 |
| June 30, 1915 | 32,474,923,456 | 199 | .000006 | 10914 | .00034 |
| June 30, 1916 | 35,220,015,651 | 239 | .000007 | 7488 | .00021 |

excludes the occurrence of any other. In this article three elementary theorems are presented which are useful in determining probabilities of events of these types. Many other theorems are given in works on probability.

**Independent events.** *If $p_1, p_2, \cdots p_r$ are the separate probabilities of $r$ independent events, the probability that they all happen on a given occasion when all of them are in question is $p_1p_2 \cdots p_r$.*

By the definition in Art. 72, out of $n$ occasions in which all the events are in question, the first event will happen $p_1n$ times; out of these $p_1n$ occasions the second event will happen $p_2(p_1n) = p_1p_2n$ times. That is, out of $n$ occasions the first two events will happen $p_1p_2n$ times. Continuing this process it is seen that out of $n$ occasions all the $r$ events will happen $p_1p_2 \cdots p_rn$ times. Hence, by Art. 72, $p_1p_2 \cdots p_r$ is the probability that all the events will happen on a given occasion.

EXERCISE. Use the definition and the fundamental principle in Art. 71 to prove this theorem.

**Dependent events.** *If $p_1$ is the probability of a first of $r$ events, $p_2$ is the probability of a second event after the first has happened, $p_3$ is the probability of a third event after the first and second have happened, and so on, then $p_1p_2 \cdots p_r$ is the probability that the $r$ events will happen in the order specified.*

The proof of this theorem is entirely analogous to that for independent events.

**Mutually exclusive events.** If *$p_1, p_2, \cdots p_r$ are the separate probabilities of $r$ mutually exclusive events, the probability that one of these events will happen on a given occasion when all of them are in question is $p_1 + p_2 \cdots + p_r$.*

By the definition in Art. 72, out of $n$ occasions in which all of the events are in question, the $r$ events will happen $p_1n, p_2n, \cdots p_rn$ times respectively. Since only one of these events can happen on a given occasion, it follows that out of $n$ occasions one or the other of $r$ events will happen $(p_1n + p_2n + \cdots + p_rn) = (p_1 + p_2 + \cdots p_r)n$ times. Hence $p_1 + p_2 + \cdots + p_r$ is the probability that one of the events will happen on a given occasion.

An illustration of the first and last of the above theorems is afforded by two lives, A and B, of ages $x$ and $y$ respectively. If $p_x$ denotes the probability that A lives one year and $p_y$ the probability that B lives one year, then $1 - p_x$ is the probability that A dies within one year and $1 - p_y$ is the probability that B dies within one year. By the first theorem the probabilities of the pairs of independent events, A lives B lives, A lives B dies, A dies B lives, A dies B dies are $p_xp_y$, $p_x(1 - p_y)$, $(1 - p_x)p_y$, $(1 - p_x)(1 - p_y)$ respectively. By the last theorem the probability that one or the other of the mutually exclusive events, A lives B lives, A dies B dies, happens, is $p_xp_y + (1 - p_x)(1 - p_y)$, and the probability that some one of the four mutually exclusive pairs of events happens is $p_xp_y + p_x(1 - p_y) + (1 - p_x)p_y + (1 - p_x)(1 - p_y)$. This last sum is unity, which it should be, since some one of the four pairs must happen.

## EXERCISES

**1.** A traveler has two railroad connections to make. The probability of making the first is $\frac{2}{3}$, of the second $\frac{2}{5}$. What is the probability of his making both connections? Ans. $\frac{4}{15}$.

**2.** If a bag contains 8 white and 4 black balls, and one ball at a time is drawn from the bag, what is the probability of drawing 2 white balls in 2 drawings; 2 black balls in 2 drawings; 1 white and 1 black in 2 drawings?

Ans. $\frac{14}{33}$; $\frac{1}{11}$; $\frac{16}{33}$.

**3.** If a bag contains 10 white and 8 black balls, and one ball at a time is drawn from the bag, what is the probability of drawing 2 white balls in 2 drawings; 3 black in 4 drawings; 2 white followed by 2 black in 4 drawings; 1 white followed by 3 black in 4 drawings; 3 black followed by 1 white in 4 drawings; 3 white in 4 drawings? Ans. $\frac{5}{17}$; $\frac{28}{153}$; $\frac{7}{102}$; $\frac{7}{153}$; $\frac{7}{153}$; $\frac{16}{51}$.

**4.** A purse contains 9 dimes and a nickel; a second purse contains 10 dimes. Nine coins are chosen at random from the first purse and placed in the second, and then nine coins are chosen at random from the second and placed in the first. Show that the probability that the nickel is in the first purse is $\frac{10}{19}$.

**5.** If two dice are thrown, show that the probability of a seven[1] is $\frac{1}{6}$; of a six is $\frac{5}{36}$; of a two is $\frac{1}{36}$; show that the probability of a three is the same as that for an eleven.

**6.** If two dice are thrown, show that the probability of a five or less is $\frac{5}{18}$.

[1] A seven is thrown when the sum of the numbers on the top of the dice is seven.

**74. Mortality Tables.** A mortality table is a table which shows for a large group of persons of the same age the numbers living at consecutive ages and the number dying between each pair of consecutive ages. The interval between consecutive ages is ordinarily one year except for infant mortality in which case it is one month. The number of persons assumed to be living at the first age given in a mortality table is called the *radix* of the table. Many mortality tables have been constructed from data obtained principally from population and insurance statistics. Glover's United States Tables, published in 1910, are based on population statistics. The American Experience Table (Table XI), published in 1868, is based on insurance statistics. This table is widely used in the United States and it will be used in what follows unless otherwise stated. In this table the radix is 100,000.

A standard notation is used with mortality tables. In this notation,

$l_x$ denotes the number living at age $x$,

$d_x$ denotes the number dying between ages $x$ and $x + 1$,

$p_x$ denotes the probability that a person aged $x$ will live one year,

${}_np_x$ denotes the probability that a person aged $x$ will live $n$ years,

$q_x$ denotes the probability that a person aged $x$ will die within one year,

$|\,{}_nq_x$ denotes the probability that a person aged $x$ will die within $n$ years,

${}_n|\,q_x$ denotes the probability that a person aged $x$ will die between the ages $x + n$ and $x + n + 1$.

The values of $l_x$, $d_x$, $p_x$, and $q_x$ are given in Table XI. The expressions for the values of the above symbols in terms of the values of $l_x$ can be written at once from their definitions and from the definition of probability given in Art. 72. For example,

$$d_{20} = l_{20} - l_{21}, \qquad p_{20} = \frac{l_{21}}{l_{20}}, \qquad q_{20} = \frac{l_{20} - l_{21}}{l_{20}}, \qquad {}_{|10}q_{20} = \frac{l_{20} - l_{30}}{l_{20}},$$

$${}_{10|}q_{20} = \frac{l_{30} - l_{31}}{l_{20}}.$$

By means of the theorems in Art. 73 various other probabilities can be readily expressed in terms of the above symbols. For example, the probability that two persons aged $x$ and $y$ will both survive $n - 1$ years but will not both survive $n$ years is ${}_{n-1}p_x \cdot {}_{n-1}p_y - {}_np_x \cdot {}_np_y$. This may be seen as follows: By the theorem on independent events, Art. 73, $\frac{l_{x+n-1}}{l_x} \cdot \frac{l_{y+n-1}}{l_y}$ is the probability that both will survive $n - 1$ years, or that they will attain the ages $x + n - 1$, $y + n - 1$. Having attained these ages, by the same theorem, the probability that both will live another year is $\frac{l_{x+n}}{l_{x+n-1}} \cdot \frac{l_{y+n}}{l_{y+n-1}}$. So that $1 - \frac{l_{x+n}}{l_{x+n-1}} \cdot \frac{l_{y+n}}{l_{y+n-1}}$ is the probability that both will not live another year. By the theorem on dependent events, Art. 73, it now follows that the desired probability is

$$\frac{l_{x+n-1}}{l_x} \cdot \frac{l_{y+n-1}}{l_y}\left(1 - \frac{l_{x+n}}{l_{x+n-1}} \cdot \frac{l_{y+n}}{l_{y+n-1}}\right) = \frac{l_{x+n-1}}{l_x} \cdot \frac{l_{y+n-1}}{l_y} - \frac{l_{x+n}}{l_x} \cdot \frac{l_{y+n}}{l_y}$$

$$= {}_{n-1}p_x \cdot {}_{n-1}p_y - {}_np_x \cdot {}_np_y.$$

For other methods of proof see Exercise 11 below.

## EXERCISES

**1.** Compute the values of $p_{25}$ and of $q_{25}$. Use Table XI in this exercise and in those Exercises following.

**2.** Compute the probability that a person aged 25 will live 14 years; 15 years. Ans. .885771; .877280.

**3.** Compute the probability that a person aged 25 will die within 14 years; within 15 years. Ans. .114229; .122720.

**4.** Compute the probability that a person aged 25 will die in the 15th year. Ans. .008491.

**5.** Same as Exercises 2, 3, and 4 for a person aged 30.
Ans. .877623; .868120; .122377; .131880; .009504.

**6.** Compute the probability that two persons aged 25 and 30 will live (*a*) 14 years; (*b*) 15 years. Ans. .777374; .761584.

**7.** Compute the probability that two persons aged 39 and 44 will not both live one year. Ans. .020311.

**8.** Compute the probability that a person aged 25 will live 15 years and a person aged 30 will die in the 15th year. Ans. .008337.

**9.** Compute the probability that a person aged 30 will live 15 years and a person aged 25 will die in the 15th year. Ans. .007371.

**10.** Compute the probability that two persons aged 25 and 30 will both die in the 15th year. Ans. .000081.

**11.** Compute the probability that two persons aged 25 and 30 will both live 14 years and that at least one will die in the 15th year, by the methods:

(*a*) Divide the number of pairs living at ages 25 and 30, $(l_{25} \cdot l_{30})$, into the difference between the number of pairs living 14 and 15 years hence, $(l_{39} \cdot l_{44} - l_{40} \cdot l_{45})$,

(*b*) Find the product of the results found in Exercises 6 (*a*) and 7,

$$\frac{l_{39}}{l_{25}} \cdot \frac{l_{44}}{l_{30}} \left(1 - \frac{l_{40}}{l_{39}} \cdot \frac{l_{45}}{l_{44}}\right),$$

(*c*) Subtract the result found in Exereise 6 (*b*) from that found in 6 (*a*),

$$\frac{l_{39}}{l_{25}} \cdot \frac{l_{44}}{l_{30}} - \frac{l_{40}}{l_{25}} \cdot \frac{l_{45}}{l_{30}},$$

(*d*) Find the sum of the results found in Exercises 8, 9, and 10,

$$\frac{l_{40}}{l_{25}} \cdot \frac{l_{44} - l_{45}}{l_{30}} + \frac{l_{45}}{l_{30}} \cdot \frac{l_{39} - l_{40}}{l_{25}} + \frac{l_{44} - l_{45}}{l_{30}} \cdot \frac{l_{39} - l_{40}}{l_{25}}.$$

Ans. .015790.

**12.** Draw the graph of the curve showing the probability of dying for each year listed in Table XI.

**13.** Draw the graph of the curve showing the number of deaths for each year listed in Table XI. In what year do the greatest number of deaths take place?

**75. The expectation of life.** By the expectation of life at age $x$ is meant the average number of years to be lived by persons of age $x$. If the deaths during any year are assumed to take place at the beginning of the year, then $l_x$ persons of age $x$ will live $l_{x+1}$ years during the first year (since $l_{x+1}$ of the $l_x$ persons live throughout this year), $l_{x+2}$ years during the second, and so on to the end of the table. Under this assumption the expectation of life at age $x$, called the *curtate expectation*, $e_x$, is given by

$$e_x = \frac{l_{x+1} + l_{x+2} + \cdots}{l_x}$$

Similarly, under the assumption that the deaths during any year take place at the end of the year, the expectation of life at age $x$ is given by

$$\frac{l_x + l_{x+1} + \cdots}{l_x}$$

Under the assumption that the deaths during any year are distributed uniformly throughout the year, so that on the average each person will live half a year in the year of death the expectation of life at age $x$, called the *complete expectation* and denoted by $\mathring{e}_x$, is given by adding $\frac{1}{2}$ to the curtate expectation. It follows that

$$\mathring{e}_x = \frac{1}{2} + \frac{l_{x+1} + l_{x+2} + \cdots}{l_x}$$

It is often supposed by those unacquainted with actuarial methods, that the expectation of life is used as a basis for actuarial computations. That this is not the case will be seen in the subsequent articles. The expectation of life is sometimes used to determine the approximate value of a life interest in settling estates.

EXERCISE

Compute the complete expectation of life at the ages 35, 43, 67, and 71.
Ans. 31.78; 26.00; 10.00; 8.00.

**76. The value of an expectation.** By the value at a given time, of the expectation of receiving a sum of money, $S$, due in $n$ years, ($n \geqq 0$) from this time, is meant the value of $S$ discounted for $n$ years times the probability of receiving $S$. If a person will win \$1000 in case he throws two heads in a single throw of two coins, the value of his expectation is $1000 \cdot \frac{1}{4} = \$250$; here $n = 0$. If a person aged 30 is to receive \$1000 at age 50, the value of his expectation is $1000\, v^{20} \cdot {}_{20}p_{30}$, where $v = \frac{1}{(1+i)}$. If two persons aged 25 and 30 are to receive \$1000 at the end of 20 years in case both are then living, the value of their expectation is $1000\, v^{20} \cdot {}_{20}p_{25} \cdot {}_{20}p_{30}$. If \$1000 is to be paid at the end of 20 years in case two persons aged 25 and 30 both live 19 years but both do not live 20 years, the value of the expectation that the money will be paid is $1000\, v^{20}({}_{19}p_{25} \cdot {}_{19}p_{30} - {}_{20}p_{25} \cdot {}_{20}p_{30})$.

### EXERCISES

**1.** A person is to receive \$60 in case he throws a total of 7 in a single throw of two dice. Find the value of his expectation. Ans. \$10.

**2.** A person aged 30 is to receive \$1000 at the end of one year. Compute the value of his expectations. Use $i = .035$. Ans. \$958.04.

**3.** A person aged 30 is to receive \$1000 at the end of each year for 5 years. Compute the sum of the values of his expectations. Use $i = .035$.
Ans. \$4403.18.

**4.** A person aged 30 is to receive \$1000 at the end of each year for 5 years after he attains the age of 50. Find the sum of the values of his expectations. Use $i = .035$. Ans. \$1774.67.

**5.** If a person aged 30 dies within one year, his estate is to receive \$1000 at the end of the year. Find the value of the expectation. Use $i = .035$.
Ans. \$8.14.

**77. Life annuities.** A life annuity is a set of periodic payments, usually equal in value, during a term of years which begins at a specified age and continues during the whole or a part of the life of a person. This person is called an *annuitant.* As in annuities certain the term is from the beginning of the first period to the end of the last. When the term continues through the whole life of the person, the annuity is called a *whole life annuity:* when it ends at a stated time, even though the annuitant be still living, it is called a *temporary life annuity.* A *pure endowment* due in $n$ years on the life of a person consists of a single payment at the end of $n$ years in case the person is living at that time; that is, it is a temporary annuity having just one payment.

The classifications of whole life and temporary life annuities are analogous to those of annuities certain. One classification depends on when the term begins with respect to the age of the annuitant; it may begin at, after, or before this age. This gives rise respectively to *ordinary, deferred,* and *forborne* whole life and temporary life annuities. In what follows the word "ordinary" is omitted when there is no ambiguity in meaning. Another classification distinguishes whether the rent is paid at the end or the beginning of the rent period. This gives rise respectively to whole life and temporary life *annuities immediate,* and to whole life and temporary life *annuities due.* A third classification depends on the relative size of the rent payments; they may be equal, increasing, or

decreasing. The last two cases give rise to *increasing and decreasing* whole life and temporary life annuities.

The written contract between a company and an annuitant is called an *annuity policy*.

**78. The value of a life annuity defined.** If each of the $l_{30}$ persons living at age 30, shown in a mortality table, holds a ten year temporary life annuity policy of annual rent 1 whose first payment is due at age 31, the amounts that will be due from these annuities at the end of 1, 2, $\cdots$, 10 years are, according to the mortality table, $l_{31}$, $l_{32}$, $\cdots$, $l_{40}$ respectively. The sum of these amounts valued as of age 30, at an annual interest rate $i$, is $vl_{31} + v^2l_{32} + \cdots + v^{10}l_{40}$ where $v = \dfrac{1}{(1+i)}$. The quotient,

$$V = \frac{vl_{31} + v^2l_{32} + \cdots + v^{10}l_{40}}{l_{30}} \quad (= \$7.95 \text{ at } i = .035)$$

gives what is called the value at age 30, at an interest rate $i$, of this temporary annuity as determined by the mortality table. The sum of the amounts due the annuitants valued as of age 40, at an interest rate $i$, is $(1+i)^9l_{31} + (1+i)^8l_{32} + \cdots + l_{40}$. The quotient

$$V = \frac{(1+i)^9l_{31} + (1+i)^8l_{32} + \cdots + l_{40}}{l_{40}} \quad (= \$12.26 \text{ at } i = .035)$$

gives what is called the value at age 40, at an interest rate $i$, of this temporary annuity as determined by the mortality table. In other words the value at age 30 of this ten year temporary annuity is the amount that each of the $l_{30}$ annuitants must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay 1 to each annuitant at the end of each year of the term of the annuity. Likewise, the value at age 40 of this annuity is the amount that each of the $l_{40}$ persons would receive if the rent payments were not drawn when they are due, but were allowed to accumulate at compound interest to the time at age 40, and if the fund thus created were then equally divided among the $l_{40}$ survivors. The values at the ages 30 and 40 of this temporary life annuity as just defined are included in the

**Definition 1.** *The value at age $x$ of any life annuity of annual rent, $R$, is the quotient obtained by dividing $l_x$ into the sum of the values at age $x$ obtained by applying the compound interest formula to each sum of the set consisting of $R$ times the number living, according to the mortality table, at the time of each rent payment during the term of the annuity.*

From this definition the value at age $x$ of an ordinary or of a deferred life annuity of annual rent $R$ may be viewed as the amount that each of the $l_x$ annuitants must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay $R$ to each annuitant living at the time of each rent payment during the term of the annuity. Likewise the value at age $x$ of a forborne life annuity whose term begins $n$ or more years prior to the time at age $x$, where $n$ is the number of years in the term of the annuity, may be viewed as the amount that each of the $l_x$ survivors would receive if the rent payments were not drawn when they are due but were allowed to accumulate at compound interest to the time at age $x$, and if the fund thus created were then equally divided among the $l_x$ survivors.

When the annuity in definition 1 consists of a single sum payable at age $x + t$, the definition can be stated in the form:

*The value, $V$, at age $x$ of a one year temporary life annuity with annual rent $R$ payable at age $x + t$ is given by*

$$V = Rv^t \frac{l_{x+t}}{l_x} = R \frac{D_{x+t}}{D_x} \qquad (1_2)$$

$$= R \cdot v^t \cdot {}_tp_x \qquad (\text{when } t > 0)$$

*where $D_x = v^x l_x$ so that $D_{x+t} = v^{x+t} l_{x+t}$.* The second form of formula $(1_2)$ is obtained from the first by multiplying numerator and denominator by $v^x$. When $t$ is positive, $V$ is the present or discounted value at age $x$ of a sum $R$ to be paid at the end of $t$ years in case a person aged $x$ survives $t$ years; that is, when $t$ is positive, $V$ is the value at age $x$ of a pure endowment due in $t$ years. When $t$ is negative, $V$ is the amount at the end of $t$ years of a sum $R$ paid at age $x + t$ which is allowed to accumulate for $t$ years, or to age $x$, as a pure endowment. The value at age $x$ of a

pure endowment of 1 due in $t$ years is denoted by ${}_tE_x$. It follows that

$$ {}_tE_x = v^t \frac{l_{x+t}}{l_x} = v^t \cdot {}_tp_x = \frac{D_{x+t}}{D_x} \qquad (1_3) $$

In finding the values at two or more ages of a given life annuity, use can be made of the following

**Theorem III.** *If $V_{x+t}$ denotes the value at age $x + t$ of a life annuity, whose rent is payable annually, its value, $V_x$, at age $x$ is given by*

$$ V_x = v^t \frac{l_{x+t}}{l_x} V_{x+t} = \frac{D_{x+t}}{D_x} V_{x+t}. $$

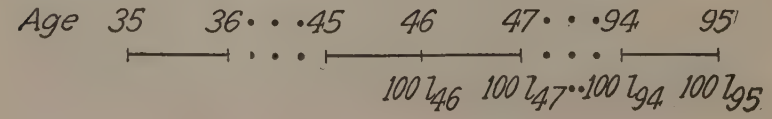
Proof: Since $V_{x+t}$ is the value per person at age $x + t$, the aggregate value at age $x + t$ is $l_{x+t} V_{x+t}$. The value of this aggregate at age $x$ is $v^t l_{x+t} V_{x+t}$ and the value per person at age $x$ is $\frac{v^t l_{x+t} V_{x+t}}{l_x}$. This theorem for life annuities is the analogue of Theorem I, Art. 15, for annuities certain. When $t$ is positive $\frac{D_{x+t}}{D_x}$ is a *discount factor;* when $t$ is negative, it is an *accumulation factor*. It should be noted that Theorem III includes formula $(1_2)$.

The symbol $D_x$ is called a commutation symbol. Table XII gives the values at $i = .035$ of this and other commutation symbols. Computation of values of life annuities and life insurances are greatly facilitated by their use.

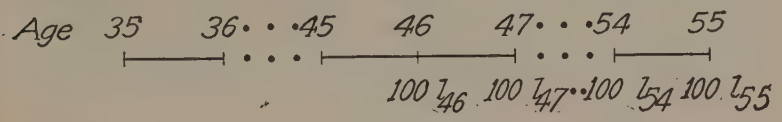
The value at age $x$ of the annuity specified by any policy is called the *net single premium* at age $x$ of the policy.

### EXERCISES

1. Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, at age 46, and at age 55 of a whole life annuity of annual rent \$100 with first payment at age 46:



**2.** Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, at age 46, and at age 55 of a ten year temporary life annuity of annual rent \$100 with first payment at age 46:




**3.** Apply definition 1 to write the expressions for the net single premiums at age 30 of the following life annuity policies, each having an annual rent of \$1000:

(*a*) Ordinary whole life annuity immediate,
(*b*) Ordinary whole life annuity due,
(*c*) Whole life annuity immediate deferred 10 years,
(*d*) Whole life annuity due deferred 10 years,
(*e*) Ordinary five year temporary life annuity immediate,
(*f*) Ordinary five year temporary life annuity due,
(*g*) Five year temporary life annuity immediate deferred 10 years,
(*h*) Five year temporary life annuity due deferred 10 years,
(*i*) Five year temporary life annuity d e forborne 5 years.

**4.** Compute the value at age 90 of an ordinary whole life annuity immediate of annual rent \$1000. Use definition 1 and $i = .035$. Ans. \$873.78.

**5.** Compute the value at age 25 of a pure endowment of \$1000 due in 25 years. Use $i = .035$. Ans. \$331.76.

**6.** Compute the present value, by use of the compound interest formula, of \$1000 due in 25 years. Use $i = .035$. Ans. \$423.15.

**7.** Compute the amount or the value at age 50 by use of formula $(1_2)$ of \$1000 paid at age 25. Use $i = .035$. Ans. \$3014.23.

**8.** Compute the amount at age 50 by use of the compound interest formula of \$1000 paid at age 25. Use $i = .035$. Ans. \$2363.25.

**79. Another definition of the value of a life annuity.** The expressions given in Art. 78 for the values at the ages 30 and 40 of a ten year temporary life annuity immediate whose term begins at age 30 can be written in the forms:

$$V = v\frac{l_{31}}{l_{30}} + v^2\frac{l_{32}}{l_{30}} + \cdots + v^{10}\frac{l_{40}}{l_{30}} \quad \text{(at age 30), and}$$

$$V = v^{-9}\frac{l_{31}}{l_{40}} + v^{-8}\frac{l_{32}}{l_{40}} + \cdots + \frac{l_{40}}{l_{40}} \quad \text{(at age 40).}$$

These forms show that the value of this annuity at either age is the sum of the values obtained by applying formula $(1_2)$ to each sum of the set consisting of 1 at the time of each rent payment during the term of the annuity. The value of the temporary life annuity defined in this way is included in the

**Definition 2.** *The value at age $x$ of any life annuity of annual rent $R$ is the sum of the values obtained by applying the formula*

$$V = Rv^t \frac{l_{x+t}}{l_x} = R \frac{D_{x+t}}{D_x}$$

*to each sum of the set consisting of $R$ at the time of each rent payment during the term of the annuity.*

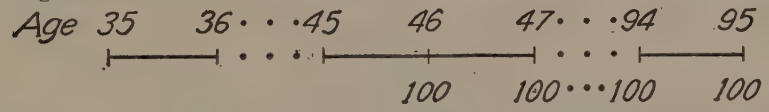
From this definition and the value of an expectation (Art. 76), the value at age $x$ of an ordinary or of a deferred annuity of annual rent $R$ is the sum of the values at age $x$ of an annuitant's expectations of receiving the rent payments during the term of the annuity; or, it is the sum of the discounted values at age $x$, obtained by use of formula $(1_2)$ when $t$ is zero or positive, of the set of sums consisting of $R$ at the time of each rent payment during the term of the annuity. Likewise, the value at age $x$ of a forborne annuity, whose term begins $n$ or more years prior to the time at age $x$, where $n$ is the number of years in the term of the annuity, is the sum of the amounts at age $x$ of the same set of sums when each is accumulated as a pure endowment, that is, by use of formula $(1_2)$ when $t$ is zero or negative.

The definitions in this and the preceding article evidently lead to the same value at age $x$ of any life annuity. By definition 1, the compound interest formula is used to find the value at age $x$ of a specified set of sums, each differing from the others, and this value is then divided by $l_x$. By definition 2, formula $(1_2)$ is used to find the value at age $x$ of a specified set of sums, each equal to $R$. In the above treatment this formula, $(1_2)$, is obtained by use of definition 1. It can also be obtained by use of the definition of the value of an expectation (Art. 76), since, when $t \geq 0$, $Rv^t$ is the value of $R$ discounted for $t$ years, and $\frac{l_{x+t}}{l_x}$ is the probability denoted by ${}_tp_x$. Other formulas for finding the value of a set of one or more

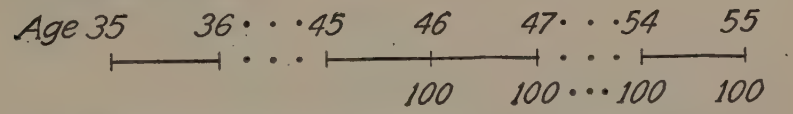
equal sums which are of fundamental importance in the theory of life annuities and of life insurances represent the values of expectations and can be written at once by one familiar with the elements of the theory of probability. Later in this chapter, analogous definitions are given for the value of any life insurance based on one life and for the values of joint life annuities and insurances.

### EXERCISES

1. Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, at age 46, and at age 55, of a whole life annuity of annual rent \$100 with first payment at age 46:




2. Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, at age 46, and at age 55, of a ten year temporary life annuity of annual rent \$100 with first payment at age 46:




3. Same as Exercise 3, Art. 78, with "definition 1" replaced by "definition 2."

**80. The value at age $x$ of any whole life annuity of annual rent $R$.** Let $x + t$ denote the age when the first rent payment of the annuity is made. By definition 1, Art. 78, the value, $V$, at age $x$ is given by

$$V = R\frac{v^t l_{x+t} + v^{t+1} l_{x+t+1} + \text{etc.}}{l_x}$$

$$= R\frac{v^{x+t} l_{x+t} + v^{x+t+1} l_{x+t+1} + \text{etc.}}{v^x l_x}$$

$$= R\frac{D_{x+t} + D_{x+t+1} + \text{etc.}}{D_x}$$

$$= R\frac{N_{x+t}}{D_x} \qquad (1_1)$$

where the commutation symbol, $N_x$* is defined by

$$N_x = D_x + D_{x+1} + \text{etc., so that } N_{x+t} = D_{x+t} + D_{x+t+1} + \text{etc.}$$

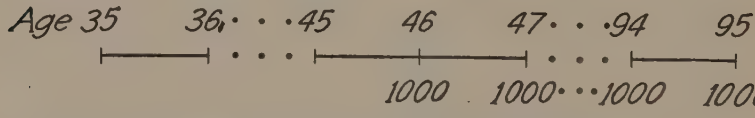
If the annual rent is 1, the values at age $x$ of a whole life annuity immediate and of a whole life annuity due are denoted respectively by $a_x$, $\mathbf{a}_x$ when the terms begin at age $x$, and by ${}_{n_1}|\,a_x$, ${}_{n_1}|\,\mathbf{a}_x$ respectively when the terms begin at age $x + n_1$, that is, when the terms are deferred $n_1$ years. From formula $(1_1)$ it follows that

$$a_x = \frac{N_{x+1}}{D_x}, \qquad \mathbf{a}_x = \frac{N_x}{D_x}$$

$${}_{n_1}|\,a_x = \frac{N_{x+1+n_1}}{D_x} \qquad {}_{n_1}|\,\mathbf{a}_x = \frac{N_{x+n_1}}{D_x}$$

## EXERCISES

**1.** Apply formula $(1_1)$ to write the expressions for the values at age 35, at age 45, at age 46, and at age 55, of the whole life annuity of annual rent \$1000 represented by the following diagram:




**2.** Find the expressions for the values at age 35, at age 46, and at age 55 of the whole life annuity in Exercise 1 by applying Theorem III to the expression for its value at age 45.

**3.** Apply formula $(1_1)$ to write the expressions for the net single premiums at age 25 of the following life annuity policies each having an annual rent of \$1000:

(*a*) Ordinary whole life annuity immediate,
(*b*) Ordinary whole life annuity due,
(*c*) Whole life annuity immediate deferred 10 years,
(*d*) Whole life annuity due deferred 10 years.

**4.** Use formula $(1_1)$ at $i = .035$ to compute the value at age 25 of an ordinary whole life annuity immediate of annual rent \$1000. Ans. \$19441.73.

**5.** Find the net single premium at age 25 of a whole life annuity due of annual rent \$1000 deferred 25 years. Ans. \$4822.03.

* According to the definition adopted by the International Congress of Actuaries

$$N_x = D_{x+1} + D_{x+2} + \text{etc.}$$

The definition here given, however, is that generally used in America, the open bar $N$ being used to distinguish it from that given for $N_x$.

**6.** The paid up value, \$25000, of a life insurance policy is used to purchase a whole life annuity policy of annual rent $R$ for a person aged 50. Find $R$ if the first rent payment is made immediately. Use $i = .035$. Ans. \$1720.02.

**7.** Solve Exercise 6 if the first instalment is made when the person aged 50 attains the age 65. Ans. \$6427.15.

**8.** Derive formula $(1_1)$ by use of definition 2, Art. 79.

**9.** Compute the value at age 67 of an ordinary whole life annuity immediate of annual rent \$100. Use $i = .035$. Ans. \$850.97.

**10.** At age 67 the complete expectation of life is 10 years. Compute the present value of the annuity certain ($R = 100$, $n = 10$, $r = 1$) at $i = .035$ and compare with the result in Exercise 9. Ans. \$831.66.

**11.** Use formula $(1_1)$ to write the expressions for $a_{25}$, $\mathbf{a}_{25}$, ${}_{10|}a_{25}$, and ${}_{10|}\mathbf{a}_{25}$.

**81. The value at age $x$ of any $n$-year temporary life annuity of annual rent $R$.** Let $x + t$ denote the age when the first rent payment of the annuity is made. By definition 1, Art. 78, the value, $V$, at age $x$ is given by

$$\begin{aligned} V &= R\frac{v^t l_{x+t} + v^{t+1}l_{x+t+1} + \cdots + v^{t+n-1}l_{x+t+n-1}}{l_x} \\ &= R\frac{v^{x+t}l_{x+t} + v^{x+t+1}l_{x+t+1} + \cdots + v^{x+t+n-1}l_{x+t+n-1}}{v^x l_x} \\ &= R\frac{D_{x+t} + D_{x+t+1} + \cdots + D_{x+t+n-1}}{D_x} \\ &= R\frac{N_{x+t} - N_{x+t+n}}{D_x} \end{aligned} \tag{1}$$

When the term of the temporary annuity continues throughout the life of the annuitant, $l_{x+t+n} = 0$ and hence $D_{x+t+n} = N_{x+t+n} = 0$. It follows that formula (1) includes formula $(1_1)$. When $n = 1$, $N_{x+t} - N_{x+t+n} = D_{x+t}$ so that formula (1) includes formula $(1_2)$.

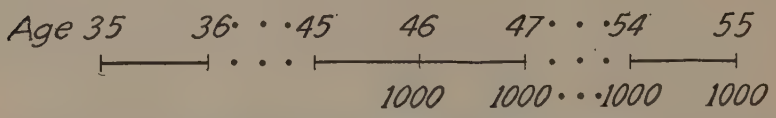
If the annual rent is 1, the values at age $x$ of an $n$-year temporary life annuity immediate and of an $n$-year temporary life annuity due are denoted respectively by $a_{x\overline{n}|}$, $\mathbf{a}_{x\overline{n}|}$ when the terms begin at age $x$, and by ${}_{n_1|}a_{x\overline{n}|}$, and ${}_{n_1|}\mathbf{a}_{x\overline{n}|}$ respectively when the terms begin at age $x + n_1$, that is, when the terms are deferred $n_1$ years. From formula (1) it follows that

$$a_{x\overline{n}|} = \frac{N_{x+1} - N_{x+1+n}}{D_x}, \qquad \mathbf{a}_{x\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}$$

$${}_{n_1|}a_{x\overline{n}|} = \frac{N_{x+1+n_1} - N_{x+1+n_1+n}}{D_x}, \qquad {}_{n_1|}\mathbf{a}_{x\overline{n}|} = \frac{N_{x+n_1} - N_{x+n_1+n}}{D_x}$$

## EXERCISES

**1.** Apply formula (1) to write the expressions for the values at age 35, at age 45, and at age 55, of the ten year temporary life annuity of annual rent \$1000 represented by the following diagram:




**2.** Find the expressions for the values at age 35, at age 45, and at age 55, of the temporary annuity in Exercise 1 by applying Theorem III to the expression for its value at age 46.

**3.** Apply formula (1) to write the expressions for the net single premiums at age 25 of the following life annuity policies each having an annual rent of \$1000:

(*a*) Ordinary ten year temporary life annuity immediate,
(*b*) Ordinary ten year temporary life annuity due,
(*c*) Ten year temporary life annuity immediate deferred 5 years,
(*d*) Ten year temporary life annuity due deferred 5 years,
(*e*) Ten year temporary life annuity due forborne 10 years.

**4.** Use formula (1) at $i = .035$ to compute the value at age 25 of an ordinary ten year temporary life annuity due of annual rent \$100. Ans. \$831.46.

**5.** Find the net single premium at age 25 of a ten year temporary life annuity immediate of annual rent \$1000 deferred 10 years. Ans. \$5158.91.

**6.** Compute the value at age 35 of a \$1000 ten year life annuity due forborne 10 years. Ans. \$12762.12.

**7.** The paid up value, \$25000, of a life insurance policy is used to purchase a twenty year temporary life annuity of annual rent $R$ for a person aged 50. Find $R$ if the first rent payment is made immediately. Use $i = .035$.
Ans. \$2006.90.

**8.** Solve Exercise 7 if the first instalment is made when the person aged 50 attains the age of 65. Ans. \$6549.47.

**9.** Use formula $(1_t)$ to derive formula (1) by resolving the $n$-year temporary life annuity whose first payment is made at age $x + t$ into the difference between two whole life annuities, one having its first payment at age $x + t$ and the other at age $x + t + n$.

**10.** Derive formula (1) by use of definition 2, Art. 78.

**11.** Use formula (1) to write the expressions for $a_{25:\overline{10}|}$, $\mathbf{a}_{25:\overline{10}|}$, ${}_{10|}a_{25:\overline{10}|}$, and ${}_{10|}\mathbf{a}_{25:\overline{10}|}$.

**12.** Prove Theorem III by use of formula 1.

**82. Some important special cases of formula (1).** By means of formula (1) the value of age $x$ of any life annuity of annual rent $R$ can be written at once. In this article important special cases are listed for the purpose of comparison.

| | Whole Life Annuity | | $n$-Year Temporary Life Annuity | |
|---|---|---|---|---|
| | Immediate | Due | Immediate | Due |
| Term begins at age $x$ | $R\frac{N_{x+1}}{D_x}$ | $R\frac{N_x}{D_x}$ | $R\frac{N_{x+1}-N_{x+1+n}}{D_x}$ | $R\frac{N_x-N_{x+n}}{D_n}$ |
| Term deferred $n_1$ years | $R\frac{N_{x+1+n_1}}{D_x}$ | $R\frac{N_{x+n_1}}{D_x}$ | $R\frac{N_{x+1+n_1}-N_{x+1+n+n_1}}{D_x}$ | $R\frac{N_{x+n_1}-N_{x+n+n_1}}{D_x}$ |
| Term forborne $n$ years | $R\frac{N_{x+1-n}}{D_x}$ | $R\frac{N_{x-n}}{D_x}$ | $R\frac{N_{x+n-1}}{D_x}$ | $R\frac{N_{x-n}-N_x}{D_x}$ |

An $n$-year temporary life annuity whose term is forborne $n$ years is usually called a *forborne life annuity*. In the application of forborne annuities it is frequently convenient to have $x$ for the age at the beginning of the term rather than at the end. When this is done, the formula listed above for the value of a *forborne temporary life annuity due* becomes, upon replacing $x$ by $x+n$,

$$V = R\frac{N_x - N_{x+n}}{D_{x+n}}.$$

When $R = 1$, the value of $V$ is denoted by ${}_nu_x$. Hence

$${}_nu_x = \frac{N_x - N_{x+n}}{D_{x+n}} \tag{2}$$

When $n = 1$, ${}_nu_x$ is denoted by $u_x$. From formula (2) it follows that

$$u_x = \frac{D_x}{D_{x+1}}.$$

Values of $u_x$ based on the American Experience Table at $3\frac{1}{2}\%$ are given in Table XIII.

EXERCISES

1. Write the expressions for the values of $u_{20}$ and of ${}_{10}u_{20}$.
2. Compute the value of $u_{20}$.

DEPARTMENT OF MATHEMATICS
CASE SCHOOL OF APPLIED SCIENCE

**83. Relations connecting the symbols for the values of life annuities.** Important and interesting relations among the symbols for the values of life annuities can be readily determined. This can be done in various ways. One fruitful method is that of equating the value at age $x$ of a given annuity to the sum of the values at this age of any set of annuities into which it can be resolved. By application of the method it is seen, for example, that

$\mathbf{a}_x = 1 + a_x$ (Whole life annuity due = 1 plus a whole life immediate.)

${}_{n_1}|\,\mathbf{a}_x = {}_{n_1}|\,a_x + {}_{n_1}E_x$ (Deferred whole life annuity due = deferred whole life annuity immediate plus a pure endowment.)

$\mathbf{a}_{x\overline{n}|} = a_{x\overline{n}|} + 1 - {}_nE_x$ (Temporary life annuity due = temporary life annuity immediate plus 1 minus a pure endowment.)

These relations can be written in the forms: $\mathbf{a}_x - a_x = 1$, ${}_{n_1}|\,\mathbf{a}_x - {}_{n_1}|\,a_x = {}_{n_1}E_x$ and $\mathbf{a}_{x\overline{n}|} - a_{x\overline{n}|} = 1 - {}_nE_x$. A more general relation which includes each of these is that which expresses the difference, $\Delta$, between the values at age $x$ of an $n$-year temporary life annuity of annual rent $R$ whose first payment is made at age $x + k$, and a like annuity whose first payment is made at age $x + k + 1$. Since these two annuities have all their rent payments occurring at the same ages except that at age $x + k$ in the first and that at age $x + k + n$ in the second, it follows that

$$\Delta = R\frac{D_{x+k} - D_{x+k+n}}{D_x}. \tag{3}$$

Formula (3) is used in Art. 84 in finding the values of annuities whose rent payments are made more than once a year.

Theorem III, Art. 78, is also useful in determining relations. An illustration is the following: The value at age $x$ of a whole life annuity immediate of annual rent 1 is denoted by $a_x$, and the value at age $x + 1$ of this same annuity is denoted by $\mathbf{a}_{x+1}$ or by $1 + a_{x+1}$. By the use of Theorem III it now follows that

$$a_x = v\frac{l_{x+1}}{l_x}(1 + a_{x+1}) = \frac{D_{x+1}}{D_x}(1 + a_{x+1})$$

This relation can be used to compute a table of values for $a_x$.

EXERCISES

1. Show that
   (a) $\mathbf{a}_{x\overline{n}|} = 1 + a_{x\overline{n-1}|}$
   (b) $a_x = a_{x\overline{n}|} + {}_n|a_x$
   (c) $a_{x+1} = u_x a_x - 1$
   (d) $\mathbf{a}_x = \frac{D_{x+1}}{D_x}\mathbf{a}_{x+1} + 1$
   (e) $\mathbf{a}_{x+1} = (\mathbf{a}_x - 1)u_x$
   (f) ${}_nu_x = ({}_{n-1}u_x + 1)u_{x+n-1}$

2. Interpret each of the relations in Exercise 1 verbally. (Note that, by Theorem III, $u_x$ is an accumulation factor for one year and $\frac{D_{x+1}}{D_x}$ is a discount factor for one year.)

**84. The value at age $x$ of a life annuity whose rent is payable more than once a year.** To find the value of a life annuity whose rent is payable more than once a year by direct application of the method given in Art. 78 would require a mortality table which shows the number living at each age, both fractional and integral, at which a rent payment is made. With such tables the computations would be quite tedious, especially if the number of payments each year were large. Satisfactory values for annuities payable $m$ times a year, $(m>1)$, can be readily found, however, by use of the values of annuities payable once a year and the method of simple interpolation. From formula (3), Art. 83, it follows that if each rent payment of an $n$-year temporary annuity whose annual rent is $\frac{R}{m}$ and whose first payment is made at age $x+k$ were made one year later, the value at age $x$ of this annuity would be decreased by $\frac{\Delta}{m}$. By simple interpolation, the value of this annuity would be decreased by $\frac{1}{m}\cdot\frac{\Delta}{m}$ if each rent payment were made $\frac{1}{m}$th years later, by $\frac{2}{m}\cdot\frac{\Delta}{m}$ if each were made $\frac{2}{m}$th years later, and so on. An $n$-year temporary annuity due of total annual rent $R$ payable in equal instalments $m$ times a year whose term begins at age $x+k$ is made up of $m$, $n$-year temporary annuities each of rent $\frac{R}{m}$, pay-

able annually, whose rent payments begin at the ages, $x + k$, $x + k + \frac{1}{m}$, and so on. By use of formula (1) and simple interpolation, the values at age $x$ of these annuities are given by

$$\frac{R}{m}\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right)$$

$$\frac{R}{m}\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{1}{m} \cdot \frac{\Delta}{m},$$

$$\frac{R}{m}\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{2}{m} \cdot \frac{\Delta}{m},$$

. . . . . .

$$\frac{R}{m}\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{m-1}{m} \cdot \frac{\Delta}{m}.$$

Summing these values by using the formula for the sum of an arithmetic progression, gives $R\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{m-1}{2m}\Delta$. An analogous procedure for an $n$-year temporary annuity immediate whose term begins at age $x + k$ gives $R\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{m+1}{2m}\Delta$. It follows that the values at age $x$ for these $m$ payment $n$-year temporary life annuities of total annual rent $R$ whose *terms begin at age* $x + k$ are given by

$$\left.\begin{aligned} V &= R\left[\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{m-1}{2m}\,\frac{D_{x+k} - D_{x+k+n}}{D_x}\right] \text{for the annuity due,} \\ V &= R\left[\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{m+1}{2m}\,\frac{D_{x+k} - D_{x+k+n}}{D_x}\right] \text{for the annuity immediate.} \end{aligned}\right\} \quad (4)$$

The symbols for the values of life annuities of annual rent 1 payable $m$ times a year are formed by writing the number $m$ in parenthesis to the upper right of the symbols for the values of annuities of annual rent 1 payable once a year. For example, $a_x^{(m)}$ denotes the value at age $x$ of a whole life annuity immediate of annual rent 1 payable $m$ times a year. The expression for any one of these symbols can be written at once by use of formulas (4).

### EXERCISES

**1.** Compute the value at age 30 of an ordinary whole life annuity immediate having a yearly rent of \$1200 payable in equal instalments (*a*) annually, (*b*) semi-annually, (*c*) quarterly, and (*d*) monthly.

Ans. \$22326.46; \$22626.46; \$22776.46; \$22876.46.

**2.** Same as Exercise 1 except that the term is deferred 10 years.

Ans. \$12789.68; \$12984.10; \$13081.30; \$13146.11.

**3.** Same as Exercise 1 for a ten year temporary life annuity due.

**4.** Write the expressions for

(*a*) $a_x^{(m)}$ and $\mathbf{a}_x^{(m)}$

(*b*) $a_{x\overline{n}|}^{(m)}$ and $\mathbf{a}_{x\overline{n}|}^{(m)}$

(*c*) ${}_{n_1}|\,a_x^{(m)}$ and ${}_{n_1}|\,\mathbf{a}_{x\overline{n}|}$

**5.** Show that

(*a*) $a_x^{(m)} = a_x + \dfrac{m-1}{2m}$ and $\mathbf{a}_x^{(m)} = \mathbf{a}_x - \dfrac{m-1}{2m}$

(*b*) $a_{x\overline{n}|}^{(m)} = a_{x\overline{n}|} + \dfrac{m-1}{2m}(1 - {}_nE_x)$ and $\mathbf{a}_{x\overline{n}|}^{(m)} = \mathbf{a}_{x\overline{n}|} - \dfrac{m-1}{2m}(1 - {}_nE_x)$

(*c*) ${}_{n_1}|\,a_x^{(m)} = {}_{n_1}|\,a_x + \dfrac{m-1}{2m}\,{}_nE_x$ and ${}_{n_1}|\,\mathbf{a}_x^{(m)} = {}_{n_1}|\,\mathbf{a}_x - \dfrac{m-1}{2m}\,{}_nE_x$

**85. Life Insurance.** A life annuity policy is a contract which provides for the payment to the annuitant of stated amounts at periodic intervals during the whole or a part of his life. A life insurance policy on a person provides for the payment of a stated amount upon his death provided death takes place within a specified term of years. The amount to be paid is called the *face* of the policy or the insurance, the person on whose life it is taken is called the *insured*, and the person to whom the insurance is paid is called the *beneficiary*. The term during which a person is insured begins at a specified age and continues during the whole or a part of his life. When the term continues during the whole life of the person, the insurance is called *whole life insurance;* when it ends at a stated time, even though the insured be still living, it is called *term* life insurance.*

Both whole life and term insurances have classifications similar to those of annuities. One classification depends on when the term begins with respect to the age of the insured; it may begin

* The word "term" in term insurances takes the place of the word "temporary" in temporary annuities.

at, after, or before this age. This gives rise respectively to ordinary, deferred, and forborne whole life and term insurances. In referring to ordinary insurances it is customary to omit the word ordinary. The classification of life annuities into annuities immediate and annuities due does not carry over to life insurances since the insurance is paid to the beneficiary in a single payment, or in the form of an equivalent benefit, upon receipt of proof of death of the insured. In computing the net values of life insurances, however, it is usually assumed that payment of the insurance is made at the end of the year of death. Another classification depends on the relative size of the amounts to be paid. Some policies provide for amounts depending on the age at which death occurs. When these amounts increase with the age, the insurance is called *increasing insurance;* when they decrease, it is called *decreasing insurance.*

**86. The value of life insurance defined.** If each of the $l_{30}$ persons living at age 30, shown in a mortality table, holds a ten year term life insurance policy of face value 1 whose term begins at age 30, the amounts that will be due their beneficiaries at the end of 1, 2, $\cdots$, 10 years are, according to the mortality table, $d_{30}$, $d_{31}$, $\cdots$, $d_{39}$ respectively. The sum of these amounts valued as of age 30, at an interest rate $i$, is $vd_{30} + v^2d_{31} + \cdots + v^{10}d_{39}$ where $v = \dfrac{1}{(1+i)}$. The quotient

$$V = \frac{vd_{30} + v^2d_{31} + \cdots + v^{10}d_{39}}{l_{30}} \quad (= .07129 \text{ at } i = .035)$$

gives what is called the value at age 30, at an interest rate $i$, of this term insurance, as determined by the mortality table. The sum of the amounts due the beneficiaries valued as of age 40 at an interest rate $i$ is $(1+i)^9d_{30} + (1+i)^8d_{31} + \cdots + d_{39}$. The quotient

$$V = \frac{(1+i)^9d_{30} + (1+i)^8d_{31} + \cdots + d_{40}}{l_{40}} \quad (= .1100 \text{ at } i = .035)$$

gives what is called the value at age 40, at an interest rate $i$, of this term insurance as determined by the mortality table. In

other words the value at age 30 of this ten year term insurance is the amount that each of the $l_{30}$ insured must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay 1 to each beneficiary at the end of each year of the term of the insurance. Likewise, the value at age 40 of this insurance is the amount that each of the $l_{40}$ persons would receive if the benefits were not drawn when they are due but were allowed to accumulate at compound interest to the time at age 40, and if the fund thus created were then equally divided among the $l_{40}$ survivors. The values at the ages 30 and 40 of this term life insurance as just defined are included in the

**Definition 1.** *The value at age $x$ of any life insurance of face value, $F$, is the quotient obtained by dividing $l_x$ into the sum of the values at age $x$ obtained by applying the compound interest formula to each sum in the set consisting of $F$ times the number dying, according to the mortality table during each year of the term of the insurance.*

From this definition the value at age $x$ of an ordinary or of a deferred life insurance of face value, $F$, may be viewed as the amount that each of the $l_x$ insured must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay $F$ to each beneficiary at the end of each year during the term of the insurance. Likewise the value at age $x$ of a forborne life insurance, whose term begins $n$ or more years prior to the time at age $x$, where $n$ is the number of years in the term of the insurance, may be viewed as the amount that each of the $l_x$ survivors would receive if the benefits were not drawn when they are due but were allowed to accumulate at compound interest to the time at age $x$, and if the fund thus created were then equally divided among the $l_x$ survivors.

When the insurance in definition 1 is for a one year term beginning at the age $x + t - 1$, the definition can be stated in the form:

*The value, $V$, at age $x$ of an insurance of face value, $F$, for the year beginning at age $x + t - 1$ is given by*

$$V = Fv^t \frac{d_{x+t-1}}{l_x} = F \frac{C_{x+t-1}}{D_x} \qquad (5_2)$$

$$= Fv^t \cdot {}_{t-1}|\, q_x \qquad \text{(when } t > 0\text{)}$$

*where* $C_x = v^{x+1}d_x$, *so that* $C_{x+t-1} = v^{x+t}d_{x+t-1}$. The second form of formula $(5_2)$ is obtained from the first by multiplying numerator and denominator by $v^x$. When $t$ is positive, $V$ is the present or discounted value at age $x$ of a benefit $F$ to be paid at the end of $t$ years in case a person aged $x$ dies during the year beginning at age $x + t - 1$. When $t$ is negative, $V$ is the amount at age $x$ of the benefit $F$ paid at age $x + t$, at the end of a one year term insurance, which is allowed to accumulate for $t$ years in accordance with formula $(5_2)$.

In finding the values at two or more ages of a given life insurance use can be made of the following

**Theorem IV.** *If* $V_{x+t}$ *denotes the value at age* $x + t$ *of a life insurance, its value* $V_x$ *at age* $x$ *is given by*

$$V_x = v^t \frac{l_{x+t}}{l_x} V_{x+t} = \frac{D_{x+t}}{D_x} V_{x+t}.$$

The proof is the same as that for Theorem III, Art. 78.

In writing the equations needed for finding the unknowns in problems involving life annuities or life insurances use can be made of the

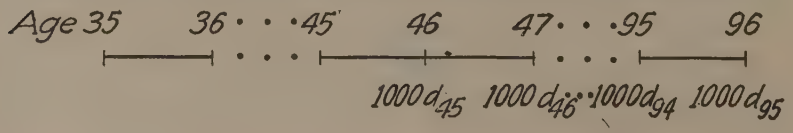
**Theorem V.** *If two life annuities, two life insurances, or a life annuity and a life insurance have equal values at age* $x + t$, *they have equal values at age* $x$.

This theorem is seen to be true by noting that Theorems III and IV have the same accumulation or discount factor, $\frac{D_{x+t}}{D_x}$. Theorem V is the analogue of Theorem II, Art. 16.

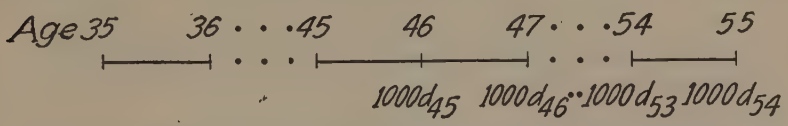
The value at age $x$ of the insurance specified by any polic called the *net single premium* at this age, of the policy.

### EXERCISES

**1.** Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, and at age 55 of a whole life insurance of face value \$1000 whose term begins at age 45:



**2.** Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, and at age 55 of a ten year term life insurance of face value \$1000, whose term begins at age 45:




**3.** Apply definition 1 to write the expressions for the net single premium at age 30 of the following \$1000 life insurance policies:

(*a*) Ordinary whole life,
(*b*) Whole life deferred 10 years,
(*c*) Ordinary ten year term,
(*d*) Ten year term deferred 10 years.

**4.** Compute the value at age 90 of an ordinary whole life insurance of face value \$1000. Use definition 1 and $i = .035$. Ans. \$936.64.

**5.** Compute the value at age 85 of a whole life insurance of face value \$1000 deferred 5 years. Use Theorem IV and the result found in Exercise 4. Ans. \$121.78

**6.** Compute the value at age 60 by use of formula $(5_2)$ of a \$1000 one year term insurance which matured at age 40; also the value at age 39. Ans. \$25.97; \$9.26.

**87. Another definition of the value of life insurance.** The expressions given in Art. 86 for the value at the ages 30 and 40 of a ten year term insurance whose term begins at age 30 can be written in the forms:

$$V = v\frac{d_{30}}{l_{30}} + v^2\frac{d_{31}}{l_{30}} + \cdots + v^{10}\frac{d_{39}}{l_{30}} \quad \text{(at age 30) and}$$

$$V = v^{-9}\frac{d_{30}}{l_{40}} + v^{-8}\frac{d_{30}}{l_{40}} + \cdots + \frac{d_{40}}{l_{40}} \quad \text{(at age 40).}$$

These forms show that the value of this insurance at either age is the sum of the values obtained by applying formula $(5_2)$ to each sum of the set consisting of 1 payable at the end of each year of the term of the insurance. The value of the term life insurance defined in this way is included in the

**Definition 2.** *The value at age $x$ of any life insurance of face value $F$ is the sum of the values obtained by applying the formula*

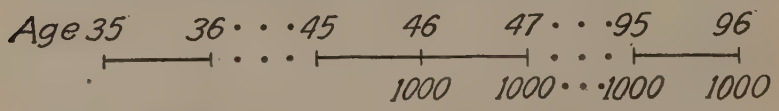
$$V = F\frac{v^t d_{x+t-1}}{l_x} = F\frac{C_{x+t-1}}{D_x}$$

*to each sum in the set consisting of $F$ payable at the end of each year of the term of the insurance.*

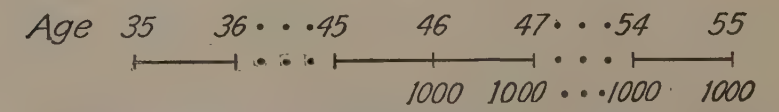
From this definition and the value of an expectation (Art. 76), the value at age $x$ of an ordinary or of a deferred insurance of face value $F$ is the sum of the values at age $x$ (of the insured) of a beneficiary's expectations of receiving $F$ at the end of each year during the term of the insurance; or it is the sum of the discounted values at age $x$, obtained by use of formula $(5_2)$ when $t$ is positive, of the set of sums consisting of $F$ at the end of each year of the term of the insurance. Likewise, the value at age $x$ of a forborne insurance whose term begins $n$ or more years prior to the time at age $x$, where $n$ is the number of years in the term of the insurance, is the sum of the amounts at age $x$ of the same set of sums when each is accumulated by use of formula $(5_2)$ when $t$ is zero or negative.

### EXERCISES

**1.** Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, and at age 55 of a whole life insurance of face value \$1000 whose term begins at age 45:




**2.** Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, and at age 55 of a ten year term life insurance of face value \$1000, whose term begins at age 45:




**3.** Same as Exercise 3, Art. 86, with "definition 1" replaced by "definition 2."

**88. The value at age $x$ of any whole life insurance of face value $F$.** Let $x + k$ denote the age when the term of the insurance begins. By definition 1, Art. 86, the value $V$ at age $x$ is given by

$$V = F\frac{v^{k+1}d_{x+k} + v^{k+2}d_{x+k+1} + \text{etc.}}{l_x}$$

$$= F\frac{v^{x+k+1}d_{x+k} + v^{x+k+2}d_{x+k+1} + \text{etc.}}{v^x l_x}$$

$$= F\frac{C_{x+k} + C_{x+k+1} + \text{etc.}}{D_x}$$

$$= F\frac{M_{x+k}}{D_x} \qquad (5_1)$$

where the commutation symbol, $M_x$, is defined by

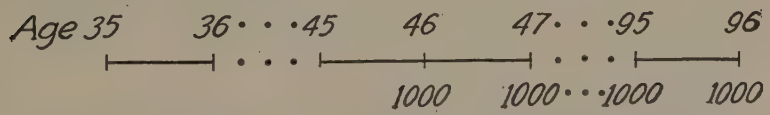
$M_x = C_x + C_{x+1} + \text{etc.}$, so that $M_{x+k} = C_{x+k} + C_{x+k+1} + \cdots$

If the annual rent is 1, the value, at age $x$, of a whole life insurance is denoted by $A_x$ when the term begins at age $x$, and by ${}_{n_1}|A_x$ when the term begins at age $x + n_1$, that is, when the term is deferred $n_1$ years. From formula $(5_1)$ it follows that

$$A_x = \frac{M_x}{D_x}, \qquad {}_{n_1}|A_x = \frac{M_{x+n_1}}{D_x}$$

### EXERCISES

**1.** Apply formula $(5_1)$ to write the expressions for the values at age 35, at age 45, and at age 55, of the whole life insurance of face value \$1000 whose term begins at age 45, represented by the following diagram:




**2.** Find the expressions for the values at age 35 and at age 55 of the whole life insurance in Exercise 1 by applying Theorem IV to the expression for its value at age 45.

**3.** Apply formula $(5_1)$ to write the expressions for the net single premiums at age 25 of the following \$1000 life insurance policies:

(*a*) Ordinary whole life,

(*b*) Whole life deferred 10 years.

4. Use formula $(5_1)$ at $i = .035$ to compute the value at age 25 of a \$1000 ordinary whole life insurance. Ans. \$308.73.

5. Find the net single premium at age 25 of a \$1000 whole life insurance deferred 10 years. Ans. \$241.41.

6. Derive formula $(5_1)$ by use of definition 2, Art. 85.

7. Use formula $(5_1)$ to write the expressions for the values of $A_{25}$ and ${}_{10|}A_{25}$.

**89. The value at age $x$ of any $n$-year term insurance of face value $F$.** Let $x + k$ denote the age when the term of the insurance begins. By definition 1, Art. 86, the value, $V$, at age $x$ is given by

$$V = F\frac{v^{k+1}d_{x+k} + v^{k+2}d_{x+k+1} + \cdots + v^{k+n}d_{x+k+n-1}}{l_x}$$

$$= F\frac{v^{x+k+1}d_{x+k} + v^{x+k+2}d_{x+k+1} + \cdots + v^{x+k+n}d_{x+k+n-1}}{v^x l_x}$$

$$= F\frac{C_{x+k} + C_{x+k+1} + \cdots + C_{x+k+n-1}}{D_x}$$

$$= F\frac{M_{x+k} - M_{x+k+n}}{D_x} \qquad (5)$$

When the term of the insurance continues throughout the life of the insured, $d_{x+k+n} = 0$ and hence $C_{x+k+n} = M_{x+k+n} = 0$. It follows that formula (5) includes formula $(5_1)$. When $k = t - 1$, and $n = 1$, $M_{x+k} - M_{x+k+n} = C_{x+t-1}$, so that formula (5) includes formula $(5_2)$. Formula (5) should be thoroughly mastered.

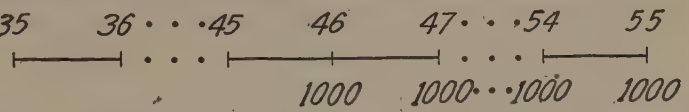
It should be especially noted that in formula (1), $x + t$ is the age when the first rent payment is made, while in formula (5), $x + k$ is the age when the term begins.

If the annual rent is 1, the value at age $x$ of an $n$-year term insurance is denoted by $A^1_{x\overline{n}|}$ when the term begins at age $x$, and by ${}_{n_1|}A^1_{x\overline{n}|}$ when the term begins at age $x + n_1$. From formula (5) it follows that

$$A^1_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}, \quad {}_{n_1|}A^1_{x\overline{n}|} = \frac{M_{x+n_1} - M_{x+n_1+n}}{D_x}$$

### EXERCISES

**1.** Apply formula (5) to write the expressions for the values at age 35, at age 45, and at age 55, of the ten year term life insurance of face value \$1000 whose term begins at age 45 represented by the following diagram:




**2.** Find the expressions for the values at age 35 and at age 55 of the term insurance in Exercise 1 by applying Theorem IV to the expression for its value at age 45.

**3.** Apply formula (5) to write the expressions for the net single premiums at age 25 of the following \$1000 life insurance policies:

(*a*) Ordinary ten year term,

(*b*) Ten year term deferred 10 years.

**4.** Use formula (5) at $i = .035$ to compute the value at age 25 of a \$1000 ordinary ten year term insurance. Ans. \$67.32.

**5.** Find the net single premium at age 25 of a \$1000 ten year term insurance deferred 5 years. Ans. \$57.60.

**6.** Derive formula (5) by use of definition 2, Art. 87.

**7.** Use formula (5) to write the expressions for the values of $A^{1}_{25\,\overline{10}|}$ and ${}_{10|}A^{1}_{25\,\overline{10}|}$.

**8.** Use formula ($5_1$) to derive formula (5) by resolving the $n$-year term insurance whose term begins at age $x + k$ into the difference between two whole life insurances whose terms begin at the ages $x + k$ and $x + k + n$.

**9.** Prove Theorem IV by use of formula (5).

**90. Some important special cases of formula (5).** By means of formula (5) the value at age $x$ of any life insurance of face value $F$ can be written at once. In this article important special cases are listed for the purpose of comparison.

| | Whole Life Insurance | $n$-Year Term Life Insurance |
|---|---|---|
| Term begins at age $x$ | $F\dfrac{M_x}{D_x}$ | $F\dfrac{M_x - M_{x+n}}{D_x}$ |
| Term deferred $n_1$ years | $F\dfrac{M_{x+n_1}}{D_x}$ | $F\dfrac{M_{x+n_1} - M_{x+n_1+n}}{D_x}$ |
| Term forborne $n$ years | $F\dfrac{M_{x+n}}{D_x}$ | $F\dfrac{M_{x-n} - M_x}{D_x}$ |

An $n$-year term life insurance whose term is forborne $n$ years is usually called a *forborne life insurance.* In the application of forborne insurance it is frequently convenient to have $x$ for the age at the beginning of the term rather than at the end. When this is done, the formula listed above for the value of a forborne $n$-year term insurance becomes, upon replacing $x$ by $x + n$,

$$V = F\frac{M_x - M_{x+n}}{D_{x+n}}$$

When $F = 1$, the value of $V$ is denoted by ${}_nk_x$. Hence

$${}_nk_x = \frac{M_x - M_{x+n}}{D_{x+n}} \tag{6}$$

When $n = 1$, ${}_nk_x$ is denoted by $k_x$. From formula (6) it follows that

$$k_x = \frac{C_x}{D_{x+1}}$$

Values of $k_x$ based on the American Experience Table at $3\frac{1}{2}\%$ are given in Table XIII.

EXERCISES

1. Write the expressions for the values of $k_{20}$ and ${}_{10}k_{20}$.
2. Compute the value of $k_{20}$.

**91. Life annuities and life insurances combined; $n$-year endowment insurance.** In the preceding articles in this chapter the values at age $x$ of life annuities and of life insurances have been determined. The value at age $x$ of any combination of a life annuity and of a life insurance may be found by summing the separate values. A common form among such combinations is an *$n$-year endowment insurance,* which combines an $n$-year term insurance with a pure endowment due in $n$ years. If the face value of each of these is $F$, the value at age $x$ of the term insurance is $F\frac{M_x - M_{x+n}}{D_x}$, and that of the pure endowment is $F\frac{D_{x+n}}{D_x}$. It follows that the value, $V$, at age $x$ of an $n$-year endowment insurance of face value, $F$, is given by

$$V = F\frac{M_x - M_{x+n} + D_{x+n}}{D_x} \tag{7}$$

When $F = 1$, the value of $V$ is denoted by $A_{x\overline{n}|}$, so that

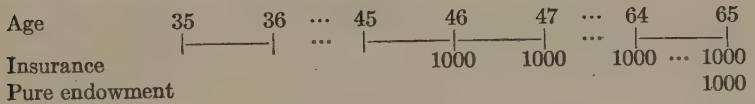
$$A_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

The values of other combinations of life annuities and life insurances can be written at once by use of formulas (1) and (5).

The value of $V$ for an $n$-year endowment insurance policy is called the *net single premium* of the policy. A modified form of an $n$-year endowment insurance policy specifies the age at maturity. For example, an *endowment at age* 65 issued at age 22 is the same as a 43-year endowment insurance.

### EXERCISES

**1.** Apply formulas (5) and $(1_2)$ to write the expressions for the values at age 35, and at age 45, of the \$1000 twenty year endowment insurance whose term begins at age 45, represented by the following diagram:




**2.** Compute the values at $i = .035$ of the expressions found in Exercise 1. Ans. \$360.03; \$560.23.

**3.** Compute the net single premium at age 25 of a \$1000 ten year endowment insurance. Use $i = .035$.

**4.** If the net single premium of a 20-year endowment insurance policy issued at age 30 is \$1000, what is the face value of the policy? Use $i = .035$. *Ans.* \$1855.90.

**5.** Write the expression for the value at age 30 of a \$1000 ordinary 35-year term insurance combined with a \$1000 whole life annuity immediate deferred 35 years. Draw a diagram similar to that in Exercise 1. Compute the value of this expression at $i = .035$. Ans. \$1641.64.

**6.** Find the net single premium at age 27 of a \$1000 endowment at age 60. Ans. \$398.73.

**92. Relations connecting the symbols denoting the values of (1) life insurances, (2) life annuities and life insurances.** The methods used in Art. 83 to find relations among life annuity symbols lead to similar relations among life insurance symbols. For example, the relation among the insurance symbols which corresponds to the formula at the end of Art. 83 can be obtained as

follows: The value at age $x$ of a whole life insurance of face value 1 is denoted by $A_x$. The value of this same insurance at age $x+1$ is $\frac{d_x}{l_{x+1}} + A_{x+1}$. This is seen at once by resolving the insurance into a one year term insurance whose term begins at age $x$ and a whole life insurance whose term begins at age $x+1$; $A_{x+1}$ denotes the value at age $x+1$ of the whole life component and, by formula $(5_2)$, $\frac{d_x}{l_{x+1}}$ is the value at age $x+1$ of the one year term insurance. Hence by Theorem IV

$$\begin{aligned} A_x &= v\frac{l_{x+1}}{l_x}\left(\frac{d_x}{l_{x+1}} + A_{x+1}\right) \\ &= v(q_x + p_x A_{x+1}) \end{aligned}$$

This relation can be used to compute a table of values for $A_x$.

The values of life insurances are based on the $d_x$ column in a mortality table. The values of life annuities are based on the $l_x$ column. By use of the relation $d_x = l_x - l_{x+1}$, it can be readily proved that $C_x = vD_x - D_{x+1}$ and $M_x = vN_x - N_{x+1}$. The proofs are as follows:

$$\begin{aligned} C_x &= v^{x+1}d_x \\ &= v^{x+1}(l_x - l_{x+1}) \\ &= vD_x - D_{x+1} \\ M_x &= C_x + C_{x+1} + \cdots + C_{95} \\ &= (vD_x - D_{x+1}) + (vD_{x+1} - D_{x+2}) + \cdots + (vD_{95} - D_{96}) \\ &= v(D_x + D_{x+1} + \cdots + D_{95}) - (D_{x+1} + D_{x+2} + \cdots + D_{95}) \\ &= vN_x - N_{x+1} \end{aligned}$$

From this value of $M_x$ it follows that the value at age $x$ of any life insurance can be expressed in terms of values of life annuities. For example:

$$\begin{aligned} A_x &= \frac{M_x}{D_x} \\ &= \frac{vN_x - N_{x+1}}{D_x} \\ &= va_x - a_x = v(1 + a_x) - a_x \\ &= 1 - d(1 + a_x) \qquad \text{(where } v = 1 - d\text{)} \\ &= 1 - da_x \end{aligned}$$

### EXERCISES

1. Show that

(a) $A^1_{x\overline{n}|} = va_{x\overline{n}|} - a_{x\overline{n}|}$ (c) $A_x = A^1_{x\overline{n}|} + {}_n|A_x$

(b) $A^1_{x\overline{n}|} = v(q_x + p_x A^1_{x+1:\overline{n-1}|})$ (d) $A_{x+1} = u_x A_x - k_x$

2. Interpret each of the relations in Exercise 1 verbally. Also interpret the relation, $A_x = 1 - d(1 + a_x)$, verbally.

**93. The value of a set of life insurances or a set of life annuities. Increasing and decreasing insurances; increasing and decreasing annuities.** The value at age $x$ of any set of life annuities or of life insurances can be found by summing the values of the separate sets. The values of some sets can be found more easily, however, by basing the computations on simplified forms of the expressions for the sum of the values of the given sets. Illustrations are afforded by sets of $n$ life insurances each of face value $F$, or of $n$ life annuities each of rent $R$, having terms of $n$, $n-1$, $\cdots$, 1 years respectively, all of which either begin at age $x$ or end at age $x+n$. When the terms all begin at the same age $x$, the sum forms a decreasing insurance or a decreasing annuity; when the terms all end at the same age $x+n$, the sum forms an increasing insurance or an increasing annuity. In the increasing insurance the face value increases by $F$ each year; in the decreasing insurance it decreases by $F$ each year. Likewise in the increasing annuity the rent increases by $R$ each year; in the decreasing annuity it decreases by $R$ each year.

The value, $V$, at age $x$ of an increasing insurance whose face value begins with $F$ and increases by $F$ each year, and whose term begins at age $x$ can be found by the following process: An increasing insurance of $F$ whose term begins at age $x$ and ends at age $x+n$ is the sum of $n$ term insurances whose terms begin at the ages $x$, $x+1$, $\cdots$, $x+n-1$ respectively and end at age $x+n$. Hence

$$V = F\left(\frac{M_x - M_{x+n}}{D_x} + \frac{M_{x+1} - M_{x+n}}{D_x} + \cdots + \frac{M_{x+n-1} - M_{x+n}}{D_x}\right)$$

$$= F\frac{M_x + M_{x+1} + \cdots + M_{x+n-1} - nM_{x+n}}{D_x}$$

$$= F\frac{R_x - R_{x+n} - nM_{x+n}}{D_x} \text{ where } R_x = M_x + M_{x+1} + \cdots, \quad (8)$$

When $F = 1$, $V$ is denoted by $(IA^1_{x\overline{n}|})$; in this case formula (8) becomes

$$(IA^1_{x\overline{n}|}) = \frac{R_x - R_{x+n} - nM_{x+n}}{D_x} \tag{8_1}$$

The expression for the value at age $x$ of an increasing annuity whose first payment is at age $x$, can be found in an analogous manner. If the rent begins with $R$ and increases by $R$ each year, the value, $V$, is given by

$$\begin{aligned} V &= R\left(\frac{N_x - N_{x+n}}{D_x} + \frac{N_{x+1} - N_{x+n}}{D_x} + \frac{N_{x+2} - N_{x+n}}{D_x} + \cdots + \frac{N_{x+n-1} - N_{x+n}}{D_x}\right) \\ &= R\frac{N_x + N_{x+1} + \cdots + N_{x+n-1} - nN_{x+n}}{D_x} \\ &= R\frac{S_x - S_{x+n} - nN_{x+n}}{D_x} \end{aligned} \tag{9}$$

where $S_x = N_x + N_{x+1} + N_{x+2} + \cdots$

When $R = 1$, $V$ is denoted by $(Ia_{x\overline{n}|})$; in this case formula (9) becomes

$$(Ia_{x\overline{n}|}) = \frac{S_x - S_{x+n} - nN_{x+n}}{D_x} \tag{9_1}$$

Values of the commutation symbols $R_x$ and $S_x$ are not included in the tables at the end of this book.* Their values can be found by use of the $M$- and $N$-columns.

### EXERCISES

**1.** Compute the value at age 35 of an increasing insurance of \$1000 issued at age 35 for a term of 10 years. Ans. \$411.45.

**2.** Same as Exercise 1 for an increasing annuity. Ans. \$42518.72.

**3.** Show that the value at age $x$ of a whole life increasing insurance of $F$ issued at age $x$ is given by

$$V = F\frac{R_x}{D_x}$$

**4.** Compute the value at age 25 of the insurance in Exercise 1.
Ans. \$268.06.

* The values of $R_x$ and $S_x$ at .03, 035, and .04, are given in Glover's Tables of Applied Mathematics.

**94. Joint life annuities.** In a single life annuity the rent payments continue as long during the term as the single life survives. In a joint life annuity the rent payments continue as long during the term as *all* the lives survive; when any life fails, the payments cease. Joint life annuities are classified in a manner analogous to single life annuities. In the next three articles formulas are developed for the values of joint life annuities involving two lives, which are analogous to those for single lives. The methods, however, are applicable to any number of lives. In Art. 104 an approximate method for computing values of joint life annuities is presented.

**95. The value of a joint life annuity defined.** If each of the $l_{28}$ persons living at age 28, shown in the mortality table, is paired with each of the $l_{32}$ persons living at age 32, there are $l_{28}l_{32}$ such pairs. Of these pairs $l_{29}l_{33}$ survive one year, $l_{30}l_{34}$ survive two years, and so on. If each of the $l_{28}l_{32}$ pairs of persons of ages 28, 32 holds a ten year temporary joint life annuity policy of annual rent 1, the amounts that will be received on these policies at the end of 1, 2, ···, 10 years are $l_{29}l_{33}$, $l_{30}l_{34}$, $\cdots$ $l_{38}l_{42}$ respectively. The sum of these amounts, valued as of the ages 28, 32 at an annual interest rate $i$, is $vl_{29}l_{33} + v^2l_{30}l_{34} + \cdots + v^{10}l_{38}l_{42}$ where $v = \dfrac{1}{(1+i)}$. The quotient

$$V = \frac{vl_{29}l_{33} + v^2l_{30}l_{34} + \cdots + v^{10}l_{38}l_{42}}{l_{28}l_{32}}$$

gives what is called the value at the ages 28, 32, at an interest rate $i$ of this temporary joint life annuity as determined by the mortality table. The sum of the amounts due the pairs of annuitants valued as of ages 38, 42 at an interest rate $i$, is $(1+i)^9l_{29}l_{33} + (1+i)^8l_{30}l_{34} + \cdots + l_{38}l_{42}$. The quotient

$$V = \frac{(1+i)^9l_{29}l_{33} + (1+i)^8l_{30}l_{34} + \cdots + l_{38}l_{42}}{l_{38}l_{42}}$$

gives what is called the value at the ages 38, 42, at an interest rate $i$ of this temporary joint life annuity as determined by the mortality table. In other words, the value at ages 28, 32 of this

ten year temporary joint life annuity is the amount that each of the $l_{28}l_{32}$ pairs of annuitants must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay 1 to each pair of annuitants at the end of each year of the term of the annuity. Likewise, the value at ages 38, 42 of this joint life annuity is the amount that each of the $l_{38}l_{42}$ pairs of persons would receive if the rent payments were not drawn when they are due but were allowed to accumulate at compound interest to the time at the ages 38, 42, and if the fund thus created were then equally divided among the $l_{38}l_{42}$ pairs of survivors. The values at the pairs of ages 28, 32 and 38, 42 of this temporary joint life annuity as just defined are included in the

**Definition 1.** *The value at the ages $x$, $y$ of any joint life annuity of annual rent, $R$, is the quotient obtained by dividing $l_x l_y$ into the sum of the values obtained by applying the compound interest formula to each sum of the set consisting of $R$ times the number of pairs living, according to the mortality table, at the time of each rent payment during the term of the annuity.*

From this definition the value at the ages $x$, $y$ of an ordinary or of a deferred joint life annuity of annual rent $R$ may be viewed as the amount that each of the $l_x l_y$ pairs of annuitants must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay $R$ to each pair of annuitants living at the time of each rent payment during the term of the annuity. Likewise, the value at the ages $x$, $y$ of a forborne joint life annuity whose term begins $n$ or more years prior to the ages $x$, $y$, where $n$ is the number of years in the term of the annuity, may be viewed as the amount that each of the $l_x l_y$ pairs of survivors would receive if the rent payments were not drawn when they are due but were allowed to accumulate at compound interest to the time at the ages $x$, $y$ and if the fund thus created were then equally divided among the $l_x l_y$ pairs of survivors.

When the joint life annuity in definition 1 consists of a single sum payable at the ages $x + t$, $y + t$, the definition can be stated in the form:

*The value, $V$, at the ages $x$, $y$ of a one year temporary joint life annuity with annual rent $R$ payable at the ages $x + t$, $y + t$ is given by*

$$V = Rv^t \frac{l_{x+t}l_{y+t}}{l_x l_y} = R\frac{D_{x+t:y+t}}{D_{xy}} \qquad (10_2)$$

$$= Rv^t \cdot {}_tp_{xy} \qquad \text{(when } t > 0\text{)}$$

*where* $D_{xy} = v^{\frac{x+y}{2}} l_x l_y$ *so that* $D_{x+t:y+t} = v^{\frac{x+y}{2}+t} l_{x+t}l_{y+t}$. The second form of formula $(10_2)$ is obtained from the first by multiplying numerator and denominator by $v^{\frac{x+y}{2}}$. When $t$ is positive, $V$ is the present or discounted value at the ages $x$, $y$ of a sum $R$ to be paid at the end of $t$ years in case two persons aged $x$ and $y$ survive $t$ years; that is, when $t$ is positive, $V$ is the value at the ages $x$, $y$ of a joint life pure endowment due in $t$ years. When $t$ is negative, $V$ is the amount at the end of $t$ years of a sum $R$ paid at the ages $x + t$, $y + t$ which is allowed to accumulate for $t$ years, or to the ages $x$, $y$, as a joint life pure endowment. The value at the ages $x$, $y$ of a joint life pure endowment of 1 due in $t$ years is denoted by ${}_tE_{xy}$. It follows that

$${}_tE_{xy} = v^t \frac{l_{x+t}l_{y+t}}{l_x l_y} = v^t \cdot {}_tp_{xy} = \frac{D_{x+t:y+t}}{D_{xy}} \qquad (10_3)$$

In finding the values at two or more pairs of ages of a joint life annuity use can be made of the following

**Theorem VI.** If $V_{x+t:y+t}$ *denotes the value at the ages* $x + t$, $y + t$ *of a joint life annuity, its value* $V_{xy}$ *at the ages* $x$, $y$ *is given by*

$$V_{xy} = v^t \frac{l_{x+t}l_{y+t}}{l_x l_y} V_{x+t:y+t} = \frac{D_{x+t:y+t}}{D_{xy}} V_{x+t:y+t}$$

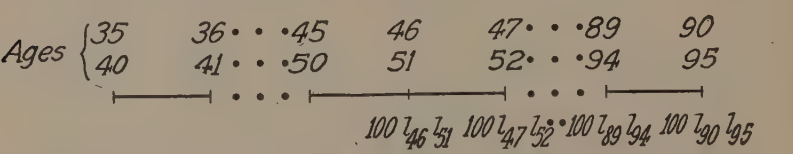
This theorem is an extension of Theorem III. The proof is left as an exercise. When $t$ is positive, $\frac{D_{x+t:y+t}}{D_{xy}}$ is a *discount factor;* when $t$ is negative, it is an *accumulation factor.* It should be noted that Theorem VI includes formula $(10_2)$.

The value at the ages $x$, $y$ of the joint life annuity specified by any policy is called the *net single premium* of the policy.

In the following articles $l_x l_y$, $l_{x+1}l_{y+1}$, and so on will often be written $l_{xy}$, $l_{x+1:y+1}$, and so on for brevity.

### EXERCISES

**1.** Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 46, 51, of a joint whole life annuity of annual rent 100 with first payment at the ages 46, 51:




**2.** Compute the value at the ages 25, 30 of a joint life pure endowment of \$1000 due in 25 years. Use $i = .035$. Ans. \$250.69.

**3.** Use definition 1 at $i = .035$ to compute the value at the ages 90, 92 of an ordinary joint whole life annuity immediate of annual rent \$1000.

Ans. \$217.06.

**96. Another definition of a joint life annuity.** The expressions given in Art. 95 for the values at the pairs of ages 28, 32 and 38, 42 of a ten year temporary joint life annuity immediate whose term begins at the ages 28, 32 can be written in the forms:

$$V = v\frac{l_{29}l_{33}}{l_{28}l_{32}} + v^2\frac{l_{30}l_{34}}{l_{28}l_{32}} + \cdots + v^{10}\frac{l_{38}l_{42}}{l_{28}l_{42}} \quad \text{(at ages 28, 32) and}$$

$$V = v^{-9}\frac{l_{29}l_{33}}{l_{38}l_{42}} + v^{-8}\frac{l_{30}l_{34}}{l_{38}l_{42}} + \cdots + \frac{l_{38}l_{42}}{l_{38}l_{42}} \quad \text{(at ages 38, 42)}$$

These forms show that the value of this annuity at either pair of ages is the sum of the values obtained by applying formula $(10_2)$ to each sum of the set consisting of 1 at the time of each rent payment during the term of the annuity. The value of the temporary joint life annuity defined in this way is included in the

**Definition 2.** *The value at the ages $x$, $y$ of any joint life annuity of annual rent $R$ is the sum of the values obtained by applying the formula*

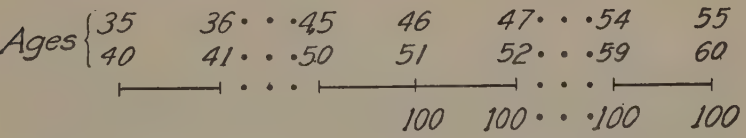
$$V = Rv^t\frac{l_{x+t}l_{y+t}}{l_x l_y} = R\frac{D_{x+t:y+t}}{D_{xy}}$$

*to each sum of the set consisting of $R$ at the time of each rent payment during the term of the annuity.*

From this definition and the value of an expectation (Art. 76), the value at the ages $x$, $y$ of an ordinary or of a deferred joint life annuity of annual rent $R$ is the sum of the values at the ages $x$, $y$ of the expectations of a pair of annuitants of receiving the rent payments during the term of the annuity; or it is the sum of the discounted values at the ages $x$, $y$, obtained by use of formula $(10_2)$ when $t$ is zero or positive of the set of sums consisting of $R$ at the time of each rent payment during the term of the annuity. Likewise, the value at the ages $x$, $y$ of a forborne joint life annuity, whose term begins $n$ or more years prior to the time at the ages $x$, $y$, where $n$ is the number of years in the term of the annuity, is the sum of the amounts at the ages $x$, $y$ of the same set of sums when each is accumulated as a joint life pure endowment, that is, by use of formula $(10_2)$ when $t$ is zero or negative.

### EXERCISE

Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 55, 60, of a 10 year temporary joint life annuity of annual rent 100 with first payment at the ages 46, 51:




**97. The value at the ages $x$, $y$ of any joint whole life annuity of annual rent $R$.** Let $x + t$, $y + t$ denote the ages when the first rent payment is made. By definition 1, Art. 95, the value, $V$, at the ages $x$, $y$ is given by

$$V = R\frac{v^t l_{x+t:y+t} + v^{t+1} l_{x+t+1:y+t+1} + \text{etc.}}{l_{xy}}$$

$$= R\frac{v^{\frac{x+y}{2}+t} l_{x+t:y+t} + v^{\frac{x+y}{2}+t+1} l_{x+t+1:y+t+1} + \text{etc.}}{v^{\frac{x+y}{2}} l_{xy}}$$

$$= R\frac{D_{x+t:y+t} + D_{x+t+1:y+t+1} + \text{etc.}}{D_{xy}}$$

$$= R\frac{N_{x+t:y+t}}{D_{xy}} \qquad (10_1)$$

where the commutation symbol $N_{xy}$ is defined by

$$N_{xy} = D_{xy} + D_{x+1:y+1} + \text{etc.},$$

so that $$N_{x+t:y+t} = D_{x+t:y+t} + D_{x+t+1:y+t+1} + \text{etc.}$$

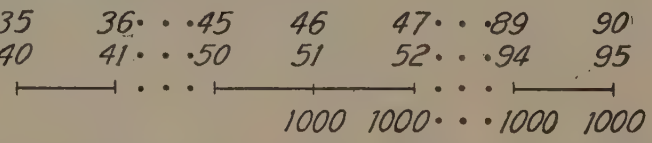
The symbols for the values of the joint whole life annuities of annual rent 1 are analogous to those for single lives given in Art. 80. From formula $(10_1)$ it follows that

$$a_{xy} = \frac{N_{x+1:y+1}}{D_{xy}}, \qquad \mathbf{a}_{xy} = \frac{N_{xy}}{D_{xy}}$$

$${}_{n_1}|\,a_{xy} = \frac{N_{x+1+n_1:y+1+n_1}}{D_{xy}} \qquad {}_{n_1}|\,\mathbf{a}_{xy} = \frac{N_{x+n_1:y+n_1}}{D_{xy}}$$

### EXERCISES

**1.** Apply formula $(10_1)$ to write the expressions for the values at the ages 35, 40, and at the ages 46, 51, of the joint whole life annuity of annual rent 1000 represented by the following diagram:




**2.** Find the expression for the value at the ages 35, 40 of the joint whole life annuity in Exercise 1 by applying Theorem VI to its value at the ages 46, 51.

**3.** Apply formula $(10_1)$ to write the expressions for the net single premiums at the ages 25, 35 of the following joint life annuity policies each having an annual rent of \$1000:

(*a*) Ordinary joint whole life annuity immediate,

(*b*) Ordinary joint whole life annuity due,

(*c*) Joint whole life annuity immediate deferred 10 years,

(*d*) Joint whole life annuity due deferred 10 years.

**4.** Use formula $(10_1)$ to write the expressions for $a_{25:35}$, $\mathbf{a}_{25:35}$, ${}_{10}|\,a_{25:35}$ and ${}_{10}|\,\mathbf{a}_{25:35}$.

**5.** Extend formula $(10_1)$ to three lives.

**98. The value at the ages $x$, $y$ of any $n$-year temporary joint life annuity.** Let $x + t$, $y + t$ denote the ages when the first rent payment is made. By definition 1, Art. 95, the value, $V$, at the ages $x$, $y$ is given by

$$V = R\frac{v^t l_{x+t:y+t} + v^{t+1} l_{x+t+1:y+t+1} + \cdots + v^{t+n-1} l_{x+t+n-1:y+t+n-1}}{l_{xy}}$$

$$= R\frac{v^{\frac{x+y}{2}+t} l_{x+t:y+t} + v^{\frac{x+y}{2}+t+1} l_{x+t+1:y+t+1} + \cdots + v^{\frac{x+y}{2}+n-1} l_{x+t+n-1:y+t+n-1}}{v^{\frac{x+y}{2}} l_{xy}}$$

$$= R\frac{D_{x+t:y+t} + D_{x+t+1:y+t+1} + \cdots + D_{x+t+n-1:y+t+n-1}}{D_{xy}}$$

$$= R\frac{N_{x+t:y+t} - N_{x+t+n:y+t+n}}{D_{xy}} \qquad (10)$$

The symbols for the values of temporary joint life annuities of annual rent 1 are analogous to those for single lives given in Art. 81. From formula (10), it follows that

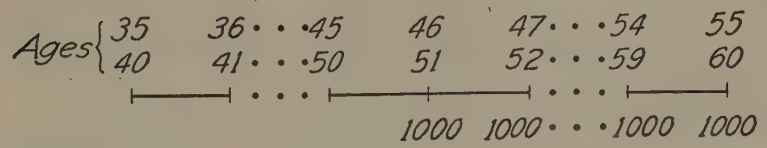
$$a_{xy\overline{n}|} = \frac{N_{x+1:y+1} - N_{x+1+n:y+1+n}}{D_{xy}}, \quad \mathbf{a}_{xy\overline{n}|} = \frac{N_{x+y} - N_{x+n:y+n}}{D_{xy}}$$

$${}_{n_1}|\, a_{xy\overline{n}|} = \frac{N_{x+1+n_1:y+1+n_1} - N_{x+1+n_1+n:y+1+n_1+n}}{D_{xy}},$$

$${}_{n_1}|\, \mathbf{a}_{xy\overline{n}|} = \frac{N_{x+n_1:y+n_1} - N_{x+n_1+n:y+n_1+n}}{D_{xy}}$$

## EXERCISES

**1.** Apply formula (10) to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 55, 60, of the 10-year temporary joint life annuity of annual rent \$1000 represented by the following diagram:




**2.** Apply formula (10) to write the expressions for the net single premiums at the ages 25, 35 of the following joint life annuity policies each having an annual rent of \$1000:

(*a*) Ordinary 20-year temporary joint life annuity due,

(*b*) Ten year temporary joint life annuity immediate deferred 5 years.

**3.** Use formula (10) to write the expressions for $a_{35:40\,\overline{10}|}$, $\mathbf{a}_{35:40\,\overline{10}|}$, ${}_{10}|\, a_{35:40\,\overline{10}|}$, and ${}_{10}|\, \mathbf{a}_{35:40\,\overline{10}|}$.

**4.** Show that formula (10) includes formulas ($10_1$) and ($10_2$).

5. Show that the value of a forborne $n$-year temporary joint life annuity due is given by

$$V = R\frac{N_{xy} - N_{x+n:y+n}}{D_{x+n:y+n}}$$

where $x$, $y$ are the ages at the beginning of the term of the annuity. (See formula (2), Art. 82.)

**99. Joint life insurance.** In a single life insurance the benefit is paid when the insured dies, providing death takes place during the term of the insurance. In a joint life insurance the benefit is paid when any one of the insured dies, providing death takes place during the term of the insurance. The discussion of joint life insurances is entirely analogous to that of joint life annuities.

**100. The value of joint life insurance defined.** If each of the $l_{28}$ persons living at age 28 shown in the mortality table is paired with each of the $l_{32}$ persons living at age 32, there are $l_{28}l_{32}$ such pairs. Of these pairs $l_{29}l_{33}$ survive one year, $l_{30}l_{34}$ survive two years, and so on. It follows that $l_{28}l_{32}-l_{29}l_{33}$ pairs fail during the first year, $l_{29}l_{33}-l_{30}l_{34}$ fail during the second, and so on. If each of the $l_{28}l_{32}$ pairs of persons of ages 28, 32 holds a ten year term joint life insurance policy of face value 1, the amounts that will be received on these policies at the end of 1, 2, $\cdots$, 10 years are $l_{28}l_{32}-l_{29}l_{33}$, $l_{29}l_{33}-l_{30}l_{34}$, $\cdots$, $l_{37}l_{41}-l_{38}l_{42}$ respectively. The sum of these amounts, valued as of the ages 28, 32 at an annual interest rate $i$, is $v(l_{28}l_{32}-l_{29}l_{33}) + v^2(l_{29}l_{33}-l_{30}l_{34}) + \cdots + v^{10}(l_{37}l_{41}-l_{38}l_{42})$ where $v = \dfrac{1}{(1+i)}$. The quotient

$$V = \frac{v(l_{28}l_{32}-l_{29}l_{33})+v^2(l_{29}l_{33}-l_{30}l_{34})+ \cdots +v^{10}(l_{37}l_{41}-l_{38}l_{42})}{l_{28}l_{32}}$$

gives what is called the value at the ages 28, 32, at an interest rate $i$, of this term joint life insurance as determined by the mortality table. The sum of the amounts due the pairs of beneficiaries valued as of the ages 38, 42 at an interest rate $i$ is $(1+i)^9(l_{28}l_{32}-l_{29}l_{33})+ \cdots + (l_{37}l_{41}-l_{38}l_{42})$. The quotient

$$V = \frac{(1+i)^9(l_{28}l_{32}-l_{29}l_{33})+(1+i)^8(l_{29}l_{33}-l_{30}l_{34})+ \cdots +(l_{37}l_{41}-l_{38}l_{42})}{l_{38}l_{42}}$$

gives what is called the value at the ages 38, 42, at an interest rate $i$, of this term joint life insurance as determined by the mortality

table. In other words, the value at ages 28, 32 of this ten year term joint life insurance is the amount that each of the $l_{28}l_{32}$ pairs of insured must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay 1 to each pair of beneficiaries at the end of each year of the term of the insurance. Likewise, the value at ages 38, 42 of this joint life insurance is the amount that each of the $l_{38}l_{42}$ pairs of persons would receive if the benefits were not drawn when they are due but were allowed to accumulate at compound interest to the time at the ages 38, 42 and if the fund thus created were then equally divided among the $l_{38}l_{42}$ survivors. The values at the pairs of ages 28, 32, and 38, 42 of this term joint life insurance as just defined are included in the

**Definition 1.** *The value at the ages $x$, $y$ of the insured of any joint life insurance of face value, $F$, is the quotient obtained by dividing $l_xl_y$ into the sum of the values obtained by applying the compound interest formula to each sum of the set consisting of $F$ times the number of pairs that fail, according to the mortality table, during each year of the insurance term.*

From this definition the value at the ages $x$, $y$ of an ordinary or of a deferred joint life insurance of face value $F$ may be viewed as the amount that each of the $l_xl_y$ pairs of insured must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay $F$ to each pair of beneficiaries at the end of each year during the term of the insurance. Likewise the value at the ages $x$, $y$ of a forborne joint life insurance whose term begins $n$ or more years prior to the ages $x$, $y$, where $n$ is the number of years in the term of the insurance, may be viewed as the amount that each of the $l_xl_y$ pairs of survivors would receive if the benefits were not drawn when they are due but were allowed to accumulate at compound interest to the time at the ages $x$, $y$ and if the fund thus created were then equally divided among the $l_xl_y$ pairs of survivors.

When the joint life insurance in definition 1 is for a one year term beginning at the ages $x + t - 1$, $y + t - 1$, the definition can be stated in the form:

*The value, $V$, at the ages $x$, $y$ of a joint life insurance of face value, $F$, for the year beginning at the ages $x + t - 1$, $y + t - 1$ is given by*

$$V = Fv^t \cdot \frac{l_{x+t-1}l_{y+t-1} - l_{x+t}l_{y+t}}{l_x l_y} = F\frac{C_{x+t-1:y+t-1}}{D_{xy}} \qquad (11_2)$$

$$= Fv^t \cdot {}_{t-1}|\, q_{xy} \qquad \text{(when } t > 0\text{)}$$

*where* $$C_{xy} = v^{\frac{x+y}{2}+1}(l_x l_y - l_{x+1}l_{y+1}),$$

*so that* $$C_{x+t-1:y+t-1} = v^{\frac{x+y}{2}+t}(l_{x+t-1}l_{y+t-1} - l_{x+t}l_{y+t}).$$

The second form of formula $(11_2)$ is obtained from the first by multiplying numerator and denominator by $v^{\frac{x+y}{2}}$. When $t$ is positive, $V$ is the present or discounted value at the ages $x$, $y$ of a benefit, $F$, to be paid at the end of $t$ years in case two persons, aged $x$ and $y$, both survive $t - 1$ years but both do not survive $t$ years. When $t$ is negative, $V$ is the amount at the ages $x$, $y$ of the benefit, $F$, paid at the ages $x + t$, $y + t$, at the end of a one year term joint life insurance, which is allowed to accumulate for $t$ years in accordance with formula $(11_2)$.

In finding the values at two or more pairs of ages of a joint life insurance use can be made of the following

**Theorem VII.** *If $V_{x+t:y+t}$ denotes the value at the ages $x + t$, $y + t$ of a joint life insurance, its value $V_{xy}$ at the ages $x$, $y$ is given by*

$$V_{xy} = v^t \frac{l_{x+t}l_{y+t}}{l_x l_y} V_{x+t:y+t} = \frac{D_{x+t:y+t}}{D_{xy}} V_{x+t:y+t}.$$

This theorem is an extension of Theorem V. The proof is left as an exercise. When $t$ is positive, $\frac{D_{x+t:y+t}}{D_{xy}}$ is a *discount factor;* when $t$ is negative, it is an *accumulation factor.* It should be noted that Theorem VII includes formula $(11_2)$.

In writing the equations needed for finding the unknowns in problems involving life annuities or life insurances, use can be made of

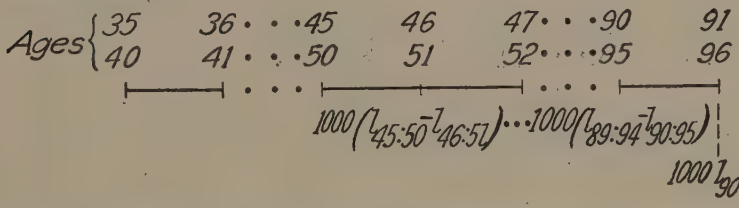
**Theorem VIII.** *If two joint life annuities, two joint life insurances, or a joint life annuity and a joint life insurance have equal values at the ages $x + t$, $y + t$, they have equal values at the ages $x$, $y$.*

This theorem is seen to be true by noting that Theorems VI and VII have the same discount or accumulation factor, $\frac{D_{x+t:y+t}}{D_{xy}}$.

The value at the ages $x$, $y$ of the joint life insurance specified by any policy is called the net single premium of the policy.

### EXERCISES

**1.** Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at the ages 35, 40, and at the ages 45, 50 of a joint whole life insurance of face value \$1000 whose term begins at the ages 45, 50:




**2.** Use definition 1 and $i = .035$ to compute the value at the ages 90, 92 of an ordinary joint whole life insurance of face value \$1000. Ans. \$958.84.

**3.** A one year term joint life insurance of \$1000 begins at the ages 40, 50. By use of formula $(11_2)$ find its value at the ages 60, 70 and at the ages 20, 30. Ans. \$109.99; \$7.89.

**101. Another definition of a value of a joint life insurance.** The expressions given in Art. 100 for the values at the pairs of ages 28, 32 and 38, 42 of a ten year term joint life insurance whose term begins at the ages 28, 32 can be written in the forms:

$$V = v\frac{l_{28}l_{32}-l_{29}l_{33}}{l_{28}l_{32}} + v^2\frac{l_{29}l_{33}-l_{30}l_{34}}{l_{28}l_{32}} + \cdots + v^{10}\frac{l_{37}l_{41}-l_{38}l_{42}}{l_{28}l_{32}} \text{ (at ages 28, 32)}$$

$$V = v^{-9}\frac{l_{28}l_{32}-l_{29}l_{33}}{l_{38}l_{42}} + v^{-8}\frac{l_{29}l_{33}-l_{30}l_{34}}{l_{38}l_{42}} + \cdots + \frac{l_{37}l_{41}-l_{38}l_{42}}{l_{38}l_{42}} \text{ (at ages 38, 42)}$$

These forms show that the value of this insurance at either pair of ages is the sum of the values obtained by applying formula $(11_2)$ to each sum of the set consisting of 1 payable at the end of each year of the term of the insurance. The value of the term life insurance defined in this way is included in the

**Definition 2.** *The value at the ages $x$, $y$ of any joint life insurance of face value, $F$, is the sum of the values obtained by applying the formula*

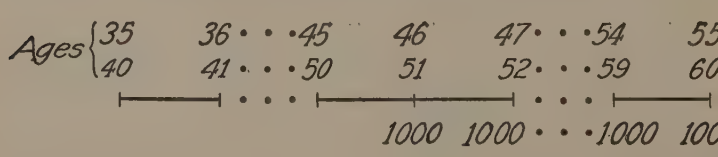
$$V = Fv^t \frac{l_{x+t-1}l_{y+t-1} - l_{x+t}l_{y+t}}{l_x l_y} = F\frac{C_{x+t-1:y+t-1}}{D_{xy}}$$

*to each sum of the set consisting of $F$ payable at the end of each year of the term of the insurance.*

From this definition and the value of an expectation (Art. 76), the value at the ages $x$, $y$ of an ordinary or of a deferred joint life insurance of face value, $F$, is the sum of the values at the ages $x$, $y$ of the insured, of the expectations of a pair of beneficiaries of receiving $F$, at the end of each year during the term of the insurance; or, it is the sum of the discounted values at the ages $x$, $y$ obtained by use of formula $(11_2)$ when $t$ is positive, of the set of sums consisting of $F$ at the end of each year of the term of the insurance. Likewise, the value at the ages $x$, $y$ of a forborne joint life insurance whose term begins $n$ or more years prior to the time at the ages $x$, $y$, where $n$ is the number of years in the term of the insurance, is the sum of the amounts at the ages $x$, $y$ of the same set of sums when each is accumulated by use of formula $(11_2)$ when $t$ is zero or negative.

EXERCISE

Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 55, 60, of a 10-year term joint life insurance of face value \$1000 whose term begins at the ages 45, 50.




**102. The value at the ages $x$, $y$ of any joint whole life insurance of face value $F$.** Let $x + k$, $y + k$ denote the ages when the term of the insurance begins. By definition 1, Art. 100, the value, $V$, at the ages $x$, $y$ is given by

$$V = F\frac{v^{k+1}(l_{x+k:y+k} - l_{x+k+1:y+k+1}) + \dfrac{v^{k+2}(l_{x+k+1:y+k+1} - l_{x+k+2:y+k+2}) + \text{etc.}}{l_{xy}}}{}$$

$$= F\frac{v^{\frac{x+y}{2}+k+1}(l_{x+k:y+k} - l_{x+k+1:y+k+1}) + \dfrac{v^{\frac{x+y}{2}+k+2}(l_{x+k+1:y+k+1} - l_{x+k+2:y+k+2}) + \text{etc.}}{v^{\frac{x+y}{2}} l_{xy}}}{}$$

$$= F\frac{C_{x+k:y+k} + C_{x+k+1:y+k+1} + \text{etc.}}{D_{xy}}$$

$$= F\frac{M_{x+k:y+k}}{D_{xy}} \qquad (11_1)$$

where the commutation symbol $M_{xy}$ is defined by

$$M_{xy} = C_{xy} + C_{x+1:y+1} + \text{etc.},$$

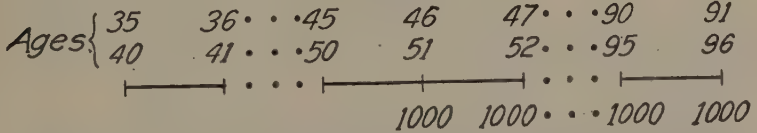
so that $$M_{x+k:y+k} = C_{x+k:y+k} + C_{x+k+1:y+k+1} + \text{etc.}$$

The symbols for the values of joint whole life insurances of face value 1 are analogous to those for single lives given in Art. 88. From formula $(11_1)$ it follows that

$$A_{xy} = \frac{M_{xy}}{D_{xy}}, \qquad {}_{n_1}|A_{xy} = \frac{M_{x+n_1:y+n_1}}{D_{xy}}$$

## EXERCISES

**1.** Apply formula $(11_1)$ to write the expressions for the values at the ages 35, 40, and at the ages 45, 50, of the joint whole life insurance of face value \$1000 represented by the following diagram:




**2.** Find the expression for the value at the ages 35, 40, of the joint whole life insurance in Exercise 1 by applying Theorem VII to its value at the ages 45, 50.

**3.** Apply formula $(11_1)$ to write the expressions for the net single premiums at the ages 25, 35, of the following joint life policies each having a face value of \$1000:

(*a*) Ordinary joint whole life insurance,

(*b*) Joint whole life insurance deferred 10 years.

**4.** Use formula $(11_1)$ to write the expressions for

$$A_{25:35} \quad \text{and} \quad {}_{10}|A_{25:35}.$$

**5.** Extend formula $(11_2)$ to three lives.

**103. The value at the ages $x$, $y$ of any joint $n$-year term insurance of face value $F$.** Let $x + k$, $y + k$ denote the ages when the term of the insurance begins. By definition 1, Art. 100, the value, $V$, at the ages $x$, $y$ is given by

$$V = F\frac{v^{k+1}(l_{x+k:y+k} - l_{x+k+1:y+k+1}) + \cdots + v^{k+n}(l_{x+k+n-1:y+k+n-1} - l_{x+k+n:y+k+n})}{l_{xy}}$$

$$= F\frac{v^{\frac{x+y}{2}+k+1}(l_{x+k:y+k} - l_{x+k+1:y+k+1}) + \cdots + v^{\frac{x+y}{2}+k+n}(l_{x+k+n-1:y+k+n-1} - l_{x+k+n:y+k+n})}{v^{\frac{x+y}{2}} l_{xy}}$$

$$= F\frac{C_{x+k:y+k} + \cdots + C_{x+k+n-1:y+k+n-1}}{D_{xy}}$$

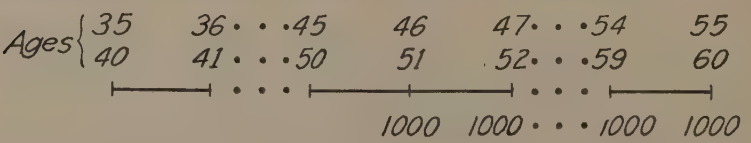
$$= F\frac{M_{x+k:y+k} - M_{x+k+n:y+k+n}}{D_{xy}} \qquad (11)$$

The symbols for the values of joint life term insurances of face value 1 are analogous to those for single lives given in Art. 89. From formula (11) it follows that

$$A_{xy\overline{n}|} = \frac{M_{xy} - M_{x+n:y+n}}{D_{xy}}, \qquad {}_{n_1|}A_{xy\overline{n}|} = \frac{M_{x+n_1:y+n_1} - M_{x+n_1+n:y+n_1+n}}{D_{xy}}$$

## EXERCISES

**1.** Apply formula (11) to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 55, 60, of the 10 year term joint life insurance of face value \$1000, represented by the following diagram:




**2.** Apply formula (11) to write the expressions for the net single premiums at the ages 25, 35, of the following joint life insurance policies each having a face value of \$1000:

(*a*) Ordinary 20 year term life insurance.

(*b*) Ten year term joint life insurance deferred 5 years.

**3.** Use formula (11) to write the expressions for

$$A_{35:40\ \overline{10}|} \text{ and } {}_{10|}A_{35:40\ \overline{10}|}.$$

**4.** Show that formula (11) includes formula ($11_1$) and formula ($11_2$).

**5.** Show that the value of a forborne $n$-year term joint life insurance is given by

$$V = F\frac{M_{xy} - M_{x+n:y+n}}{D_{x+n:y+n}}$$

where $x$, $y$ are the ages at the beginning of the term of the insurance. (See formula 6, Art. 90.)

**6.** Use formulas (11) and ($10_3$) to write the expressions for the value at the ages 25, 30 of a 20-year joint life endowment insurance.

**104. Computation of the values of joint life annuities and of joint life insurances.** A table giving the values at a given interest rate of any one of the commutation symbols $D_x$, $N_x$, $C_x$, $M_x$, $R_x$, $S_x$ for each of the 86 ages in the American Experience Table requires only 86 entries. Tables giving the values of any one of the corresponding symbols for two, three, four, or more lives would require large numbers of entries. Such tables would be impractical to prepare. There are two practical methods in common use for computing the value of any joint life annuity or insurance. One of these is based on replacing lives of unequal ages by an equal number of lives of equivalent equal ages; the other is based on replacing lives of unequal ages by a single life in which case an appropriate change in the interest rate is needed.* An important property of the method of equal ages may be stated in the form (using two ages): *If $w$, $w$ are the equal ages which are equivalent to the ages $x$, $y$ then $w + t$, $w + t$ are the equal ages which are equivalent to the ages $x + t$, $y + t$.* This property is called the *Law of Uniform Seniority*. From this property it follows that formulas (10) and (11) may be written in the forms:

$$V = R\frac{N_{w+t:w+t} - N_{w+t+n:w+t+n}}{D_{ww}} \qquad (10')$$

$$V = F\frac{M_{w+k:w+k} - M_{w+k+n:w+k+n}}{D_{ww}} \qquad (11')$$

* For brief expositions of these two methods reference may be made to a paper by H. L. Rietz in Volume XXVIII of the American Mathematical Monthly, page 158 (1921), or to Mathematical Theory of Life Insurance by C. H. Forsyth — John Wiley and Son (1924).

In this article some examples are solved to show how values of joint life annuities and insurances can be computed by the use of tables giving the values of joint life commutations symbols for equal ages and of $\mu_x$, the " force of mortality " at the age $x$. By $\mu_x$ is meant the instantaneous yearly death rate at the age $x$. In the yearly death rate, $q_x = \frac{d_x}{l_x}$, $d_x$ is the actual number of deaths among the $l_x$ persons during the year following the age $x$; in the force of mortality the denominator is $l_x$ but the numerator is the number of deaths that would take place among the $l_x$ persons during the year following the age $x$, if the rate throughout the year remained the same as the instantaneous rate at age $x$.

In Table XIV are given Hunter's Makehamized American Experience Mortality Table, values of the force of mortality, and values at $3\frac{1}{2}\%$ of the commutation symbols for two equal ages. In the table below are given for certain ages the values at 5% of the commutation symbols for a single life according to the American Experience Table and for two lives of equal ages according to Hunter's Makehamized American Experience Table:

| Age $x$ | $D_x$ | $N_x$ | $M_x$ | $\mu_x$ | $D_{xx}$ | $N_{xx}$ | $M_{xx}$ |
|---|---|---|---|---|---|---|---|
| 25 | 26291.40 | 435657.09 | 5545.82 | .00804 | 23396.19 | 339371.57 | 72356.38 |
| 26 | 24837.49 | 409365.69 | 5343.89 | .00809 | 21925.55 | 315975.38 | 68791.01 |
| 27 | 23462.44 | 384528.20 | 5151.57 | .00814 | 20545.12 | 294049.84 | 65427.49 |
| 28 | 22162.03 | 361065.76 | 4968.42 | .00821 | 19249.49 | 273504.71 | 62254.58 |
| 29 | 20932.25 | 338903.73 | 4793.98 | .00827 | 18033.56 | 254255.22 | 59261.65 |
| 30 | 19769.12 | 317971.48 | 4627.62 | .00835 | 16891.33 | 236221.66 | 56426.79 |
| 31 | 18669.07 | 298202.35 | 4468.93 | .00843 | 15819.24 | 219330.33 | 53749.39 |
| 32 | 17628.76 | 279533.28 | 4317.65 | .00853 | 14812.74 | 203511.09 | 51217.36 |
| 33 | 16644.79 | 261904.52 | 4173.14 | .00863 | 13867.27 | 188698.35 | 48816.35 |
| 34 | 15713.98 | 245259.74 | 4034.94 | .00875 | 12979.59 | 174831.08 | 46543.01 |
| 35 | 14833.53 | 229545.76 | 3902.78 | .00888 | 12145.40 | 161851.49 | 44381.88 |
| 36 | 14000.79 | 214712.22 | 3776.40 | .00902 | 11361.92 | 149706.09 | 42330.61 |
| . | . | . | . | . | . | . | . |
| 47 | 7318.515 | 97755.471 | 2663.49 | .01215 | 5297.803 | 58734.708 | 25009.12 |
| 48 | 6886.371 | 90436.956 | 2579.85 | .01265 | 4922.039 | 53436.905 | 23774.25 |
| 49 | 6476.407 | 83550.585 | 2497.81 | .01321 | 4567.884 | 48514.866 | 22576.52 |
| 50 | 6087.169 | 77074.178 | 2416.97 | .01384 | 4234.369 | 43946.982 | 21416.56 |
| 51 | 5717.409 | 70987.009 | 2337.07 | .01453 | 3919.986 | 39712.613 | 20289.09 |
| 52 | 5365.975 | 65269.600 | 2257.90 | .01531 | 3623.609 | 35792.627 | 19191.98 |
| 53 | 5031.809 | 59903.625 | 2179.25 | .01617 | 3344.111 | 32169.018 | 18122.53 |
| 54 | 4713.927 | 54871.816 | 2100.98 | .01712 | 3080.668 | 28824.907 | 17080.54 |

The value of $w$ for a given set of $n$ ages $x$, $y$, $\cdots$ is found as follows:

(1) Find $\mu_w$ by means of the relation $\mu_w = \dfrac{\mu_x + \mu_y + \text{etc.}}{n}$;

(2) Find $w$ by simple interpolation in the table for $x$ and $\mu_x$.

EXAMPLE 1. Find the equal ages equivalent to the ages 25, 27, 34.

SOLUTION. In this example, $\mu_w = \dfrac{\mu_{25} + \mu_{27} + \mu_{34}}{3} = .00831$. Simple interpolation in the table for $x$ and $\mu_x$ shows that $w = 29.75$.

EXAMPLE 2. Find the equal ages equivalent to the ages 28, 32.

SOLUTION. In this example $\mu_w = \dfrac{\mu_{28} + \mu_{32}}{2} = .00837$. Simple interpolation in the table for $x$ and $\mu_x$ shows that $w = 30.25$.

EXAMPLE 3. Find the value of $a_{28:32}$ at 5%.

SOLUTION. By formula (10′)

$$a_{28:32} = a_{ww} = \frac{N_{w+1:w+1}}{D_{ww}} \text{ where, by Example 2, } w = 30.25$$

$$= \frac{215375.52}{16623.31} \text{ (by interpolation)}$$

$$= 12.956$$

EXAMPLE 4. Find the value of $A_{28:32}$ at $3\frac{1}{2}\%$.

SOLUTION. By formula (11′)

$$A_{28:32} = A_{ww} = \frac{M_{ww}}{D_{ww}} \text{ where, by Example 2, } w = 30.25$$

$$= \frac{11339.33}{25685.03}$$

$$= .44148$$

## EXERCISES

1. Solve Example 3 at $3\frac{1}{2}\%$.
2. Solve Example 4 at 5%. Ans. .33542.
3. Find the value at 5% of $a_{30:50}$.
4. Find the value at 5% of $A_{25:\overline{10}|}$.
5. Find the value at 5% of $a_{25:\overline{10}|}$.
6. Find the value at 5% of $A_{30:53}$.

**105. Survivorship annuities and insurances.** The annuities and insurances considered in Arts. 94 to 104 inclusive are based on the joint lives of two or more persons. Combinations of these can also be made which are analogous to those considered in Arts. 91 and 93 for single life annuities and insurances. Other annuities and insurances of importance are based on one or more *survivors* of a set of joint lives. A brief treatment is given in this article of a few simple annuities and insurances which are based on the survivor of two persons $(x)$ and $(y)$ of ages $x$ and $y$ respectively. The values of these annuities and insurances, as well as of many others, can be written at once by resolving them into annuities and insurances based on a single life, or on joint lives.

The value, $V$, of a life annuity immediate of annual rent, $R$, payable during the joint lives of $(x)$ and $(y)$ and during the life of the survivor is given by

$$V = R(a_x + a_y - a_{xy})$$

Such an annuity is called an annuity for the life of the last survivor. When $R = 1$, $V$ is denoted by $a_{\overline{xy}}$ so that

$$a_{\overline{xy}} = a_x + a_y - a_{xy}.$$

The value, $V$, of a life annuity immediate of annual rent, $R$, payable during the life of a designated one $[(x)$ or $(y)]$ after the death of the other $[(y)$ or $(x)]$ is given by

$V = R(a_x - a_{xy})$ if $(x)$ is the survivor, and by

$V = R(a_y - a_{xy})$ if $(y)$ is the survivor.

In the first case the annuity is called a survivorship or reversionary annuity on the life of $(x)$ beginning at the death of $(y)$. When $R = 1$, $V$ is denoted by $a_{y|x}$, so that

$$a_{y|x} = a_x - a_{xy}.$$

The value, $V$, of a life annuity immediate of annual rent, $R$, payable during the life of either one but after the death of the other is given by

$$V = R(a_x + a_y - 2\,a_{xy})$$

This is a combination of the two cases in the preceding survivorship annuity.

The value, $V$, of a joint life annuity immediate of annual rent, $\frac{R}{2}$, combined with a survivorship annuity of annual rent, $R$, is given by $V = \frac{R}{2}\, a_{xy} + R(a_x - a_{xy})$, if $x$ is the survivor; that is,

$$V = R(a_x - \tfrac{1}{2}\, a_{xy}) \text{ if } (x) \text{ is the survivor.}$$

Similarly $V = R(a_y - \frac{1}{2}\, a_{xy})$ if $(y)$ is the survivor.

The value, $V$, of a life insurance of face value, $F$, payable upon the death of the survivor of $(x)$ and $(y)$ is given by

$$V = F(A_x + A_y - A_{xy}).$$

Corresponding formulas can be written for more than two lives.

## EXERCISES

**1.** Compute the values at $3\frac{1}{2}\%$ and at $5\%$ of $a_{30:53}$.

**2.** Compute the values at $3\frac{1}{2}\%$ and at $5\%$ of $a_{30\,|\,53}$.

**3.** Interpret each of the following:

(*a*) $\mathbf{a}_{x|y} = \mathbf{a}_y - \mathbf{a}_{xy}$,

(*b*) $\mathbf{a}_{\overline{xy}} = \mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_{xy}$,

(*c*) $a_{\overline{xyz}} = a_x + a_y + a_z - a_{yz} - a_{xz} - a_{xy} + a_{xyz}$,

(*d*) $A_{\overline{xyz}} = A_x + A_y + A_z - A_{yz} - A_{xz} - A_{xy} + A_{xyz}$.

**4.** Show that $\mathbf{a}_{x|y} = a_{x|y}$.

# CHAPTER V

## APPLICATION OF LIFE ANNUITIES AND OF LIFE INSURANCES

**106. Introduction.** In Chapter IV formulas are derived for finding the values at age $x$ of life annuities and of life insurances and a few simple applications are given. In this chapter the applications are treated more fully and systematically. The suggestions given in Art. 33, Chapter III, for solving problems based on annuities certain are helpful for solving problems based on life annuities and life insurances.

The applications are arranged under three headings as follows: first, Premiums; second, Valuation of Policies, Cost of Insurance, Dividends, Policy Options; third, Life Estates, Remainders, Inheritance Taxes.

### Premiums

**107. Premiums defined.** As stated in Arts. 78, 86, and 91, Chapter IV, the value of a life annuity, of a life insurance, or of a combined life annuity and life insurance which is specified by a policy, is called the *net single premium* of the policy. The net single premium is the net cost of the policy at the time it is purchased. A purchaser of a policy usually finds it more convenient to make periodic payments, during the whole or a part of his life, which are equivalent to the net cost, than to pay this cost in a lump sum. Such payments are called net premiums, and they are usually made at the beginning of the payment periods. In other words the *net premiums* of a policy are the rent payments of a whole or temporary life annuity due whose value at the time the policy is purchased equals that of the net single premium of the policy at this time. These premiums are called *net level premiums* when they are equal in value. Other forms of net premiums are defined in Arts. 112, 120, 121, and 122. Net premiums provide companies with funds for the payment of policy benefits. In

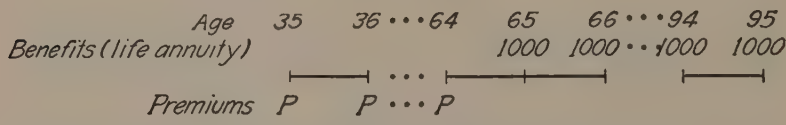
fact they usually provide more than is necessary for this purpose. This is due largely to two factors: first, companies are usually able to invest premiums at a higher rate than that used in computing them, and, second, the mortality shown by the table used in computing premiums is larger in the early years of policies than the actual mortality. In addition to funds for the payment of policy benefits companies must also have funds for the payment of interest on capital invested, for rent, taxes, salaries, and wages, agent's commissions, medical fees, and other expenses connected with the administration of its affairs. Net premiums are not sufficient to provide these additional funds. They are obtained by the addition of certain amounts, called *loadings*, to the net premiums. The net premiums plus the loadings are called the *gross* or *office* premiums. The gross premiums are those actually paid by the policyholders. The gross premiums must be ample to enable companies to pay policy benefits and expenses, as well as to build up and maintain an adequate surplus. Such a surplus is necessary since companies cannot conduct business on an exact theoretical basis. Gross premiums are considered further in Arts. 113 and 114.

The definition of a net level premium of any policy leads at once to an equation which determines its value. One member of this equation is the value at a known age of the annuity, corresponding to the form of payment of the net level premiums; the other member is the value at this age of the net single premium of the policy. Otherwise stated, one member of this equation is the value at a given age of the policy benefits and the other member is the value at this age of the premium payments. Formulas (1) and (5), Chapter IV, give these values for single lives and formulas (10) and (11) for two lives. Theorems V and VIII may be used in writing these equations. In Arts. 108 to 111 inclusive these premiums are determined for some important types of policies.

**108. Net level premiums of life annuity policies.** Both members of the equation which determine the net level annual premiums of life annuity policies on single lives can be written by the use of formula (1), Chapter IV. When two or more lives are involved, formula (10), Chapter IV, is needed.

EXAMPLE 1. Find the net level annual premium, $P$, of a \$1000 whole life annuity policy purchased at age 35 if the first benefit of the policy is payable at age 65 and if the premiums are payable annually, as long as the annuitant survives, during the term which begins at age 35 and ends at age 65.

SOLUTION. In the following diagram the benefits are above the line and the premiums below the line:




Equating the value of the premiums at age 35 to the value of the benefits at this age gives

$$P\frac{N_{35}-N_{65}}{D_{35}} = 1000\frac{N_{65}}{D_{35}}$$

Solving,

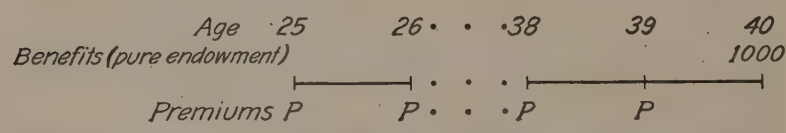
$$P = 1000\frac{N_{65}}{N_{35}-N_{65}} = \$119.08$$

EXERCISE 1. Solve Example 1 by equating values at age 65.

EXERCISE 2. Solve Example 1 if the premiums are payable annually from age 35 to age 50.

EXAMPLE 2. Find the net level annual premium, $P$, of a \$1000 pure endowment policy purchased at age 25 and due in 15 years if the premiums are payable annually, as long as the annuitant survives, during the term which begins at age 25 and ends at age 40.

SOLUTION. In the following diagram the benefits are above the line and the premiums below the line:




Equating the value of the premiums at age 25 to the value of the benefit at this age gives,

$$P\frac{N_{25}-N_{40}}{D_{25}} = 1000\frac{D_{40}}{D_{25}}$$

Solving,

$$P = 1000\frac{D_{40}}{N_{25}-N_{40}} = \$42.40$$

EXERCISE 1. Solve Example 2 by equating values at age 40.

EXERCISE 2. Solve Example 2 if the premiums are paid annually from age 25 to age 35.

EXAMPLE 3. Find the net level annual premium, $P$, of a \$1000 annuity policy purchased at age 35 whose first \$1000 is paid at age 65 if the policy provides a ten-year annuity certain followed by a whole life annuity, and if the premiums are payable annually from age 35 to age 65. Use $i = .035$.

SOLUTION. In this case the policy benefits consist of a ten-year annuity certain, followed by a whole life annuity. Equating the value of the premiums at age 65 to that of the benefits at this age gives

$$P\frac{N_{35} - N_{65}}{D_{65}} = 1000\,(1 + a_{\overline{9}|.035}) + 1000\,\frac{N_{75}}{D_{65}}$$

Solving, $P = \$651.22$

## EXERCISES

1. Compute the net level annual premium, $P$, payable as a life annuity due from age 25 to age 50 of a pure endowment policy of \$1000 due at age 50.

2. Same as Exercise 1 except the \$1000 is due at age 60.

3. Same as Exercise 1 for a whole life annuity policy of \$1000 annual rent with first payment at age 50.

4. Same as Exercise 1 except that the premiums are payable quarterly instead of annually. Use formula (4), Art. 84.

5. Same as Exercise 1 for a 20-year temporary life annuity policy of \$1000 annual rent with first payment at age 50.

6. Compute the net level annual premium, $P$, payable as a joint life annuity due from the ages 25, 30 to the ages 50, 55 of a joint life pure endowment policy of \$1000 due at the ages 50, 55.

7. Same as Exercise 6 for a joint whole life annuity policy of \$1000 annual rent with first payment at the ages 26, 31.

8. Same as Example 6 with first payment at ages 50, 55.

**109. Net level premium formulas for life annuity policies.** A general formula for the net level premium at age $x$, corresponding to a prescribed form of payment, can readily be found for any life annuity policy by the method of equating the value of the premiums at age $x$ to that of the benefits at this age. For an $n$-payment, $n$-year pure endowment policy of rent, $R$, issued at age $x$, whose premiums, $P$, are payable annually as a life annuity due from age $x$ to age $x + n$, the value of $P$ is given by

$$P = R\,\frac{D_{x+n}}{N_x - N_{x+n}}$$

In this case the equation of value at age $x$ is $P\,\dfrac{N_x - N_{x+n}}{D_x} = R\dfrac{D_{x+n}}{D_x}$;

the value of $P$ given above is found by solving this equation for $P$.

When $R = 1$, $P$ is denoted by $\frac{1}{{}_nu_x}$ (Art. 82, Chap. IV), so that for an *n-payment, n-year pure endowment policy of face value* 1

$$P = \frac{1}{{}_nu_x} = \frac{D_{x+n}}{N_x - N_{x+n}}$$

Similarly for an *n-payment whole life annuity of annual rent*, $R$, issued at age $x$ with first payment at age $x + n$, whose premiums, $P$, are payable annually as a life annuity due from age $x$ to age $x + n$, the value of $P$ is given by

$$P = R\frac{N_{x+n}}{N_x - N_{x+n}}$$

Since these and like formulas for the net level premiums of other life annuity and of life insurance policies can be derived so easily, the student is advised to write the equation which determines the value of $P$ in each case rather than to substitute into a formula.

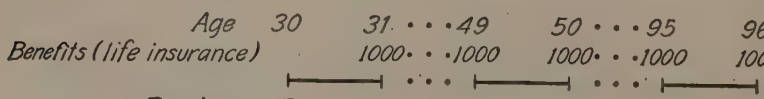
### EXERCISE

Derive the formula for the net level annual premium, $P$, at age $x$ of an $n$-year temporary life annuity policy of annual rent, $R$, with first payment at age $x + t$ if the premiums are payable as a life annuity due from age $x$ to age $x + t$; from age $x$ to age $x + m$.

**110. Net level premiums of life insurance policies. Notation.** The benefits of these policies consist of life insurances so that the formulas for the values of life insurances and of annuities are needed to write the equations which determine the net level premiums of these policies.

EXAMPLE 1. Find the net level annual premium, $P$, of a 20-payment whole life insurance policy of \$1000 issued at age 30.

SOLUTION. In the following diagram the benefits are above the line and the premiums below the line:



Equating the value of the premiums at age 30 to the value of the benefit at this age gives

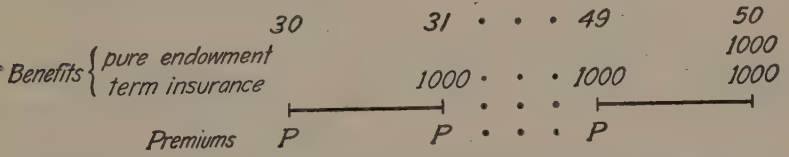
$$P\,\frac{N_{30}-N_{50}}{D_{30}} = 1000\,\frac{M_{30}}{D_{30}}$$

Solving,

$$P = 1000\,\frac{M_{30}}{N_{30}-N_{50}} = \$24.71$$

EXAMPLE 2. Find the net level annual premium, $P$, of an ordinary 20-year endowment insurance policy of \$1000 issued at age 30.

SOLUTION. In the following diagram the benefits are above the line and the premiums below the line:




Equating the value of the premiums at age 30 to the value of the benefits at this age gives

$$P\,\frac{N_{30}-N_{50}}{D_{30}} = 1000\,\frac{M_{30}-M_{50}+D_{50}}{D_{30}}$$

Solving,

$$P = 1000\,\frac{M_{30}-M_{50}+D_{50}}{N_{30}-N_{50}} = \$39.51$$

EXAMPLE 3. Same as Example 2 except that the premiums are payable quarterly.

SOLUTION. Equating values at age 30, using formula, (4) Chapter IV, and solving for $P$, the quarterly premium, gives

$$P = \frac{1000}{4}\,\frac{M_{30}-M_{50}+D_{50}}{N_{30}-N_{50}-\frac{3}{8}(D_{30}-D_{50})} = \$10.04$$

EXAMPLE 4. Find the net level annual premium, $P$, of a 20-payment increasing insurance of \$1000 issued at age 25 for a term of 20 years.

SOLUTION. Equating values at age 25, using formula (8), Chapter IV, and solving for $P$, gives

$$P = 1000\,\frac{R_{25}-R_{45}-20\,M_{45}}{N_{25}-N_{45}} = \$82.08$$

EXAMPLE 5. Find the net level annual premium, $P$, of a \$1000 ordinary joint whole life policy issued at the ages 28, 32.

SOLUTION. Equating values at the ages 28, 32 and solving for $P$ gives

$$P = 1000\,\frac{M_{28:32}}{N_{28:32}} = \$26.73$$

### EXERCISES

1. Compute the net level annual premium of a $1000 ordinary whole life insurance policy issued at age 35. Ans. $19.91.
2. Same as Exercise 1 for a 20-payment whole life insurance policy. Ans. $27.40.
3. Same as Exercise 1 for a 20-year endowment insurance policy. Ans. $40.12.
4. Same as Exercise 1 for a 10-payment, 20-year endowment insurance policy.
5. Same as Exercise 1 for a 10-payment, 10-year term insurance policy.
6. Same as Exercise 1 for a 1-payment, 1-year term insurance policy. Ans. $8.64.

**111. Net level premium formulas for life insurance policies. Notation.** A general formula for the net level premium at age $x$ corresponding to a prescribed form of payment, can readily be found for any life insurance policy by the method of equating the value of the premiums at age $x$ to the value of the benefits at this age. There is a standard notation for the net level premiums of the various forms of life insurance policies of face value 1 issued at age $x$. In what follows in this article this notation is given for the common forms of policies, and the formulas for the values of the premiums are written. In the ordinary whole life insurance policy the premiums are payable annually as a whole life annuity due beginning at age $x$; in the *limited payment* policies the premiums are payable annually as a temporary life annuity due beginning at age $x$ and continuing during a term which is equal to the number of payments. In each of the following policies, the age of issue is $x$ and the face value is 1.

*Ordinary whole life insurance policy*

$$P_x = \frac{M_x}{N_x};$$

*m-payment whole life insurance policy*

$$_mP_x = \frac{M_x}{N_x - N_{x+m}};$$

*n-payment, n-year term insurance policy*

$$P^1_{\overline{x\,n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}};$$

*m-payment, n-year endowment insurance policy*

$$ {}_mP_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+m}}; $$

*n-payment, n-year endowment insurance policy*

$$ P_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}} $$

The last policy is usually called an $n$-year endowment insurance policy.

EXERCISE. Verify each of the above formulas by equating the value of the premiums at age $x$ to that of the benefits at this age and solving for the premium.

The above notation is extended to premiums of joint life insurance policies by replacing $x$ by $xy$, and to premiums payable $m$ times a year by affixing $(m)$ in parenthesis to the upper right of the symbols. For example, $P_{xy}$ denotes the net level premium payable annually of a joint whole life insurance policy of face value 1 for two lives, and $P_x^{(m)}$ denotes the net level premium payable $m$ times a year of a whole life insurance policy of face value 1 for a single life.

## EXERCISES

**1.** Show that $A_x = P_x(1 + a_x) = P_x\mathbf{a}_x$. Interpret this relation verbally.

**2.** Use the relations

$A_x = 1 - d\mathbf{a}_x$ (Art. 92)

$A_x = P_x\mathbf{a}_x$ (Exercise 1)

to show that

(*a*) $P_x = \dfrac{1}{\mathbf{a}_x} - d$

(*b*) $P_x = \dfrac{A_x}{1 - dA_x}$

(*c*) $A_x = \dfrac{P_x}{P_x + d}$

(*d*) $\mathbf{a}_x = \dfrac{1 - A_x}{d}$

(*e*) $\mathbf{a}_x = \dfrac{1}{P_x + d}$

Interpret each of these formulas verbally.

**3.** Write the formulas for $P_{25:30}$ and $P_{25:30\,\overline{20}|}$.

**4.** Write the formulas for $P_{25}^{(4)}$, ${}_{20}P_{25}^{(4)}$, and $P_{25\,\overline{20}|}^{(4)}$.

**112. Natural Premiums. Full preliminary term net premiums.** The net single premium for a one-year term insurance issued at age $x$ is called the *natural premium* at age $x$ of this insurance. By putting $F = t = 1$ in formula $(5_2)$, Chapter IV, it is seen that the natural premium, $A^1_{x\,\overline{1}|}$,* of an insurance of 1 issued at age $x$ is given by

$$A^1_{x\overline{1}|} = \frac{C_x}{D_x}$$

The natural premiums for insurance issued one year at a time through a term of years evidently increase with the age and so differ from the net level premiums considered in Arts. 110 and 111.

The *full preliminary term net premiums* of any given life insurance policy issued at age $x$ are defined as follows: The net premium for the first year is the natural premium at age $x$ for the face of the policy; the net premium for each subsequent year of the premium payment period is the net level premium at age $x + 1$ of a like policy issued at age $x + 1$ for a term which ends at the same time as that of the given policy. The premiums subsequent to the first are called *renewal* premiums. For example, under the full preliminary term plan the first net premium of any life insurance policy of face value 1 issued at age 30 is denoted by $A^1_{30:\overline{1}|}$; when the policy is ordinary whole life each renewal premium is denoted by $P_{31}$; when it is a 20-payment life, each renewal premium is denoted by ${}_{19}P_{31}$; when it is a 20-year endowment, each renewal is denoted by $P_{31\,\overline{19}|}$.

Gross premiums of policies are usually level and so it follows that the full preliminary term premiums allow a larger loading for the first year than for the later years. This is usually desirable, particularly for new companies, since the expenses on a policy for the first year are heavier than for the later years. For limited payment policies, however, having large gross premiums, as for example 10-year endowment, or 10-payment whole life insurance policies, the full preliminary term net premiums provide loadings for the first year which are larger than are needed to cover the extra expenses for the year. To avoid this situation various states have

* $P^1_{x\overline{1}|}$ is also used to represent the natural premiums.

adopted laws modifying the full preliminary term net premiums. Some of these modified forms of preliminary term premiums are discussed in Arts. 120, 121, and 122 after the subject of reserves has been considered.

EXAMPLE 1. Compute the natural premium at age 30 for a $1000 policy.

SOLUTION. The natural premium in this case is $1000 \frac{C_{30}}{D_{30}} = \$8.14$.

EXAMPLE 2. Compute the value of each renewal full preliminary term net premium of a $1000 ordinary whole life insurance policy issued at age 30.

SOLUTION. Equating the value of the renewal premiums at age 31 to the value of the insurance at this age gives

$$P_{31} \frac{N_{31}}{D_{31}} = 1000 \frac{M_{31}}{D_{31}}$$

Solving,

$$P_{31} = 1000 \frac{M_{31}}{M_{31}} = \$17.68$$

### EXERCISES

**1.** Write the formulas for the full preliminary term premiums of the following life insurance policies issued at age 25:

(*a*) Ordinary whole life, (*c*) 10-year endowment,
(*b*) 30-payment life, (*d*) 10-payment 20-year endowment.

**2.** Compute the values of the premiums in Exercise 1 (*b*) and 1 (*d*).

**113. Gross premiums based on net level and full preliminary net term premiums.** As stated in Art. 107 gross premiums are formed by adding loadings to the net premiums, and, when invested, they must provide a company with funds for the payment of policy benefits, expenses, and surplus. A portion of the surplus provided is held for unforeseen contingencies, and the remainder is given to those to whom it belongs. In stock companies issuing only non-participating policies it goes to the stockholders. In strictly mutual companies, the insured or their beneficiaries participate in this surplus; it is paid to the insured in the form of dividends, or to the beneficiaries in a lump sum or some other form of equivalent benefit. In mixed companies, part goes to the stockholders and part to the policyholders. The loadings in the gross premiums of participating policies are usually larger than of non-participating since any excess is returned to the insured. In this article no attempt is made to discuss the general theory

underlying methods of loading. A few examples will be given, however, to illustrate some of these methods for participating policies. The study of actual loading formulas affords a good way of learning the principles used. An attempt by the student to discover these formulas for policies issued by the various companies is an excellent exercise.

The following table of net premiums on $1000 policies may be used in applying loading formulas:

| | | Whole Life | | 20-Payment Life | | 20-Year Endowment | |
|---|---|---|---|---|---|---|---|
| Natural Premium | Age | Net Level | Preliminary Term Renewals | Net Level | Preliminary Term Renewals | Net Level | Preliminary Term Renewals |
| 7.54 | 20 | 13.48 | 13.77 | 20.72 | 21.76 | 38.90 | 41.36 |
| 7.79 | 25 | 15.10 | 15.48 | 22.53 | 23.68 | 39.14 | 41.61 |
| 8.14 | 30 | 17.19 | 17.68 | 24.71 | 26.02 | 39.51 | 41.99 |
| 8.64 | 35 | 19.91 | 20.55 | 27.40 | 28.89 | 40.12 | 42.63 |
| 9.46 | 40 | 23.50 | 24.36 | 30.75 | 32.47 | 41.18 | 43.75 |
| 10.79 | 45 | 28.35 | 29.51 | 35.07 | 37.09 | 43.08 | 45.77 |
| 13.31 | 50 | 34.99 | 36.59 | 40.82 | 43.22 | 46.46 | 49.35 |
| 17.94 | 55 | 44.13 | 46.34 | 48.70 | 51.60 | 52.21 | 55.43 |
| 25.79 | 60 | 56.83 | 59.92 | 59.85 | 63.44 | 61.65 | 65.43 |

In the formulas given in the exercises which follow $G$ denotes the gross premium on a $1000 policy. In some of these formulas the loadings are based on net premiums; in others, they are based on gross premiums.

### EXERCISES

**1.** Compute the gross premiums for the ages 25 and 50 of a $1000 ordinary whole life policy by use of the formulas:

(*a*) $G = 1000\,P_x\,(1 + .3)$

(*b*) $G = 1000\,P_x\,(1 + .2)$

(*c*) $G = 1000\,P_x\,(1 + .2) + 1$

**2.** Compute the gross premiums for the ages 25 and 50 of a \$1000 20-payment whole life policy by use of the formulas

$$(a)\ G = 1000\,[{}_{20}P_x(1 + .2) + P_x(.1)],$$
$$(b)\ G = 1000\,[{}_{20}P_x(1 + .15) + P_x(.15)],$$
$$(c)\ G = 1000\,[{}_{20}P_x(1 + .1) + P_x(.1)] + 1,$$
$$(d)\ G = 1000\,[{}_{20}P_x(1 + .105) + P_x(.105)].$$

**3.** Compute the gross premiums for the ages 25 and 50 of a \$1000 20-year endowment insurance policy by use of the formulas obtained by replacing ${}_{20}P_x$ in each of the formulas given in Exercise 2 by $P_{x\,\overline{20}|}$.

**4.** Compute the gross premiums for the ages 25 and 50 of a \$1000 whole life policy by use of the formula, obtained by solving $G = 1000\,P_x + .14\,G + 1$ for $G$,

$$G = 1000\,\frac{P_x + 1}{.86}$$

**5.** Compute the gross premiums for the ages 25 and 50 of a \$1000 10-payment whole life policy by use of the formula

$$G = 1000\,\frac{{}_{10}P_x + 1}{.86}$$

**6.** Same as Exercises 1, 2, 3, 4, and 5 when the full preliminary term renewal premiums are used in place of the net level premiums.

**7.** Determine the amount of loading in each gross premium in Exercises 1 to 5 inclusive.

**8.** Determine the amount of loading for the first year and for each subsequent year in each gross premium in Exercise 6.

**9.** Same as Exercises 1 and 4 when $P_x$ is replaced by $P_x^{(2)}$.

**10.** Compute the gross premiums payable semi-annually for the ages 25 and 50 of a \$1000 whole life policy by multiplying the results found in Exercise 1 by .52. (Note that this method is simpler than that of Exercise 9.)

**11.** Interpret each of the formulas in Exercise 2 verbally.

**114. Gross premiums on return premium policies.** The benefits provided by some policies include the return of the gross premiums without interest. These gross premiums constitute an increasing insurance. In this article the gross premiums are found for a few policies which provide for the return of the gross premiums.

EXAMPLE 1. Find the gross premium for an increasing insurance of \$1000 issued at age 35 for a term of 30 years if the loading is 13 per cent of the gross premium.

SOLUTION. By the method in Example 4, Art. 110, the net level premium on this increasing insurance is \$206.79. Hence if $G$ denotes the gross premium, $G = 206.79 + .13\,G$. Solving, $G = \$237.69$.

EXAMPLE 2. A ten-year "term-endowment" policy provides that if the insured dies during the ten years, the beneficiary receives the face of the policy plus the premiums paid; but if the insured survives the ten-year term, he gets the amount of the premiums paid. The gross premium at age 35 on the company's regular 10-year term policy for \$1000 is \$13.10. Find the gross premium, assuming a 30 per cent loading on increasing insurance and a 15 per cent loading on 10-year pure endowment.

SOLUTION. Let $G$ denote the gross premium. By the method in Example 2, Art. 108, the net level annual premium for a pure endowment of $10\,G$ issued at age 35 and due in 10 years is $.07765\ (10\,G)$. The gross premium for this pure endowment is then $.07765\ (10\,G)(1.15)$. By the method in Example 4, Art. 110, the net annual premium for an increasing insurance of $G$ issued at age 35 for a term of 10 years is $.04972\,G$. The gross premium for this increasing insurance is then $.04972\,G\ (1.3)$.

Hence $G = 13.10 + .04972\ (1.3)\ G + .07765\ \ (1.15)\ \ 10\,G$. Solving, $G = \$308.96$.

EXAMPLE 3. A policy issued at age 35 provides a pure endowment of \$1300 at age 65 with the return of the premiums paid in the event of death before age 65. Find the gross premium on this policy if the net level premium for the increasing insurance is loaded as in Example 1, and if the gross premium $P'$ for the \$1300 pure endowment is given by $P' = \dfrac{P}{.87} + 1$, where $P$ is the net level premium for this pure endowment.

SOLUTION. Let $G$ denote the gross premium on this policy. By the result found in Example 1 the gross premium for the increasing insurance of $G$ is $.23769\,G$. By the method in Example 2, Art. 108, the level net premium $P$, on the \$1300 pure endowment, is \$16.7918. The gross premium $P'$ for this pure endowment is then $P' = \dfrac{P}{.87} + 1 = \$20.301$. Hence $G = 20.30 + .23769\,G$. Solving for $G$ gives $G = \$26.63$.

## EXERCISES

**1.** Show that the gross premium, $G$, for an ordinary whole life insurance of \$1000 issued at age $x$ with return of all gross premiums paid prior to death is given by

$$G = \frac{1000\,M_x(1+p) + cN_x}{N_x - R_x(1+p)}$$

the loading being a constant $c$, plus $p$ times the net annual premium $P$.

[HINT. Eliminate $P$ from $P\dfrac{N_x}{D_x} = \dfrac{1000\,M_x + GR_x}{D_x}$ and $G = P(1+p) + c$ and solve for $G$.]

**2.** Compute $G$ in Exercise 1 for $p = .25$, $c = 1$ and $x = 25$.

**3.** Compute the net annual premium for an ordinary whole life insurance of \$1000 issued at age 25 with return of all net premiums paid prior to death.

## Valuation of Policies, Cost of Insurance, Dividends, Policy Options

**115. Reserves defined.** The table of premiums in Art. 113 shows that the net level premium of a \$1000 ordinary whole life insurance policy issued at age 20 is \$13.48. The same table shows that the natural premiums are smaller than this net level premium for ages under 51 and larger for ages over 54. In other words the net level premiums of this policy provide more than is needed to pay death losses during each of the earlier years and less than is needed during each of the later years. The parts of the premiums of the early years not needed for mortality are accumulated at compound interest and held by the company to meet the heavier mortality during the later years. The amount so held at any age after the age of issue is called the reserve, or value of the policy, at that age. For example, the reserve on the above policy at the end of two years is given by

$$\frac{l_{20}(13.48 - 7.54)(1.035)^2 + l_{21}(13.48 - 7.59)(1.035)}{l_{22}} = \$12.61$$

where 7.54 and 7.59 are the natural premiums at the ages 20, 21, respectively. This method of computation becomes more tedious as the age at which the reserve is to be found becomes larger.

A simple method of computation is obtained by viewing the reserve on the policy at age $20 + n$ as the value at age $20 + n$ of the premiums paid from age 20 to age $20 + n$ less the value at age $20 + n$ of the insurance (death losses) whose term begins at age 20 and ends at age $20 + n$. These values can be written at once by use of formulas (1) and (5), Chapter IV. By this method the reserve on the \$1000 ordinary whole life policy at the end of two years is given by

$$13.48\frac{N_{20} - N_{22}}{D_{22}} - 1000\frac{M_{20} - M_{22}}{D_{22}} = \$12.61$$

Another simple method of computation is obtained by viewing the reserve on this policy as the value at age $20 + n$ of the whole life insurance whose term begins at age $20 + n$ less the value at age $20 + n$ of the premiums (13.48) paid from age $20 + n$ through

life. The value of the whole life insurance at age $20 + n$ is the net single premium at this age, and this net single premium would of itself be just enough to enable the company to pay all future death losses. The company receives, however, the premiums (13.48) from age $20 + n$ through life. It follows that the net single premium at age $20 + n$ less the value at this age of the future net annual premiums (13.48) is the reserve at age $20 + n$ on the ordinary whole life policy issued at age 20. By this method the reserve on the \$1000 ordinary whole life policy at the end of two years is given by

$$1000\frac{M_{22}}{D_{22}} - 13.48\frac{N_{22}}{D_{22}} = \$12.61.$$

The value of any policy at the end of $n$ years after the age of issue may be found by methods analogous to those just given for a whole life insurance policy issued at age 20. The definitions of reserve upon which the last two methods of computation are based may be stated in the general forms:

1. *The value, or terminal reserve, at age $x + n$ of a policy issued at age $x$ is the value at age $x + n$ of the net premiums of the policy paid from age $x$ to age $x + n$ less the value at age $x + n$ of the benefits provided by the policy from age $x$ to age $x + n$.*

2. *The value, or terminal reserve, at age $x + n$ of a policy issued at age $x$ is the value at age $x + n$ of the benefits provided by the policy from age $x + n$ to the end of its term less the value at age $x + n$ of the net premiums of the policy from age $x + n$ to the end of the term during which the premiums are paid.*

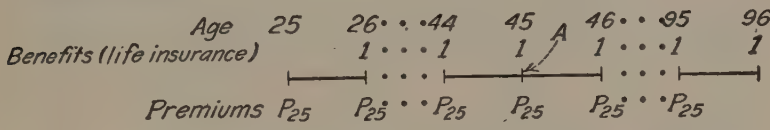
The method of computation based on the first form of definition is called the *retrospective method;* that based on the second form is called the *prospective method.* In Arts. 116 and 117 these methods are applied to find expressions for the reserves of the common forms of policies.

**116. Retrospective and prospective methods of valuation. Notation.** By use of the definitions in Art. 115 and the formulas for the values of life annuities and life insurances the expressions for the terminal reserve under the retrospective and the prospective

methods can be written at once. The two expressions give the same value of a given policy under a given set of net premiums, but one may be simpler to compute than the other.

EXAMPLE 1. Use the net level premiums, $P_{25}$, to find the terminal reserve, $V$, at the end of 20 years of a whole life insurance policy of face value 1 issued at age 25.

SOLUTION. The policy benefits and premiums are represented by the following diagram.




The reserve is to be found at the point $A$ in this diagram, that is, at age 45. By the retrospective method this reserve is the value at age 45 of the premiums to the left of the vertical line through $A$ less the value at age 45 of the benefits to the left of this line. By the prospective method this reserve is the value at age 45 of the benefits to the right of this line less the value at age 45 of the premiums to the right of it. That is, using formulas (1) and (5), Chapter IV, by the

Retrospective method: $V = P_{25}\dfrac{N_{25} - N_{45}}{D_{45}} - \dfrac{M_{25} - M_{45}}{D_{45}} = .21304$

Prospective method: $V = \dfrac{M_{45}}{D_{45}} - P_{25}\dfrac{N_{45}}{D_{45}} = .21304$

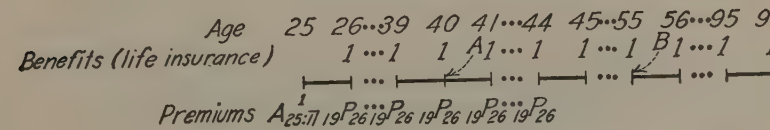
EXERCISE 1. By replacing $P_{25}$ by its value in commutation symbols, show that the two expressions for $V$ are equal.

EXERCISE 2. Write the expression for the reserve of this policy at age 95. (Use the prospective method.)

EXERCISE 3. Write the expression for the reserve of this policy at age 26. (Use the retrospective method.)

EXAMPLE 2. Use the full preliminary term net premiums, $A^1_{25:\overline{1}|}$ and ${}_{19}P_{26}$, to find the terminal reserves, $V$, at the end of 15 years and at the end of 30 years of a \$1 20-payment life insurance policy issued at age 25.

SOLUTION. The policy benefits and premiums are represented by the following diagram:



The reserves are to be found at the points $A$ and $B$ in this diagram. By the retrospective method the reserve at $A$ is given by

$$V = {}_{19}P_{26}\frac{N_{26} - N_{40}}{D_{40}} - \frac{M_{26} - M_{40}}{D_{40}} = .30148$$

In this case the reserve at age 26 is zero. By the prospective method the reserve at $B$ is given by $V = \dfrac{M_{55}}{D_{55}} = .56615$.

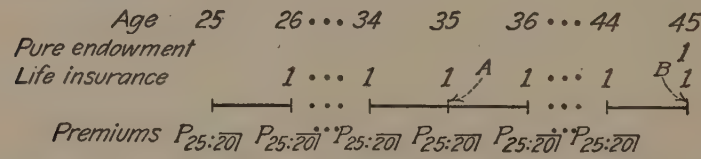
EXERCISE 1. Use the prospective method to write the expression for the reserve at $A$ and show that it is the same as that found by the retrospective method.

EXERCISE 2. Use the retrospective method to write the expression for the reserve at $B$ and show that it is the same as that found by the prospective method.

EXERCISE 3. Find the reserves at $A$ and $B$ on a \$1000 policy of the type in Example 2.

EXAMPLE 3. Use the net level premiums, $P_{25:\overline{20}|}$, to find the terminal reserves, $V$, at the end of 10 years and at the end of 20 years, of a \$1 20-year endowment insurance policy issued at age 25.

SOLUTION. The policy benefits and the premiums are represented by the following diagram:


Age 25 26 ··· 34 35 36 ··· 44 45
Pure endowment
Life insurance
1 ··· 1 1 A 1 ··· 1 B 1
Premiums $P_{25:\overline{20}|}$ $P_{25:\overline{20}|}$ $P_{25:\overline{20}|}$ $P_{25:\overline{20}|}$ $P_{25:\overline{20}|}$ $P_{25:\overline{20}|}$

The reserves are to be found at the points $A$ and $B$ in this diagram. By the retrospective method the reserve at $A$ is given by

$$V = P_{25:\overline{20}|}\frac{N_{25} - N_{35}}{D_{35}} - \frac{M_{25} - M_{35}}{D_{35}} = .39621$$

By the prospective method the reserve at $B$ is 1.

EXERCISE 1. Use the prospective method to write the expression for the reserve at $A$ and show that it is the same as that found by the retrospective method.

EXERCISE 2. Use the retrospective method to write the expression for the reserve at $B$ and show that it equals 1.

## EXERCISES

**1.** Use net level premiums to compute the terminal reserve, at the end of 5 years, of a \$1000 10-year term insurance policy issued at age 25.

**2.** Use net level premiums to compute terminal reserve, at end of 10 years, of a \$1000 ordinary joint whole life insurance policy issued at ages 28, 32.

**117. Reserve formulas based on net level and on full preliminary term net premiums. Notation.** A general formula for the terminal reserve at age $x + n$, under a given set of premiums, can readily be obtained for any policy issued at age $x$ by either the retrospective or the prospective method. There is a standard notation for this reserve under the net level annual premiums for the various forms of life insurance policies of face value 1 issued at age $x$. In what follows in this article this notation is given for the common form of policies, and the formulas for the values of the terminal reserve at age $x + n_1$, that is, at the end of the $n_1{}^{th}$ policy year, are written.

*Ordinary whole life insurance policy*

Retrospective method:

$$ {}_{n_1}V_x = P_x \frac{N_x - N_{x+n_1}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}} $$

Prospective method:

$$ {}_{n_1}V_x = \frac{M_{x+n_1}}{D_{x+n_1}} - P_x \frac{N_{x+n_1}}{D_{x+n_1}} $$

*m-payment whole life insurance policy*

Retrospective method:

$$ n_1 \leq m, \quad {}_{n_1:m}V_x = {}_mP_x \frac{N_x - N_{x+n_1}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}} $$

$$ n_1 \geq m, \quad = {}_mP_x \frac{N_x - N_{x+m}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}} $$

Prospective method:

$$ n_1 \leq m, \quad {}_{n_1:m}V_x = \frac{M_{x+n_1}}{D_{x+n_1}} - {}_mP_x \frac{N_{x+n_1} - N_{x+m}}{D_{x+n_1}} $$

$$ n \geq m, \quad = \frac{M_{x+n_1}}{D_{x+n_1}} $$

*n-year endowment insurance policy*

Retrospective method:

$$ {}_{n_1}V_{x\overline{n}|} = P_{x\overline{n}|} \frac{N_x - N_{x+n_1}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}} $$

Prospective method:

$$ {}_{n_1}V_{x\overline{n}|} = \frac{M_{x+n_1} - M_{x+n} + D_{x+n}}{D_{x+n_1}} - P_{x\overline{n}|}\frac{N_{x+n_1} - N_{x+n}}{D_{x+n_1}} $$

It should be noted that when $n_1 = n$, ${}_nV_{x\overline{n}|} = 1$.

*n-payment, n-year term insurance policy*

Retrospective method:

$$ {}_{n_1}V^1_{x\overline{n}|} = P^1_{x\overline{n}|}\frac{N_x - N_{x+n_1}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}} $$

Prospective method:

$$ {}_{n_1}V^1_{x\overline{n}|} = \frac{M_{x+n_1} - M_{x+n}}{D_{x+n_1}} - P^1_{x\overline{n}|}\frac{N_{x+n_1} - N_{x+n}}{D_{x+n_1}} $$

EXERCISE. Verify that the two methods lead to the same terminal reserve for each of the above insurance policies.

The notation for the net level premium reserves of a given policy can be used also to represent the full preliminary term premium reserves of the policy. For example, the full preliminary term premium reserve at the end of fifteen years of an ordinary whole life policy of face value 1 issued at age 30 is represented by ${}_{14}V_{31}$ and that of a 20-payment life policy is represented by ${}_{14:19}V_{31}$.

Under full preliminary term valuation the first year's insurance of a given policy is treated as a one-year term policy and the subsequent years as a policy of the same kind whose term ends at the same time as that of the given policy but begins at an age one year higher. In this method of valuation there is no reserve at the end of the first year. By the prospective method it is seen that the full preliminary term premium reserve and the net level premium reserve of a given policy are equal in value at the end of the premium payment term or at any subsequent time. The two reserves are not equal, however, at any age prior to that at the end of the premium payment term, the reserve based on the net level premiums being greater in this case than that based on the full preliminary term premiums. This follows at once by the prospective method upon noting that the full preliminary term renewal premium is greater than the net level premium. For some policies the differences between these reserves are large during the early policy years. For example, on a $1000 10-year endowment

insurance policy issued at age 25, the net level premium reserve at the end of the second year is \$167.66 and the full preliminary term premium reserve is \$93.24. In this case the difference is \$74.42. On this policy the differences range from \$82.08 at the end of the first year to zero at the end of the tenth year. For such policies the modified preliminary term plans of valuation discussed in Arts. 120 to 123 inclusive are more satisfactory than the full preliminary term plan.

### EXERCISES

**1.** Write the symbols which represent the net level premium reserves at the end of 1, 2, 10, and 15 years of the following policies of face value 1 issued at age 35: ordinary whole life; 25-payment life; 15-year endowment; 15-year term.

**2.** Same as Exercise 1 except that the net level premiums are replaced by the full preliminary term premiums.

**3.** Show that $_{n_1}V_x = A_{x+n_1} - P_x a_{x+m_1} = 1 - \dfrac{a_{x+n_1}}{a_x}$

[HINT. Use Exercise 2, Art. 111.]

**4.** The following table shows the net level premium reserves and the full preliminary term premium reserves after 1, 2, 10, 15, 20, and 25 years for some \$1000 insurance policies issued at age 35; the first row for each year gives the net level premium reserves and the second row gives the full preliminary term reserves:

| YEAR | ORDINARY LIFE | 20-PAYMENT LIFE | 10-PAYMENT LIFE | 15-YEAR ENDOWMENT | 20-YEAR ENDOWMENT |
|---|---|---|---|---|---|
| 1 | 11.76<br>0.00 | 19.58<br>0.00 | 37.73<br>0.00 | 48.80<br>0.00 | 32.86<br>0.00 |
| 2 | 23.91<br>12.29 | 39.90<br>21.01 | 77.01<br>42.79 | 99.63<br>53.44 | 67.06<br>35.37 |
| 10 | 135.76<br>125.48 | 232.19<br>219.96 | 456.00<br>456.00 | 592.44<br>571.53 | 395.99<br>375.46 |
| 15 | 219.15<br>209.85 | 384.02<br>377.22 | 508.49<br>508.49 | 1000.00<br>1000.00 | 664.10<br>652.58 |
| 20 | 310.75<br>302.54 | 566.15<br>566.15 | 566.15<br>566.15 | | 1000.00<br>1000.00 |
| 25 | 407.30<br>400.25 | 626.92<br>626.92 | 626.92<br>626.92 | | |

Verify a few of these reserves.

**118. Fackler's accumulation formula.** By means of the formulas for the values of life annuities and of life insurances a formula can be written which is useful in computing tables of reserves of a given policy. The net level premium reserves at the end of $n_1$ years and of $n_1 + 1$ years of an ordinary whole life policy of face value 1 issued at age $x$ are represented by ${}_{n_1}V_x$ and ${}_{n_1+1}V_x$ respectively. By the retrospective method it is seen that ${}_{n_1+1}V_x$ is the value of ${}_{n_1}V_x + P_x$ accumulated as a pure endowment from age $x + n_1$ to age $x + n_1 + 1$ less the value at age $x + n_1 + 1$ of the insurance (death losses) whose term begins at age $x + n_1$ and ends at age $x + n_1 + 1$. Hence by formulas $(1_2)$ and $(5_2)$, Chapter IV,

$$\begin{aligned} {}_{n_1+1}V_x &= ({}_{n_1}V_x + P_x)\frac{D_{x+n_1}}{D_{x+n_1+1}} - \frac{C_{x+n_1}}{D_{x+n_1+1}} \\ &= ({}_{n_1}V_x + P_x)u_{x+n_1} - k_{x+n_1} \end{aligned}$$

where $u_{x+n_1}$ and $k_{x+n_1}$ are the valuation symbols defined in Arts. 82 and 90, Chapter IV. This relation connecting ${}_{n_1+1}V_x$ and ${}_{n_1}V_x$ is called *Fackler's accumulation formula.* An analogous formula can be written for any policy. Another accumulation formula is given in Exercise 3 below.

EXERCISES

1. Compute ${}_2V_{35}$ and ${}_3V_{35}$ by the use of Fackler's Formula.

2. Show that ${}_{n_1+t}V_x = {}_{n_1}V_x \dfrac{D_{x+n_1}}{D_{x+n_1+t}} + P_x \cdot {}_tu_{x+n_1} - {}_tk_{x+n_1}$.

3. Use the relation given in Exercise 3, Art. 117, to show that

$${}_{n_1+1}V_x = {}_{n_1}V_n + \frac{\mathrm{a}_{x+n_1} - \mathrm{a}_{x+n_1+1}}{\mathrm{a}_x}.$$

[Note that this formula is well suited for computing a table of reserves when life annuity tables are available.]

**119. Cost of Insurance.** If death occurs during the $n_1^{\text{th}}$ policy year of an ordinary whole life insurance policy of face value 1 issued at age $x$, $1 - {}_{n_1}V_x$ will be needed at the end of the year in addition to the net level reserve, ${}_{n_1}V_x$, at that time to pay the face value of the policy. This sum $1 - {}_{n_1}V_x$ is called the *amount at*

*risk* during the $n_1^{\text{th}}$ policy year.' The value of the expectation that the amount at risk will be paid at the end of the year is then $q_{x+n_1-1}\,(1 - {}_{n_1}V_x)$, where $q_{x+n_1-1}$ is the probability that a person aged $x + n_1 - 1$ will die within one year. The value of this expectation is the mortality cost during the year. It is called the cost of insurance for the $n_1^{\text{th}}$ policy year, and it will be represented by the symbol ${}_{n_1}K_x$. It follows that

$$ {}_{n_1}K_x = q_{x+n_1-1}\,(1 - {}_{n_1}V_x) $$

A similar formula can be written for any other insurance policy. Analogous formulas can also be written based on full preliminary term reserves and on the modified preliminary term reserves discussed in Arts. 120 to 123 inclusive.

EXAMPLE. Compute the cost of insurance during the 10th policy year of a \$1000 20-year endowment insurance policy issued at age 35.

SOLUTION. In this case the cost of insurance is given by

$$ \begin{aligned} 1000\ {}_{10}K_{35:\overline{20}|} &= 1000\ q_{44}\,(1 - {}_{10}V_{35:\overline{20}|}) \\ &= 1000\ (.010829)(.60401) \\ &= \$6.54 \end{aligned} $$

EXERCISE. Solve this example if the full preliminary term reserve is used instead of the net level reserve.

If the reserve, ${}_{n_1}V_x + P_x$, at the beginning of the $n_1 + 1^{\text{th}}$ policy year of an ordinary whole life insurance policy of face value 1 issued at age $x$ be improved at interest for one year and the cost of insurance be deducted the difference obtained is the terminal reserve at the end of the year. This leads to the important relation,

$$ {}_{n_1+1}V_x = ({}_{n_1}V_x + P_x)(1 + i) - {}_{n_1+1}K_x $$

EXERCISE 1. Verify this relation by substituting expressions for the values of the symbols occurring in it.

EXERCISE 2. Solving

$$ {}_{n_1+1}V_x = ({}_{n_1}V_x + P_x)(1 + i) - q_{x+n_1}(1 - {}_{n_1+1}V_x) \quad \text{for} \quad {}_{n_1+1}V_x \quad \text{gives} $$

$$ {}_{n_1+1}V_x = ({}_{n_1}V_x + P_x)\,\frac{1 + i}{1 - q_{x+n_1}} - \frac{q_{x+n_1}}{1 - q_{x+n_1}} $$

Show that this is Fackler's formula in another form.

### EXERCISE

The following table shows the cost of insurance under net level reserves during the 1st, 5th, 10th, 15th, 20th, and 25th years for some $1000 insurance policies issued at age 35:

| Year | Ordinary Life | 20-Payment Life | 10-Payment Life | 15-Year Endowment | 20-Year Endowment |
|---|---|---|---|---|---|
| 1 | 8.84 | 8.77 | 8.61 | 8.51 | 8.65 |
| 5 | 8.99 | 8.58 | 7.62 | 7.04 | 7.88 |
| 10 | 9.36 | 8.32 | 5.89 | 0.00 | 6.54 |
| 15 | 10.23 | 8.07 | 6.44 | | 4.40 |
| 20 | 11.99 | 7.55 | 7.55 | | 0.00 |
| 25 | 14.65 | 9.22 | 9.22 | | |

Verify a few of these costs.

**120. Modified preliminary term valuation. Ordinary whole life basis.** Loadings based on net level premiums are uniform in size. They were well suited for paying policy expenses when these were spread over the whole premium payment term. They are not well suited at present, however, since much of the expense, such as agents' commissions and medical and inspection fees, occurs during the first year of insurance. Under the net level plan the first-year loading is not sufficient to cover the first-year expense of a policy and it must be supplemented from the surplus funds.

Loadings based on full preliminary term net premiums are much larger for the first year and somewhat smaller for subsequent years than those based on the net level premiums of a given policy. These loadings are not satisfactory for limited payment life and for endowment insurance policies which have short premium payment terms, since for such policies the first-year loadings are greater than what is needed for first year expenses. For example, a $1000 10-year endowment insurance policy issued at age 25 has a natural premium of $7.79 and a level net premium of $86.45. If this policy carries a gross premium of $103.39, the first-year loading under the net level plan is $16.94, while that under the full

preliminary term plan is \$95.60. This last amount is greatly in excess of what should be allowed for first-year loading.

The insurance laws of each state in the United States specify the *minimum* amount which a company doing business in the state can hold as a reserve on each policy. To remove the objection to excessive first-year loading for some policies under full preliminary term valuation, certain modified preliminary term methods are used. In this article the so-called *modified preliminary term valuation* * is presented. In this method the ordinary whole life insurance policy is used as a basis for the modifications.**

If $m$ is the number of years in the premium payment term of a given policy issued at age $x$, it was seen in Art. 117 that the full preliminary term premium reserve at the end of the $m^{th}$ policy year equals the net level premium reserve at this time, but is less than the net level at the end of any earlier policy year. Under modified preliminary term valuation the reserve of this policy is brought up to the net level at the end of $m$ years by adding to the full preliminary term reserve at the end of $n_1$ years, where $n_1$ equals $1, 2, \cdots, m$, of an ordinary whole life policy issued at age $x$ and having the same face value as the given policy, the value at age $x + n_1$ of a life annuity due whose term begins at age $x$ and ends at age $x + n_1$ and whose annual rent is the net annual premium of a pure endowment issued at age $x$ and due in $m$ years with face value equal to the net level premium reserve at age $x + m$ of the given policy less the full preliminary term reserve at this age of the ordinary whole life policy. Ordinary whole life policies and ordinary term policies are not affected by this method of valuation. If the pure endowment annual premium corresponding to any policy of face value 1 issued at age $x$ is denoted by $\pi_x$, the first year premium under modified preliminary term valuation is $A^1_{x\,\overline{1}|} + \pi_x$ and each subsequent premium is $P_{x+1} + \pi_x$.

EXAMPLE 1. Use modified preliminary term valuation to find the net annual premiums of a \$1000 15-year endowment insurance policy issued at age 25.

* This is sometimes called straight modified preliminary term valuation.

** The Ohio Standard uses this method for all limited payment life and endowment policies having fewer than twenty annual premiums.

SOLUTION. In this case $1000(1 - {}_{14}V_{26})$ is the difference between the net level premium reserve of the given policy at the end of 15 years and the full preliminary term reserve of the ordinary whole life policy. By Art. 109 it follows that, for this policy,

$$1000\,\pi_{25} = 1000(1 - {}_{14}V_{26})\,\frac{D_{40}}{N_{25} - N_{40}}$$
$$= \$39.83$$

Hence the required premiums are

First year: $1000\,A^{\frac{1}{25}\,\Pi} + 39.83 = 7.79 + 39.83 = \$47.62$

Last fourteen years: $1000\,P_{26} + 39.83 = 15.48 + 39.83 = \$55.31$

EXERCISE 1. Show that, under modified preliminary term valuation, the net annual premiums of an $n$-year endowment insurance policy of face value 1 issued at age $x$ are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } n-1 \text{ years:} & P_{x+1} + \pi_x\end{array}\right\} \text{where } \pi_x = \frac{1 - {}_{n-1}V_{x+1}}{{}_n\ddot{u}_x}$$

EXERCISE 2. Show that, under modified preliminary term valuation, the net annual premiums of an $m$-payment, $n$-year endowment insurance policy of face value 1 issued at age $x$ are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } m-1 \text{ years:} & P_{x+1} + \pi_x\end{array}\right\} \text{where } \pi_x = \frac{A_{x+m:\overline{n-m}\rceil} - {}_{m-1}V_{x+1}}{{}_m\ddot{u}_x}$$

EXERCISE 3. Express the values of $\pi_x$ in Exercises 1 and 2 in commutation symbols.

EXAMPLE 2. Use modified preliminary term valuation to find the net annual premiums of a \$1000 10-payment whole life insurance policy issued at age 25.

SOLUTION. In this case $1000(A_{35} - {}_9V_{26})$ is the difference between the net level premium reserve of the given policy at the end of 10 years and the full preliminary term reserve of the ordinary whole life policy at this time. By Art. 109 it follows, that for this policy,

$$1000\,\pi_{25} = 1000(A_{35} - {}_9V_{26})\,\frac{D_{35}}{N_{25} - N_{35}}$$
$$= \$22.58$$

Hence, the required premiums are

First year: $1000\,A^{\frac{1}{25}\,\Pi} + 22.58 = 7.79 + 22.58 = \$30.37$

Last nine years: $1000\,P_{26} + 22.58 = 15.48 + 22.58 = \$38.06$

EXERCISE 4. Show that, under modified preliminary term valuation, the net annual premiums of an $m$-payment whole life insurance policy issued at age $x$ are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } m-1 \text{ years:} & P_{x+1} + \pi_x\end{array}\right\} \text{where } \pi_x = \frac{A_{x+m} - {}_{m-1}V_{x+1}}{{}_m\ddot{u}_x}$$

EXERCISE 5. Express $\pi_x$ in Exercise 4 in commutation symbols.

In computing reserves under modified preliminary term valuation, it is convenient to resolve the net premiums into the two sets of which they are composed, viz.: the pure endowment premiums, and the full preliminary term ordinary whole life premiums. Tables of net level premium reserves can then be used in the computations.

EXAMPLE 3. Compute the terminal reserves under modified preliminary term valuation after 1, 5, and 10 years of a \$1000 10-payment life insurance policy issued at age 25.

SOLUTION. The net premiums for this policy are given in the solution of Example 2. Upon resolving these premiums into the two component sets used in finding them, it is seen by the retrospective method that the terminal reserves to be found are

First year: $22.58\, u_{25} = \$23.55$
Fifth year: $1000\, {}_4V_{26} + 22.58\, {}_5u_{25} = 33.54 + 128.55 = \$162.09$
Tenth year: $1000\, {}_9V_{26} + 22.58\, {}_{10}u_{25} = 82.42 + 288.16 = \$370.58$

EXERCISE 6. Solve Example 3 by use of the prospective method of computing reserves. [Note that the reserve after 10 years is the net single premium, $A_{35}$.]

## EXERCISES

**1.** The following table shows the net premiums under modified preliminary term valuation of some \$1000 insurance policies issued at age 35:

| YEAR | 20-PAYMENT LIFE | 10-PAYMENT LIFE | 15-YEAR ENDOWMENT | 20-YEAR ENDOWMENT |
|---|---|---|---|---|
| First . . . | 16.38 | 34.32 | 44.54 | 29.10 |
| Subsequent . | 28.28 | 46.22 | 56.44 | 41.00 |

Verify the premiums for the 20-year endowment insurance policy.

**2.** Show that, for an $m$-payment life insurance policy of face value 1 issued at age $x$, the terminal reserve under modified preliminary term valuation after

1 year is $\pi_x u_x$
$t$ years $(1 < t < m)$ is ${}_{t-1}V_{x+1} + \pi_x\, {}_tu_x$
$t$ years $(t \geq m)$ is $A_{x+t}$

where $\pi_x$ is given in Exercise 4 under Example 2.

**3.** Show that, for an $n$-year endowment insurance policy of face value 1 issued at age $x$, the terminal reserve under modified preliminary term valuation after

1 year is $\pi_x u_x$
$t$ years $(1 < t < n)$ is ${}_{t-1}V_{x+1} + \pi_x\, {}_tu_x$
$n$ years is 1

where $\pi_x$ is given in Exercise 1 under Example 1.

4. The following table shows the terminal reserves after 1, 2, 5, 10, 15, and 20 years under modified preliminary term valuation for some $1000 insurance policies issued at age 35:

| Year | 20-Payment Life | 10-Payment Life | 15-Year Endowment | 20-Year Endowment |
|---|---|---|---|---|
| 1 | 8.09 | 26.81 | 37.48 | 21.36 |
| 2 | 28.79 | 67.10 | 88.92 | 55.95 |
| 5 | 95.73 | 198.22 | 256.59 | 168.40 |
| 10 | 225.00 | 456.00 | 587.58 | 388.80 |
| 15 | 380.02 | 508.49 | 1000.00 | 660.10 |
| 20 | 566.15 | 566.15 | | 1000.00 |

Verify a few of these reserves.

5. Write expressions analogous to those given in Exercise 3 for an $m$-payment, $n$-year endowment insurance policy.

**121. Illinois Standard.** In the preliminary term method of valuation presented in Art. 120 the full preliminary term ordinary whole life policy is used as a basis. Another preliminary term method uses the full preliminary term 20-payment life insurance policy as a basis. Under the Illinois Standard the 20-payment life policy is used as a basis for valuing all limited payment life insurance policies and all endowment insurance policies whose full preliminary term renewal premiums at age of issue are greater than the full preliminary term renewal premiums of a 20-payment life policy having the same face value and issued at the same age.* Other policies may be valued on the full preliminary term plan. The term policies commonly issued by insurance companies are usually valued, however, on the net level premium basis. A 15-payment life insurance policy issued at any age, a 15-year endowment insurance policy issued at any age, and a 30-year endowment insurance policy issued at any age less than 30 are illustrations of policies valued on the 20-payment life basis. An

* Fackler and Fackler. Illinois Standard Tables, Vol. I, Spectator Company, 1922.

ordinary whole life insurance policy issued at any age, a 20-payment life insurance policy issued at any age, and a 30-year endowment insurance policy issued at any age greater than 30 are illustrations of policies valued on the full preliminary term basis.

The method of valuing a policy on the 20-payment life basis is similar to that presented in Art. 120 on the ordinary whole life basis. Let $m$ denote the number of years in the premium payment term of the policy, issued at age $x$, to be valued on the 20-payment life basis under the Illinois Standard. When $m \leq 20$, the reserve of the policy is brought up to the net level premium reserve at the end of $m$ years by adding to the full preliminary term premium reserve at the end of $n_1$ years, where $n_1 = 1, 2, \ldots, m$, of a 20-payment life insurance policy issued at age $x$ and having the same face value as the given policy, the value at age $x + n_1$ of a life annuity due whose term begins at age $x$ and ends at age $x + n_1$ and whose annual rent is the net annual premium of a pure endowment issued at age $x$ and due in $m$ years with face value equal to the net level premium reserve at age $x + m$ of the given policy less the full preliminary term reserve at this age of the 20-payment life policy. When $m > 20$, the method is that obtained by replacing $m$ by 20 in the method just described when $m \leq 20$; for these policies the reserve is brought up to the net level premium reserve at the end of 20 years. If the pure endowment annual premium corresponding to any policy of face value 1 issued at age $x$ is denoted by $\pi_x$, the first year premium under the Illinois Standard is $A^1_{x\,\overline{1}|} + \pi_x$; when $m \leq 20$, each subsequent premium is ${}_{19}P_{x+1} + \pi_x$; when $m > 20$ each premium from the second to the twentieth inclusive is ${}_{19}P_{x+1} + \pi_x$ and each premium subsequent to the twentieth is the net level premium of the given policy.

EXAMPLE 1. Find the net annual premiums under the Illinois Standard of a \$1000 15-year endowment insurance policy issued at age 25.

SOLUTION. In this case, $1000(1 - {}_{14:19}V_{26})$ is the difference between the net level premium reserve of the given policy at the end of 15 years and the full preliminary term reserve of the 20-payment life policy at this time. By Art. 109 it follows that, for this policy,

$$1000\,\pi_{25} = 1000(1 - {}_{14:19}V_{26})\,\frac{D_{40}}{N_{25} - N_{40}}$$

$$= \$32.35$$

Hence the required premiums are

First year: $1000\,A^{1}_{25:\overline{1}|} + 32.35 = 7.79 + 32.35 = \$40.14$

Last fourteen years: $1000\,{}_{19}P_{26} + 32.35 = 23.68 + 32.35 = \$56.03$

EXERCISE 1. Show that when $n \leq 20$, the net annual premiums under the Illinois Standard of an $n$-year endowment insurance policy of face value 1 issued at age $x$ are

First year: $\dfrac{C_x}{D_x} + \pi_x$

Last $n-1$ years: ${}_{n-1}P_{x+1} + \pi_x$

where $\pi_x = \dfrac{1 - {}_{n-1:\,19}V_{x+1}}{{}_{n}u_x}$

EXERCISE 2. Show that when $m \leq 20$, the net annual premiums under the Illinois Standard of an $m$-payment $n$-year endowment insurance policy of face value 1 issued at age $x$ are

First year: $\dfrac{C_x}{D_x} + \pi_x$

Last $m-1$ years: ${}_{19}P_{x+1} + \pi_x$

where $\pi_x = \dfrac{A_{x+m:\overline{n-m}|} - {}_{m-1:\,19}V_{x+1}}{\cdot\, {}_{m}u_x}$

EXAMPLE 2. Find the net annual premiums under the Illinois Standard of a \$1000 30-year endowment insurance policy issued at age 30.

SOLUTION. In this case, $1000({}_{20}V_{30:\overline{30}|} - A_{50})$ is the difference between the net level premium reserve of the given policy at the end of 20 years and the full preliminary term reserve of the 20-payment life policy at this time. By Art. 109 it follows that, for this policy,

$$1000\,\pi_{30} = 1000({}_{20}V_{30:\overline{30}|} - A_{50})\,\frac{D_{50}}{N_{30} - N_{50}}$$
$$= \$0.50$$

Hence the required premiums are

First year: $1000\,A^{1}_{30\,\overline{1}|} + 0.50 = 8.14 + 0.50 = \$8.64$

Next nineteen years: $1000\,{}_{19}P_{31} + 0.50 = 26.02 + 0.50 = \$26.52$

Last ten years: $1000\,P_{30:\overline{30}|} = \$25.21$

EXERCISE 3. Show that when $n > 20$, the net annual premiums of an $n$-year endowment insurance policy of face value 1 issued at age $x$, which, under the Illinois Standard, is valued on a 20-payment life basis are

First year: $\dfrac{C_x}{D_x} + \pi_x$

Next nineteen years: ${}_{19}P_{x+1} + \pi_x$

Last $n-20$ years: $P_{x\overline{n}|}$

where $\pi_x = \dfrac{{}_{20}V_{x\overline{n}|} - A_{x+20}}{{}_{20}u_x}$

EXERCISE 4. When $\pi_x = 0$ in Exercise 3, show that $P_{x\overline{n}|} = {}_{20}P_x$. In this case the net level $n$-year endowment insurance premium at age $x$ equals the net level 20-payment life insurance premium at age $x$.

EXAMPLE 3. Find the net annual premiums under the Illinois Standard of a \$1000 10-payment life insurance policy issued at age 25.

SOLUTION. In this case, $1000\,(A_{35} - {}_{9:19}V_{26})$ is the difference between the net level premium reserve of the given policy at the end of 10 years and the full preliminary term reserve of the 20-payment life policy at this time. By Art. 109 it follows that for this policy

$$1000\,\pi_{25} = 1000\,(A_{35} - {}_{9:19}V_{26})\,\frac{D_{35}}{N_{25} - N_{35}}$$
$$= \$15.36$$

Hence the required premiums are

First year: $1000\,A^{1}_{25:\overline{1}|} + 15.36 = 7.79 + 15.36 = \$23.15$

Last nine years: $1000\,{}_{19}P_{26} + 15.36 = 23.68 + 15.36 = \$39.04$

EXERCISE 5. Show that when $m < 20$, the net annual premiums under the Illinois Standard of an $m$-payment life insurance policy of face value 1 issued at age $x$ are

First year: $\dfrac{C_x}{D_x} + \pi_x$

Last $m - 1$ years: ${}_{19}P_{x+1} + \pi_x$

$$\text{where } \pi_x = \frac{A_{x+m} - {}_{m-1:19}V_{x+1}}{{}_{m}u_x}$$

EXAMPLE 4. Compute the terminal reserves under the Illinois Standard after 1, 8, and 15 years of a $1000 15-year endowment insurance policy issued at age 25.

SOLUTION. The net annual premiums for this policy are given in the solution of Example 1. Upon resolving these premiums into the two component sets used in finding them it is seen by the retrospective method that the terminal reserves to be found are

First year: $32.35\,u_{25} = \$33.76$

Eighth year: $1000\,{}_{7:19}V_{26} + 32.35\,{}_{8}u_{25} = 130.32 + 315.34 = \$445.66$

Fifteenth year: $1000\,{}_{14:19}V_{26} + 32.35\,{}_{15}u_{25} = 301.48 + 691.51 = \$999.99$

EXERCISE 6. Solve Example 4 by use of the prospective method of computing reserves. [Note that the reserve after 15 years is $1000.]

## EXERCISES

**1.** The following table shows the net annual premiums under the Illinois Standard of some $1000 policies issued at age 35:

| YEAR | 10-PAYMENT LIFE | 15-PAYMENT LIFE | 15-YEAR ENDOWMENT | 20-YEAR ENDOWMENT | 25-YEAR ENDOWMENT | 15-PAYMENT ENDOWMENT AT 65 |
|---|---|---|---|---|---|---|
| First | 26.97 | 14.61 | 36.93 | 21.36 | 12.75 | 20.58 |
| Subsequent | 47.22 | 34.85 | 57.18 | 41.61 | (2 to 20) $33.00<br>(21 to 25) $31.50 | 40.83 |

Verify the premiums for the 25-year endowment and the 15-payment endowment at 65.

**2.** Show that, under the Illinois Standard, the net annual premiums of a 25-year endowment insurance policy are full preliminary term when the age of issue is equal to or greater than 46, and modified preliminary term (20-payment life basis) when the age of issue is less than 46. [See Fackler and Fackler, Illinois Standard Tables.]

**3.** Show that, under the Illinois Standard, the net premiums of an endowment insurance at 65 are full preliminary term when the age of issue is equal to or less than 37, and modified preliminary term when the age of issue is greater than 37.

**4.** Show that for an $m$-payment ($m < 20$), whole life insurance policy of ace value 1 issued at age $x$, the terminal reserve under the Illinois Standard after

1 year is $\pi_x u_x$
$t$ years ($1 < t < m$) is ${}_{t-1:19}V_{x+1} + \pi_x t u_x$
$t$ years ($t \geq m$) is $A_{x+t}$

where $\pi_x$ is given in Exercise 5, Example 3.

**5.** If the $m$-payment life policy in Exercise 4 is replaced by an $n$-year endowment insurance policy ($n \leq 20$), write the corresponding expressions for the terminal reserves. [See Exercise 1, Example 1 for $\pi_x$.]

**6.** Same as Exercise 5 for an $m$-payment $n$-year endowment insurance policy, ($m \leq 20$). [See Exercise 2, Example 1 for $\pi_x$.]

**7.** The following table shows the terminal reserves after 1, 2, 5, 15, 20, and 25 years under the Illinois Standard of some $1000 policies issued at age 35.

| Year | 10-Payment Life | 15-Payment Life | 15-Year Endowment | 20-Year Endowment | 25-Year Endowment | 15-Payment Endowment at 65 |
|---|---|---|---|---|---|---|
| 1 | 19.14 | 6.23 | 29.54 | 13.28 | 4.29 | 12.46 |
| 2 | 60.15 | 33.74 | 81.40 | 48.17 | 29.78 | 46.49 |
| 5 | 193.59 | 122.93 | 250.46 | 161.54 | 112.34 | 157.06 |
| 10 | 456.00 | 296.73 | 584.19 | 383.76 | 272.86 | 373.66 |
| 15 | 509.49 | 508.49 | 1000.00 | 657.29 | 467.68 | 640.03 |
| 20 | 566.15 | 566.15 | | 1000.00 | 706.27 | 735.05 |
| 25 | 626.92 | 626.92 | | | 1000.00 | 850.55 |

Verify a few of these reserves.

**122. New Jersey Standard.** The New Jersey Standard is the same as the Illinois Standard for a limited payment life and for an

endowment insurance policy whose full preliminary term renewal premiums are greater than the full preliminary term renewal premiums of a 20-payment life policy issued at the same age and having the same face value. These policies are valued on the basis of a 20-payment life preliminary term policy by the method described in Art. 121.

Under the New Jersey Standard any policy whose full preliminary term renewal premiums are less than the full preliminary term renewal premiums of a 20-payment life policy issued at the same age and having the same face value and for which the premium charged for the first year's insurance is greater than 150% of the net premium (natural) for this year's insurance, is valued as follows: The terminal reserve of any such policy issued at age $x$ is brought up to the net level premium reserve at the end of twenty years by adding to the full preliminary term premium reserve at the end of $n_1$ years, where $n_1 = 2, 3, \ldots, 20$, of the given policy the value at age $x + n_1$ of a life annuity due whose term begins at age $x + 1$ and ends at age $x + n_1$ and whose annual rent is the net annual premium of a pure endowment issued at age $x + 1$ and due in 19 years with face value equal to the net level premium reserve at age $x + 20$ of the given policy less its full preliminary term premium reserve at this age. These policies have no reserve at the end of the first year. This method applies to limited payment life and to endowment insurance policies which are not valued on the 20-payment life basis as well as to ordinary whole life policies. In what follows in this article $\pi_{x+1}$ will denote the annual pure endowment premium which corresponds to any policy of face value 1.

Policies, other than industrial, which do not come under either of the two methods described may be valued, under the New Jersey Standard, on the full preliminary term plan. As stated in Art. 121, the term policies commonly issued by companies are usually valued, however, on the net level premium basis.

EXAMPLE 1. Find the net annual premiums, under the New Jersey Standard, of a \$1000 ordinary whole life policy issued at age 25.

SOLUTION. In this case, $1000\ ({}_{20}V_{25} - {}_{19}V_{26})$ is the difference between the net level premium reserve and the full preliminary term premium reserve of

the given policy at the end of 20 years. By Art. 109 it follows that, for this policy,

$$1000\,\pi_{26} = 1000({}_{20}V_{25} - {}_{19}V_{26})\,\frac{D_{45}}{N_{26} - N_{45}} = \$0.20$$

Hence the required premiums are

First year: $1000\,A^{1}_{25:\overline{1}|} = \$7.79$

Next nineteen years: $1000\,P_{26} + 0.20 = 15.48 + 0.20 = \$15.68$

After twenty years: $1000\,P_{25} = \$15.10$

EXERCISE 1. Show that, under the New Jersey Standard, the net annual premiums of an ordinary whole life policy of face value 1 issued at age $x$ are

First year: $\dfrac{C_x}{D_x}$

Next nineteen years: $P_{x+1} + \pi_x$ where $\pi_x = ({}_{20}V_x - {}_{19}V_{x+1})\,\dfrac{D_{x+20}}{N_{x+1} - N_{x+20}}$

After twenty years: $P_x$

EXAMPLE 2. Find the net annual premiums, under the New Jersey Standard, of a \$1000 25-payment life insurance policy issued at age 25.

SOLUTION. In this case, $1000\,({}_{20:\overline{25}|}V_{25} - {}_{19:\overline{24}|}V_{26})$ is the difference between the net level premium reserve and the full preliminary term premium reserve of the given policy at the end of 20 years. By Art. 109, it follows that, for this policy

$$1000\,\pi_{26} = 1000({}_{20:\overline{25}|}V_{25} - {}_{19:\overline{24}|}V_{26})\,\frac{D_{45}}{N_{26} - N_{45}} = \$0.13$$

Hence the required premiums are

First year: $1000\,A^{1}_{25:\overline{1}|} = \$7.79$

Next nineteen years: $1000\,{}_{24}P_{26} + 0.13 = 20.58 + 0.13 = \$20.71$

Last five years: $1000\,{}_{25}P_{25} = \$19.77$

EXERCISE 2. Show that, under the New Jersey Standard, the net annual premiums of an $m$-payment ($m > 20$) life insurance policy of face value 1 issued at age $x$ are

First year: $\dfrac{C_x}{D_x}$

Next nineteen years: ${}_{m-1}P_{x+1} + \pi_{x+1}$

where $\pi_{x+1} = ({}_{20:\overline{m}|}V_x - {}_{19:\overline{m-1}|}V_{x+1})\,\dfrac{D_{x+20}}{N_{x+1} - N_{x+20}}$

After twenty years: ${}_{m}P_x$

EXAMPLE 3. Find the net annual premiums, under the New Jersey Standard, of a \$1000 30-year endowment insurance policy issued at age 35.

SOLUTION. In this case, $\pi_{36}$ is given by

$$1000\,\pi_{36} = 1000\,({}_{20}V_{35:\overline{30}|} - {}_{19}V_{36:\overline{29}|})\,\frac{D_{55}}{N_{36} - N_{55}} = \$0.27$$

and the required premiums are

First year: $1000\,A^{1}_{35:\overline{1}|} = \$8.64$

Next nineteen years: $1000\,P_{36:\overline{29}|} + 0.27 = \$27.44 + 0.27 = \$27.71$

Last ten years: $1000\,P_{35:\overline{30}|} = \$26.31$

EXERCISE 3. Show that, under the New Jersey Standard, the net annual premiums of an $n$-year endowment insurance policy of face value 1 issued at age $x$ which is valued on a full preliminary policy are

First year: $\dfrac{C_x}{D_x}$

Next nineteen years: $P_{x+1:\overline{n-1}|} + \pi_{x+1}$

where $\pi_{x+1} = ({}_{20}V_{\overline{x\,n}|} - {}_{19}V_{x+1:\overline{n-1}|}) \dfrac{D_{x+20}}{N_{x+1} - N_{x+20}}$

Last $n - 20$ years: $P_{\overline{x\,n}|}$

EXAMPLE 4. Compute the terminal reserves under the New Jersey Standard, after 1, 2, 8, and 25 years of a $1000 ordinary whole life policy issued at age 25.

SOLUTION. The net annual premiums for this policy are given in the solution of Example 1. Since the first premium is the natural premium there is no reserve at the end of one year. Using the retrospective method for the reserves after two and eight years and the prospective method for the reserves after 25 years, the reserves are found to be

Second year: $1000\ {}_1V_{26} + .20\ \dfrac{D_{25}}{D_{26}} = \$8.17$

Eighth year: $1000\ {}_7V_{26} + .20\ \dfrac{N_{26} - N_{33}}{D_{33}} = \$63.55$

Twenty-fifth year: $1000\ A_{50} - 15.10\ \dfrac{N_{50}}{D_{50}} = \$288.97$

## EXERCISES

1. The following table shows the net annual premiums under the New Jersey Standard of some $1000 policies issued at age 35.

| YEAR | ORDINARY WHOLE LIFE | 30-PAYMENT LIFE | 20-PAYMENT LIFE | 20-PAYMENT ENDOWMENT AT 65 |
|---|---|---|---|---|
| First . . . . . | 8.64 | 8.64 | 8.64 | 13.59 |
| Next nineteen . . | 20.81 | 23.37 | 28.89 | 33.84 |
| After twenty . . . | 19.91 | 22.28 | | |

Verify the premiums for the 30-payment life and the 20-payment endowment at 65.

2. Show that for an ordinary whole life insurance policy of face value 1 issued at age $x$ the terminal reserve under the New Jersey Standard after

1 year is 0

$t$ years $(1 < t < 20)$ is ${}_{t-1}V_{x+1} + \pi_{x+1}({}_{t-1}u_{x+1})$

$t$ years $(t \geq 20)$ is $A_{x+t} - P_x \dfrac{N_{x+t}}{D_{x+t}}$

where $\pi_{x+1}$ is given in Exercise 1, Example 1.

**3.** If the ordinary whole life policy in Exercise 2 is replaced by an $m$-payment life policy ($m > 20$), write the corresponding expressions for the terminal reserves. [See Exercise 2, Example 2 for $\pi_{x+1}$.]

**4.** Same as Exercise 3 for an $n$-year endowment insurance policy which, under the New Jersey Standard, is valued on a full preliminary term $n$-year endowment insurance policy. [See Exercise 3, Example 3 for $\pi_{x+1}$.]

**5.** The following table shows the terminal reserves after 1, 2, 5, 10, 20, and 25 years under the New Jersey Standard of some \$1000 policies issued at age 35:

| Year | Ordinary Whole Life | 30-Payment Life | 20-Payment Life | 20-Payment Endowment at 65 |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 5.17 |
| 2 | 12.56 | 15.24 | 21.01 | 31.58 |
| 5 | 52.74 | 64.18 | 88.88 | 117.16 |
| 10 | 128.42 | 157.40 | 219.96 | 283.73 |
| 15 | 215.06 | 266.40 | 377.22 | 486.26 |
| 20 | 310.75 | 391.61 | 566.15 | 723.05 |
| 25 | 407.30 | 528.47 | 626.92 | 850.55 |

Verify a few of these reserves.

**123. Other state standards. Select and ultimate valuation.** By the use of the methods presented in the preceding articles the net premiums and the terminal reserves of policies can be determined under the standards of valuation adopted by most of the states in the United States. New York State, however, is an exception. In this state the so-called *select* and *ultimate method of valuation* is in use.* In this method, use is made of the principle that a company is benefited through the addition of new policy holders because of the light mortality among them for several years. This light mortality is due to selection based on careful medical examination. This method requires companies to use certain specified percentages of mortality shown by the American Experience Table in finding the minimum values of a policy during

* Insurance companies operating in New York may, at their option, use the Illinois Standard of valuation or the select and ultimate method of valuation.

the first five years of insurance. These percentages range from 50% for the first year to 95% for the fifth year.

For the study of the standard of valuation used in a given state, a copy of the law relating to the valuation of policies in the state should be obtained.

**124. Initial and mean reserves.** The terminal reserve of a policy for a given policy year is the value of the policy at the end of this year. This reserve does not include the net premium payable at the beginning of the following year. The value of the policy at the beginning of a given policy year, just after a premium is paid, is called the *initial reserve* for this year. The initial reserve of a policy for its $n^{\text{th}}$ policy year is then the terminal reserve of the policy after $n - 1$ years plus the net premium payable at the beginning of the $n^{\text{th}}$ year. The initial reserve of a policy for a year subsequent to the premium payment term is evidently the same as the terminal reserve for the preceding year. If the value of an ordinary whole life policy of face value 1 issued at age $x$ at the beginning of its $n^{\text{th}}$ policy year is represented by ${}_nI_x$, then

$$ {}_nI_x = {}_{n-1}V_x + P_x $$

The definition of initial reserve leads to a similar formula for any other policy.

In submitting annual reports to state departments of insurance, each company is required to value its policies at a specified date, which is usually December 31$^{\text{st}}$. Since policy years do not usually end December 31$^{\text{st}}$, terminal reserves are not applicable in finding values at this date. The value of any policy at the end of a calendar year might be found by the use of a formula which would give its value $n + k$ years ($k < 1$) after its date of issue. This process would involve tedious computations, however. In America, it is customary to use the average of initial and terminal reserves in finding the values of policies at the end of a calendar year. This practice is based on the assumption that the dates of issue of policies issued during any calendar year are distributed uniformly throughout the year, so that on the average they are issued about the middle of the year. The value of a policy obtained by taking the arithmetic mean of its initial and terminal $n^{\text{th}}$ year

reserves is called the *mean reserve* of the policy for the $n^{\text{th}}$ policy year. The mean reserve for the $n^{\text{th}}$ year of an ordinary whole life policy of face value 1 issued at age $x$ is given by

$$\text{Mean Reserve} = \frac{{}_nI_x + {}_nV_x}{2} = \frac{{}_{n-1}V_x + P_x + {}_nV_x}{2}$$

A similar formula can be written for any policy.

### EXERCISES

1. Find the mean reserves for the tenth policy year under the Illinois Standard of the policies in Exercises 1 and 7, Art. 121.
2. Same as Exercise 1 for the policies in Exercises 1 and 5, Art. 122.
3. Same as Exercise 1 for the policies in Exercises 1 and 4, Art. 120.
4. Same as Exercise 1 for the policies in Exercise 3, Art. 117.

**125. Surplus and dividends.** The surplus provided by the gross premiums (see Art. 107) is derived principally from the following sources:

1. Interest earned in excess of that provided by the rate used in computing premiums and reserves,
2. Mortality savings,
3. Loadings in excess of expenses.

As stated in Art. 113 in stock insurance companies surplus is distributed to the stockholders in the form of dividends on the stock; in mutual insurance companies the surplus is returned to the policy holders; in mixed companies, part of the surplus goes to the stockholders and the rest goes to the policy holders. A method in common use in America of distributing surplus among the policy holders is the contribution plan. This plan was first introduced in 1863 by Sheppard Homans, actuary of the Mutual Life Insurance Company of New York. The dividend formula he introduced can be derived readily from the relation given in Art. 119 which expresses the terminal reserve for the $(n_1 + 1)^{\text{th}}$ policy year in terms of that for the $n_1{}^{\text{th}}$ year and the cost of insurance. This relation for an ordinary life policy can be written in the form,

$$({}_{n_1}V_x + P_x)(1 + i) - {}_{n_1+1}V_x - \frac{d_{x+n_1}}{l_{x+n_1}}(1 - {}_{n_1+1}V_x) = 0$$

This relation is based on the theoretical assumptions that an interest rate, $i$, will be earned and that the mortality table used will give the mortality accurately. In practice, however, the interest rate earned by a company is larger than that assumed in computing premiums and reserves, the actual mortality is more favorable than that given by the mortality table, and the gross premium less the necessary expense is usually greater than the net premium. It follows that if the actual experience of a company is used in place of the theoretical assumptions, the left-hand member of the above relation becomes positive and represents the contribution to surplus during the $(n_1 + 1)^{\text{th}}$ policy year. Let $l'_{x+n_1}$, $d'_{x+n_1}$, and $i'$ represent a company's own experience and let $e$ denote the proportionate expense of the policy for its $(n_1 + 1)^{\text{th}}$ year. Replacing $l_{x+n_1}$, $d_{x+n_1}$, $i$, and $P_x$ in the left-hand member of the above relation by $l'_{x+n_1}$, $d'_{x+n_1}$, $i'$, and $G_x - e$ respectively, where $G_x$ is the gross premium received by the company, gives

$$\textit{Contribution to surplus} = {}_{n_1}V_x(1 + i') + (G_x - e)(1 + i') - {}_{n_1+1}V_x - \frac{d'_{x+n_1}}{l'_{x+n_1}}(1 - {}_{n_1+1}V_x)$$

This is Mr. Homans' formula as first introduced. If $l$ denotes the loading, so that $G_x = P_x + l$, the formula can be written

$$\textit{Contribution to surplus} = [({}_{n_1}V_x + P_x)(1 + i') - {}_{n_1+1}V_x] - \frac{d'_{x+n_1}}{l'_{x+n_1}}(1 - {}_{n_1+1}V_x) + (l - e)(1 + i')$$

The three component parts of this formula correspond to the three sources of surplus mentioned at the beginning of this article. An analogous formula could be written for any policy.

To apply formulas of the type just discussed a participating company would need to use a mortality table based on its own experience as well as the actual interest rate earned. Most companies do not apply such formulas now, but they use in some form or other the three sources of divisible surplus exhibited by these formulas.* The forms in which these sources are used by com-

* For an excellent presentation of methods of surplus distribution actually in use see a paper in Vol. XI, Part I, No. 23, June 1922 of the Record of the American Institute of Actuaries by J. Charles Rietz, actuary of the Midland Mutual Life Insurance Company, Columbus, Ohio.

panies show wide variations. The interest element in a dividend formula is often determined as that percentage of the reserve which equals the difference between the interest rate a company is likely to earn over a period of years and that used in computing reserves. Some companies which determine the interest element in this way use initial reserves, some use terminal reserves, and some use mean reserves. The mortality element in the dividend formula is often determined as a percentage of the cost of insurance, the percentages used being graded usually to correspond to the age. The loading element is often determined by deducting an expense charge from the loading, the expense charge usually being a percentage of the gross premium plus a constant. For further information regarding the ways in which the chief sources of surplus are used in dividend formulas and for information regarding the actual constants used in these formulas reference may be made to the paper by Mr. J. Charles Rietz cited above. In this paper the dividend formulas of thirty companies are given.

**126. Policy options. Cash surrender values, paid-up insurance, extended insurance.** When a life insurance policy matures, the proceeds are paid to the beneficiary in a manner specified in the policy. In case of whole life and endowment insurance policies the proceeds are paid in cash or in some equivalent form. The equivalent form of settlement can be selected from those specified in the policy by the insured at any time during the term of the policy, or by the beneficiary upon death of the insured in case the insured has made no selection. These optional methods of settlement ordinarily provide for instalments which form annuities certain, life annuities, or both annuities certain and life annuities whose rent is payable once or more annually. In some of the options the whole cash value is paid in the form of equivalent instalments; in others, part is paid in cash and the rest in instalments. The unknowns in these optional methods of settlement can be readily computed by use of the formulas for the values of annuities and of insurances. Besides the ordinary forms of policies there are special forms each of which provides a particular kind of settlement. These special forms include the retirement annuities, the guaranteed income, and the child and educational endowments.

At the end of each year during the term of a life insurance policy, the policy has a cash or surrender value which is equal to the terminal reserve less a surrender charge whose maximum amount is regulated by state law. The surrender charge actually made varies with different companies, but it is usually small; often there is no charge at all. At any time when a policy is in force the insured has the privilege of obtaining a loan on it for an amount which may be equal or nearly equal to the cash value at the time. If the insured allows a policy to lapse, the cash value less any indebtedness is payable to him in cash or in some equivalent form. Two common equivalent forms are *extended insurance* and *paid-up insurance*. Under extended insurance, the one who surrenders his policy is insured during a stated term for an amount which equals the face of the policy, or the face of the policy less the indebtedness. Under paid-up insurance, the one who surrenders his policy is insured for a stated amount during the remainder of the term of the policy he surrenders. Policies usually specify the cash and loan values, the amount of paid-up insurance, and the term of the extended insurance for each year during the term of the policy. The term in extended insurance and the face value in paid-up insurance are easily determined by the use of the formula for the value of a life insurance.

EXAMPLE 1. Under the New Jersey Standard, the terminal reserve at the end of 25 years of a \$1000 ordinary whole life policy issued at age 35 is \$407.30. If the full amount of this reserve is allowed as a cash surrender value, how much paid-up insurance will it purchase?

SOLUTION. Substitution into the formula, $V = F\frac{M_x}{D_x}$, for the value at age $x$ of a whole life insurance issued at age $x$ gives

$$407.30 = F(.62692)$$

Solving,

$$F = \$649.68$$

EXAMPLE 2. If the cash surrender value at the end of 25 years of the policy in Example 1 is used to purchase extended insurance of \$1000 face value, find the term of the insurance.

SOLUTION. Substitution into the formula for the value at age $x$ of an $n$-year term insurance issued at age $x$, gives

$$407.30 = 1000\,\frac{M_{60} - M_{60+n}}{D_{60}}$$

Solving,

$$M_{60+n} = M_{60} - .4703\,D_{60}$$
$$= 1151.435$$

By interpolation in Table XIII,

$$60 + n = 77 \text{ yrs. } 195 \text{ days}$$
$$n = 17 \text{ yrs. } 195 \text{ days.}$$

EXAMPLE 3. Solve Example 2 if there is a surrender charge of \$5.

SOLUTION. In this case \$402.30 is the net single premium of the extended insurance and the equation in $n$ becomes,

$$402.30 = \frac{M_{60} - M_{60+n}}{D_{60}}$$

Solving,

$$M_{60+n} = M_{60} - .4023\, D_{60}$$
$$= 1651.357$$

By interpolation,

$$60 + n = 74 \text{ yrs. } 249 \text{ days,}$$
$$n = 14 \text{ yrs. } 249 \text{ days.}$$

EXAMPLE 4. Assuming that the policy in Example 1 is surrendered at the end of 25 years, and that it carries a loan of \$100 at that time, and allowing a surrender charge of \$5, find the term of extended insurance of \$900 face value. [Note that in this case the face of the policy is reduced by the amount of the loan. This is common practice. Discuss reason.]

SOLUTION. In this example the amount available for the purchase of extended insurance is \$302.30 and the equation in $n$ becomes,

$$302.30 = 900\, \frac{M_{60} - M_{60+n}}{D_{60}}$$

Solving,

$$M_{60+n} = M_{60} - \frac{302.3\, D_{60}}{900}$$
$$= 2139.588$$

By interpolation,

$$60 + n = 72 \text{ yrs. } 71 \text{ days.}$$
$$n = 12 \text{ yrs. } 71 \text{ days.}$$

EXAMPLE 5. Under the New Jersey Standard the terminal reserve at the end of 10 years of a \$1000 20-year endowment insurance policy issued at age 35 is \$383.75. If the policy is surrendered at this time, and this reserve less \$5 is used to purchase a \$1000 10-year term insurance and a pure endowment of face value $F$ due in 10 years, find $F$.

SOLUTION. The net single premium of a \$1000 10-year term insurance issued at age 55 is \$106.65. It follows that $383.75 - 106.65 - 5 = \$272.10$ is the net single premium of the pure endowment. Substitution into $V = F\, \frac{D_{x+n}}{D_x}$, gives

$$272.10 = F\, \frac{D_{55}}{D_{45}}$$

Solving,

$$F = 272.1\, \frac{D_{45}}{D_{55}}$$
$$= \$440.96$$

### EXERCISES

1. Solve Example 1 under net level reserves.
2. Solve Example 2 under net level reserves.
3. Solve Example 3 under net level reserves.
4. Solve Example 4 under net level reserves.
5. Solve Example 5 under net level reserves.
6. If the policy in Example 5 is surrendered at the end of 4 years and if the cash value at that time is \$122.23, find the term of the extended insurance of \$1000 face value that can be purchased if a surrender charge of \$5 is made.
7. Find the face value of the paid-up insurance, term 14 years, that can be purchased for the cash value of the policy in Exercise 6.

## Life Estates, Remainders, Inheritance Taxes

**127. Life estates and remainders.** Bequests are often made such that the whole or a part of the income from an estate goes to a given person during his life. The income received by such a person is called a *life estate*. Other bequests leave the whole or a part of the income from an estate to a person during a part of his life. More generally, there are bequests which leave the income from an estate to two or more persons for all or part of their lives. At the end of the terms during which the incomes are bequeathed the estate is given to one or more parties. A party who is bequeathed an estate after its income has been enjoyed by one or more persons holds what is called a *remainder* in the estate. In settling estates of this sort it is usually necessary to find the present values of the whole and the temporary life estates and also of the remainders. The values of the life estates are found by the use of the formulas for the values of annuities. The values of the remainders considered in this article can be found by the use of the formulas for the values of insurances. According to the laws of some states, however, the values of the remainders are found by deducting the values of the life estates from the appraised value of the estate. In the solutions of the examples in this article, the American Experience table at 5% is used for one life and Hunter's Makehamized American Experience table at 5% is used for two lives (Art. 104, Chapter IV). The answers are given to the nearest dollar.

EXAMPLE 1. By the terms of a will the income at 5% annually of a \$10000 estate goes to a widow aged 50 during her life. Find the value of her life estate.

SOLUTION. The value, $V$, of her life estate is given by

$$\begin{aligned} V &= 500\, a_{50} \\ &= 500\,(11.66175) \\ &= \$5831 \end{aligned}$$

EXAMPLE 2. By the terms of the will in Example 1, the estate goes to a hospital when the widow dies. Find the value of this remainder.

SOLUTION 1. In this solution the value $V$ is found by use of the insurance formula

$$\begin{aligned} V &= 10000\, A_{50} \\ &= 10000\,(.3970598) \\ &= \$3971 \end{aligned}$$

SOLUTION 2. In this solution the value $V$ is found by deducting the result found in Example 1 from \$10000. That is

$$\begin{aligned} V &= 10000 - 5830.88 \\ &= \$4169 \end{aligned}$$

(Account for the different results in these solutions.)

EXAMPLE 3. By the terms of a will the income at 5% annually of a \$10000 estate goes to a son aged 25 for 10 years or so long as he lives during the 10 years, after which the estate is bequeathed to a charitable institution. Find the value of each legacy.

SOLUTION. The value of the son's legacy is given by

$$\begin{aligned} V &= 500\, a_{25:\overline{10}|} \\ &= 500\,\frac{N_{26} - N_{36}}{D_{25}} \\ &= 500(7.4037) \\ &= \$3702 \end{aligned}$$

By the method of solution 1, Example 2, the value of the remainder is given by

$$\begin{aligned} V &= 10000\, A_{25:\overline{10}|} \\ &= 10000\,\frac{M_{25} - M_{35} + D_{35}}{D_{25}} \\ &= 10000(.626690) \\ &= \$6267 \end{aligned}$$

By the method of solution 2, Example 2, the value of the remainder is given by

$$\begin{aligned} V &= 10000 - 3701.85 \\ &= \$6298 \end{aligned}$$

EXAMPLE 4. The income at 5% annually of a \$10000 estate is bequeathed to two persons aged 28 and 32 during their joint lives, after which the estate is given to a third party. Find the values of the life estate and the remainder.

SOLUTION. The value of the life estate is given by

$$\begin{aligned} V &= 500\, a_{28:32} \\ &= 500(12.956) \\ &= \$6478 \end{aligned}$$

By the method of solution 1, Example 2, the value of the remainder is given by

$$\begin{aligned} V &= 10000\, A_{28:32} \\ &= 10000(.33542) \\ &= \$3354 \end{aligned}$$

By the method of solution 2, Example 2, the value of the remainder is given by

$$\begin{aligned} V &= 10000 - 6478 \\ &= \$3522 \end{aligned}$$

EXAMPLE 5. The income at 5% annually of a \$10000 estate is bequeathed to $(x)$ aged 28 and $(y)$ aged 32 equally as long as they both live and all to the survivor as long as he lives. Find the value of the bequest.

SOLUTION. The value of the bequest is given by

$$\begin{aligned} V &= 500(a_{28} + a_{32} - a_{28:32}) \\ &= 500(15.292 + 14.857 - 12.956) \\ &= \$8597 \end{aligned}$$

EXAMPLE 6. Find the value of each interest in the estate in Example 5.

SOLUTION. The value of the interest of $(x)$ is given by

$$\begin{aligned} V &= 500(a_{28} - \tfrac{1}{2}\, a_{28:32}) \\ &= \$4407 \end{aligned}$$

The value of the interest of $(y)$ is given by

$$\begin{aligned} V &= 500(a_{32} - \tfrac{1}{2}\, a_{28:32}) \\ &= \$4190 \end{aligned}$$

EXAMPLE 7. A son aged 30 is bequeathed \$500 at the end of each year he lives after the death of his mother, aged 53. Find the value of his legacy.

SOLUTION. The value of his legacy is given by

$$\begin{aligned} V &= 500(a_{30} - a_{30:53}) \\ &= 500(15.08425 - 10.03887) \\ &= \$2522.69 \end{aligned}$$

EXERCISE 1. If the mother is given the income during her life, find the value of the son's legacy by the method of solution 2, Example 2.

EXAMPLE 8. The income at 5% annually of a \$10000 estate goes to a hospital until one of two sons aged 28 and 32 dies. Then the income goes to the survivor during his life. Find the value of the survivor's interest.

SOLUTION. The value of this legacy is given by

$$\begin{aligned} V &= 500(a_{28} + a_{32} - 2\,a_{28:32}) \\ &= 500(4.2375) \\ &= \$2119 \end{aligned}$$

EXAMPLE 9. The estate in Example 5 is given to $(z)$ at the death of the survivor of $(x)$ and $(y)$. Find the value of the legacy to $(z)$.

SOLUTION. The value of this survivorship insurance is given by

$$\begin{aligned} V &= 10000(A_{28} + A_{32} - A_{28:32}) \\ &= 10000(.22419 + .24492 - .33542) \\ &= \$1337 \end{aligned}$$

EXERCISE 2. Find the value of this legacy by deducting the value of the bequest to $(x)$ and $(y)$ from 10000.

The above solutions of simple examples in valuing life estates and remainders are based on life annuities and life insurances involving one or two lives. Similar problems involving one, two, three, or more lives can be solved readily by use of the same methods and tables giving the values of commutation symbols for two, three, or more lives of equal ages. For valuing more complex estates additional formulas are needed for the values of contingent annuities and insurances.* In valuing estates in a given state the law of the state pertaining to such valuation should be consulted.

**EXERCISE**

Solve each of the above examples by use of $3\frac{1}{2}\%$ tables.

**128. Inheritance taxes.** The determination of inheritance taxes affords one of the important applications of the evaluation of estates. The tax laws of each state which levies inheritance taxes specify the rates and the exemptions for various classes of successors to a given estate. By means of these exemptions and rates the amount of inheritance taxes that must be paid by a given successor can be easily computed from the value of the inheritance. Under the Ohio Law, for example, there are four classes of successors, the first of which consists of a wife or a minor child. For this class the exemption is \$5000 and the rates are as follows:

*See Institute of Actuaries Textbook, Part II, for a full treatment of annuities and insurances.

1% on \$25000 or part thereof over the exemption; 2% on next \$75000 or part thereof; 3% on next \$100000 or part thereof; 4% on the balance. According to this law a widow or a minor child who inherits property whose total value is \$250000 would pay, $25000(.01) + 75000(.02) + 100000(.03) + 45000(.04) = \$6550$ in inheritance taxes.

# TABLES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100–115 TABLE I

| | Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 434 | 433 | 432 | | 431 | 430 | 429 | | 428 | 427 | 426 | | 425 | 424 | 423 |
| 1 | 43.4 | 43.3 | 43.2 | 1 | 43.1 | 43.0 | 42.9 | 1 | 42.8 | 42.7 | 42.6 | 1 | 42.5 | 42.4 | 42.3 |
| 2 | 86.8 | 86.6 | 86.4 | 2 | 86.2 | 86.0 | 85.8 | 2 | 85.6 | 85.4 | 85.2 | 2 | 85.0 | 84.8 | 84.6 |
| 3 | 130.2 | 129.9 | 129.6 | 3 | 129.3 | 129.0 | 128.7 | 3 | 128.4 | 128.1 | 127.8 | 3 | 127.5 | 127.2 | 126.9 |
| 4 | 173.6 | 173.2 | 172.8 | 4 | 172.4 | 172.0 | 171.6 | 4 | 171.2 | 170.8 | 170.4 | 4 | 170.0 | 169.6 | 169.2 |
| 5 | 217.0 | 216.5 | 216.0 | 5 | 215.5 | 215.0 | 214.5 | 5 | 214.0 | 213.5 | 213.0 | 5 | 212.5 | 212.0 | 211.5 |
| 6 | 260.4 | 259.8 | 259.2 | 6 | 258.6 | 258.0 | 257.4 | 6 | 256.8 | 256.2 | 255.6 | 6 | 255.0 | 254.4 | 253.8 |
| 7 | 303.8 | 303.1 | 302.4 | 7 | 301.7 | 301.0 | 300.3 | 7 | 299.6 | 298.9 | 298.2 | 7 | 297.5 | 296.8 | 296.1 |
| 8 | 347.2 | 346.4 | 345.6 | 8 | 344.8 | 344.0 | 343.2 | 8 | 342.4 | 341.6 | 340.8 | 8 | 340.0 | 339.2 | 338.4 |
| 9 | 390.6 | 389.7 | 388.8 | 9 | 387.9 | 387.0 | 386.1 | 9 | 385.2 | 384.3 | 383.4 | 9 | 382.5 | 381.6 | 380.7 |
| | 422 | 421 | 420 | | 419 | 418 | 417 | | 416 | 415 | 414 | | 413 | 412 | 411 |
| 1 | 42.2 | 42.1 | 42.0 | 1 | 41.9 | 41.8 | 41.7 | 1 | 41.6 | 41.5 | 41.4 | 1 | 41.3 | 41.2 | 41.1 |
| 2 | 84.4 | 84.2 | 84.0 | 2 | 83.8 | 83.6 | 83.4 | 2 | 83.2 | 83.0 | 82.8 | 2 | 82.6 | 82.4 | 82.2 |
| 3 | 126.6 | 126.3 | 126.0 | 3 | 125.7 | 125.4 | 125.1 | 3 | 124.8 | 124.5 | 124.2 | 3 | 123.9 | 123.6 | 123.3 |
| 4 | 168.8 | 168.4 | 168.0 | 4 | 167.6 | 167.2 | 166.8 | 4 | 166.4 | 166.0 | 165.6 | 4 | 165.2 | 164.8 | 164.4 |
| 5 | 211.0 | 210.5 | 210.0 | 5 | 209.5 | 209.0 | 208.5 | 5 | 208.0 | 207.5 | 207.0 | 5 | 206.5 | 206.0 | 205.5 |
| 6 | 253.2 | 252.6 | 252.0 | 6 | 251.4 | 250.8 | 250.2 | 6 | 249.6 | 249.0 | 248.4 | 6 | 247.8 | 247.2 | 246.6 |
| 7 | 295.4 | 294.7 | 294.0 | 7 | 293.3 | 292.6 | 291.9 | 7 | 291.2 | 290.5 | 289.8 | 7 | 289.1 | 288.4 | 287.7 |
| 8 | 337.6 | 336.8 | 336.0 | 8 | 335.2 | 334.4 | 333.6 | 8 | 332.8 | 332.0 | 331.2 | 8 | 330.4 | 329.6 | 328.8 |
| 9 | 379.8 | 378.9 | 378.0 | 9 | 377.1 | 376.2 | 375.3 | 9 | 374.4 | 373.5 | 372.6 | 9 | 371.7 | 370.8 | 369.9 |
| | 410 | 409 | 408 | | 407 | 406 | 405 | | 404 | 403 | 402 | | 401 | 400 | 399 |
| 1 | 41.0 | 40.9 | 40.8 | 1 | 40.7 | 40.6 | 40.5 | 1 | 40.4 | 40.3 | 40.2 | 1 | 40.1 | 40.0 | 39.9 |
| 2 | 82.0 | 81.8 | 81.6 | 2 | 81.4 | 81.2 | 81.0 | 2 | 80.8 | 80.6 | 80.4 | 2 | 80.2 | 80.0 | 79.8 |
| 3 | 123.0 | 122.7 | 122.4 | 3 | 122.1 | 121.8 | 121.5 | 3 | 121.2 | 120.9 | 120.6 | 3 | 120.3 | 120.0 | 119.7 |
| 4 | 164.0 | 163.6 | 163.2 | 4 | 162.8 | 162.4 | 162.0 | 4 | 161.6 | 161.2 | 160.8 | 4 | 160.4 | 160.0 | 159.6 |
| 5 | 205.0 | 204.5 | 204.0 | 5 | 203.5 | 203.0 | 202.5 | 5 | 202.0 | 201.5 | 201.0 | 5 | 200.5 | 200.0 | 199.5 |
| 6 | 246.0 | 245.4 | 244.8 | 6 | 244.2 | 243.6 | 243.0 | 6 | 242.4 | 241.8 | 241.2 | 6 | 240.6 | 240.0 | 239.4 |
| 7 | 287.0 | 286.3 | 285.6 | 7 | 284.9 | 284.2 | 283.5 | 7 | 282.8 | 282.1 | 281.4 | 7 | 280.7 | 280.0 | 279.3 |
| 8 | 328.0 | 327.2 | 326.4 | 8 | 325.6 | 324.8 | 324.0 | 8 | 323.2 | 322.4 | 321.6 | 8 | 320.8 | 320.0 | 319.2 |
| 9 | 369.0 | 368.1 | 367.2 | 9 | 366.3 | 365.4 | 364.5 | 9 | 363.6 | 362.7 | 361.8 | 9 | 360.9 | 360.0 | 359.1 |
| | 398 | 397 | 396 | | 395 | 394 | 393 | | 392 | 391 | 390 | | 389 | 388 | 387 |
| 1 | 39.8 | 39.7 | 39.6 | 1 | 39.5 | 39.4 | 39.3 | 1 | 39.2 | 39.1 | 39.0 | 1 | 38.9 | 38.8 | 38.7 |
| 2 | 79.6 | 79.4 | 79.2 | 2 | 79.0 | 78.8 | 78.6 | 2 | 78.4 | 78.2 | 78.0 | 2 | 77.8 | 77.6 | 77.4 |
| 3 | 119.4 | 119.1 | 118.8 | 3 | 118.5 | 118.2 | 117.9 | 3 | 117.6 | 117.3 | 117.0 | 3 | 116.7 | 116.4 | 116.1 |
| 4 | 159.2 | 158.8 | 158.4 | 4 | 158.0 | 157.6 | 157.2 | 4 | 156.8 | 156.4 | 156.0 | 4 | 155.6 | 155.2 | 154.8 |
| 5 | 199.0 | 198.5 | 198.0 | 5 | 197.5 | 197.0 | 196.5 | 5 | 196.0 | 195.5 | 195.0 | 5 | 194.5 | 194.0 | 193.5 |
| 6 | 238.8 | 238.2 | 237.6 | 6 | 237.0 | 236.4 | 235.8 | 6 | 235.2 | 234.6 | 234.0 | 6 | 233.4 | 232.8 | 232.2 |
| 7 | 278.6 | 277.9 | 277.2 | 7 | 276.5 | 275.8 | 275.1 | 7 | 274.4 | 273.7 | 273.0 | 7 | 272.3 | 271.6 | 270.9 |
| 8 | 318.4 | 317.6 | 316.8 | 8 | 316.0 | 315.2 | 314.4 | 8 | 313.6 | 312.8 | 312.0 | 8 | 311.2 | 310.4 | 309.6 |
| 9 | 358.2 | 357.3 | 356.4 | 9 | 355.5 | 354.6 | 353.7 | 9 | 352.8 | 351.9 | 351.0 | 9 | 350.1 | 349.2 | 348.3 |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 00 0000 | 0434 | 0868 | 1301 | 1734 | 2166 | 2598 | 3029 | 3461 | 3891 |
| 01 | 4321 | 4751 | 5181 | 5609 | 6038 | 6466 | 6894 | 7321 | 7748 | 8174 |
| 02 | 8600 | 9026 | 9451 | 9876 | *0300 | *0724 | *1147 | *1570 | *1993 | *2415 |
| 03 | 01 2837 | 3259 | 3680 | 4100 | 4521 | 4940 | 5360 | 5779 | 6197 | 6616 |
| 04 | 7033 | 7451 | 7868 | 8284 | 8700 | 9116 | 9532 | 9947 | *0361 | *0775 |
| 05 | 02 1189 | 1603 | 2016 | 2428 | 2841 | 3252 | 3664 | 4075 | 4486 | 4896 |
| 06 | 5306 | 5715 | 6125 | 6533 | 6942 | 7350 | 7757 | 8164 | 8571 | 8978 |
| 07 | 9384 | 9789 | *0195 | *0600 | *1004 | *1408 | *1812 | *2216 | *2619 | *3021 |
| 08 | 03 3424 | 3826 | 4227 | 4628 | 5029 | 5430 | 5830 | 6230 | 6629 | 7028 |
| 09 | 7426 | 7825 | 8223 | 8620 | 9017 | 9414 | 9811 | *0207 | *0602 | *0998 |
| 110 | 04 1393 | 1787 | 2182 | 2576 | 2969 | 3362 | 3755 | 4148 | 4540 | 4932 |
| 11 | 5323 | 5714 | 6105 | 6495 | 6885 | 7275 | 7664 | 8053 | 8442 | 8830 |
| 12 | 9218 | 9606 | 9993 | *0380 | *0766 | *1153 | *1538 | *1924 | *2309 | *2694 |
| 13 | 05 3078 | 3463 | 3846 | 4230 | 4613 | 4996 | 5378 | 5760 | 6142 | 6524 |
| 14 | 6905 | 7286 | 7666 | 8046 | 8426 | 8805 | 9185 | 9563 | 9942 | *0320 |
| 115 | 06 0698 | 1075 | 1452 | 1829 | 2206 | 2582 | 2958 | 3333 | 3709 | 4083 |

| | Proportional Parts | | |
|---|---|---|---|
| | 386 | 385 | 384 |
| 1 | 38.6 | 38.5 | 38.4 |
| 2 | 77.2 | 77.0 | 76.8 |
| 3 | 115.8 | 115.5 | 115.2 |
| 4 | 154.4 | 154.0 | 153.6 |
| 5 | 193.0 | 192.5 | 192.0 |
| 6 | 231.6 | 231.0 | 230.4 |
| 7 | 270.2 | 269.5 | 268.8 |
| 8 | 308.8 | 308.0 | 307.2 |
| 9 | 347.4 | 346.5 | 345.6 |
| | 383 | 382 | 381 |
| 1 | 38.3 | 38.2 | 38.1 |
| 2 | 76.6 | 76.4 | 76.2 |
| 3 | 114.9 | 114.6 | 114.3 |
| 4 | 153.2 | 152.8 | 152.4 |
| 5 | 191.5 | 191.0 | 190.5 |
| 6 | 229.8 | 229.2 | 228.6 |
| 7 | 268.1 | 267.4 | 266.7 |
| 8 | 306.4 | 305.6 | 304.8 |
| 9 | 344.7 | 343.8 | 342.9 |

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## TABLE I

| Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 380 | 379 | 378 | | 377 | 376 | 375 | | 374 | 373 | 372 | | 371 | 370 | 369 |
| 1 | 38.0 | 37.9 | 37.8 | 1 | 37.7 | 37.6 | 37.5 | 1 | 37.4 | 37.3 | 37.2 | 1 | 37.1 | 37.0 | 36.9 |
| 2 | 76.0 | 75.8 | 75.6 | 2 | 75.4 | 75.2 | 75.0 | 2 | 74.8 | 74.6 | 74.4 | 2 | 74.2 | 74.0 | 73.8 |
| 3 | 114.0 | 113.7 | 113.4 | 3 | 113.1 | 112.8 | 112.5 | 3 | 112.2 | 111.9 | 111.6 | 3 | 111.3 | 111.0 | 110.7 |
| 4 | 152.0 | 151.6 | 151.2 | 4 | 150.8 | 150.4 | 150.0 | 4 | 149.6 | 149.2 | 148.8 | 4 | 148.4 | 148.0 | 147.6 |
| 5 | 190.0 | 189.5 | 189.0 | 5 | 188.5 | 188.0 | 187.5 | 5 | 187.0 | 186.5 | 186.0 | 5 | 185.5 | 185.0 | 184.5 |
| 6 | 228.0 | 227.4 | 226.8 | 6 | 226.2 | 225.6 | 225.0 | 6 | 224.4 | 223.8 | 223.2 | 6 | 222.6 | 222.0 | 221.4 |
| 7 | 266.0 | 265.3 | 264.6 | 7 | 263.9 | 263.2 | 262.5 | 7 | 261.8 | 261.1 | 260.4 | 7 | 259.7 | 259.0 | 258.3 |
| 8 | 304.0 | 303.2 | 302.4 | 8 | 301.6 | 300.8 | 300.0 | 8 | 299.2 | 298.4 | 297.6 | 8 | 296.8 | 296.0 | 295.2 |
| 9 | 342.0 | 341.1 | 340.2 | 9 | 339.3 | 338.4 | 337.5 | 9 | 336.6 | 335.7 | 334.8 | 9 | 333.9 | 333.0 | 332.1 |
| | 368 | 367 | 366 | | 365 | 364 | 363 | | 362 | 361 | 360 | | 359 | 358 | 357 |
| 1 | 36.8 | 36.7 | 36.6 | 1 | 36.5 | 36.4 | 36.3 | 1 | 36.2 | 36.1 | 36.0 | 1 | 35.9 | 35.8 | 35.7 |
| 2 | 73.6 | 73.4 | 73.2 | 2 | 73.0 | 72.8 | 72.6 | 2 | 72.4 | 72.2 | 72.0 | 2 | 71.8 | 71.6 | 71.4 |
| 3 | 110.4 | 110.1 | 109.8 | 3 | 109.5 | 109.2 | 108.9 | 3 | 108.6 | 108.3 | 108.0 | 3 | 107.7 | 107.4 | 107.1 |
| 4 | 147.2 | 146.8 | 146.4 | 4 | 146.0 | 145.6 | 145.2 | 4 | 144.8 | 144.4 | 144.0 | 4 | 143.6 | 143.2 | 142.8 |
| 5 | 184.0 | 183.5 | 183.0 | 5 | 182.5 | 182.0 | 181.5 | 5 | 181.0 | 180.5 | 180.0 | 5 | 179.5 | 179.0 | 178.5 |
| 6 | 220.8 | 220.2 | 219.6 | 6 | 219.0 | 218.4 | 217.8 | 6 | 217.2 | 216.6 | 216.0 | 6 | 215.4 | 214.8 | 214.2 |
| 7 | 257.6 | 256.9 | 256.2 | 7 | 255.7 | 254.8 | 254.1 | 7 | 253.4 | 252.7 | 252.0 | 7 | 251.3 | 250.6 | 249.9 |
| 8 | 294.4 | 293.6 | 292.8 | 8 | 292.0 | 291.2 | 290.4 | 8 | 289.6 | 288.8 | 288.0 | 8 | 287.2 | 286.4 | 285.6 |
| 9 | 331.2 | 330.3 | 329.4 | 9 | 328.5 | 327.6 | 326.7 | 9 | 325.8 | 324.9 | 324.0 | 9 | 323.1 | 322.2 | 321.3 |
| | 356 | 355 | 354 | | 353 | 352 | 351 | | 350 | 349 | 348 | | 347 | 346 | 345 |
| 1 | 35.6 | 35.5 | 35.4 | 1 | 35.3 | 35.2 | 35.1 | 1 | 35.0 | 34.9 | 34.8 | 1 | 34.7 | 34.6 | 34.5 |
| 2 | 71.2 | 71.0 | 70.8 | 2 | 70.6 | 70.4 | 70.2 | 2 | 70.0 | 69.8 | 69.6 | 2 | 69.4 | 69.2 | 69.0 |
| 3 | 106.8 | 106.5 | 106.2 | 3 | 105.9 | 105.6 | 105.3 | 3 | 105.0 | 104.7 | 104.4 | 3 | 104.1 | 103.8 | 103.5 |
| 4 | 142.4 | 142.0 | 141.6 | 4 | 141.2 | 140.8 | 140.4 | 4 | 140.0 | 139.6 | 139.2 | 4 | 138.8 | 138.4 | 138.0 |
| 5 | 178.0 | 177.5 | 177.0 | 5 | 176.5 | 176.0 | 175.5 | 5 | 175.0 | 174.5 | 174.0 | 5 | 173.5 | 173.0 | 172.5 |
| 6 | 213.6 | 213.0 | 212.4 | 6 | 211.8 | 211.2 | 210.6 | 6 | 210.0 | 209.4 | 208.8 | 6 | 208.2 | 207.6 | 207.0 |
| 7 | 249.2 | 248.5 | 247.8 | 7 | 247.1 | 246.4 | 245.7 | 7 | 245.0 | 244.3 | 243.6 | 7 | 242.9 | 242.2 | 241.5 |
| 8 | 284.8 | 284.0 | 283.2 | 8 | 282.4 | 281.6 | 280.8 | 8 | 280.0 | 279.2 | 278.4 | 8 | 277.6 | 276.8 | 276.0 |
| 9 | 320.4 | 319.5 | 318.6 | 9 | 317.7 | 316.8 | 315.9 | 9 | 315.0 | 314.1 | 313.2 | 9 | 312.3 | 311.4 | 310.5 |
| | 344 | 343 | 342 | | 341 | 340 | 339 | | 338 | 337 | 336 | | 335 | 334 | |
| 1 | 34.4 | 34.3 | 34.2 | 1 | 34.1 | 34.0 | 33.9 | 1 | 33.8 | 33.7 | 33.6 | 1 | 33.5 | 33.4 | |
| 2 | 68.8 | 68.6 | 68.4 | 2 | 68.2 | 68.0 | 67.8 | 2 | 67.6 | 67.4 | 67.2 | 2 | 67.0 | 66.8 | |
| 3 | 103.2 | 102.9 | 102.6 | 3 | 102.3 | 102.0 | 101.7 | 3 | 101.4 | 101.1 | 100.8 | 3 | 100.5 | 100.2 | |
| 4 | 137.6 | 137.2 | 136.8 | 4 | 136.4 | 136.0 | 135.6 | 4 | 135.2 | 134.8 | 134.4 | 4 | 134.0 | 133.6 | |
| 5 | 172.0 | 171.5 | 171.0 | 5 | 170.5 | 170.0 | 169.5 | 5 | 169.0 | 168.5 | 168.0 | 5 | 167.5 | 167.0 | |
| 6 | 206.4 | 205.8 | 205.2 | 6 | 204.6 | 204.0 | 203.4 | 6 | 202.8 | 202.0 | 201.6 | 6 | 201.0 | 200.4 | |
| 7 | 240.8 | 240.1 | 239.4 | 7 | 238.7 | 238.0 | 237.3 | 7 | 236.6 | 235.9 | 235.2 | 7 | 234.5 | 233.8 | |
| 8 | 275.2 | 274.4 | 273.6 | 8 | 272.8 | 272.0 | 271.2 | 8 | 270.4 | 269.6 | 268.8 | 8 | 268.0 | 267.2 | |
| 9 | 309.6 | 308.7 | 307.8 | 9 | 306.9 | 306.0 | 305.1 | 9 | 304.2 | 303.3 | 302.4 | 9 | 301.5 | 300.6 | |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 115 | 06 0698 | 1075 | 1452 | 1829 | 2206 | 2582 | 2958 | 3333 | 3709 | 4083 |
| 16 | 4458 | 4832 | 5206 | 5580 | 5953 | 6326 | 6699 | 7071 | 7443 | 7815 |
| 17 | 8186 | 8557 | 8928 | 9298 | 9668 | *0038 | *0407 | *0776 | *1145 | *1514 |
| 18 | 07 1882 | 2250 | 2617 | 2985 | 3352 | 3718 | 4085 | 4451 | 4816 | 5182 |
| 19 | 5547 | 5912 | 6276 | 6640 | 7004 | 7368 | 7731 | 8094 | 8457 | 8819 |
| 120 | 07 9181 | 9543 | 9904 | *0266 | *0626 | *0987 | *1347 | *1707 | *2067 | *2426 |
| 21 | 08 2785 | 3144 | 3503 | 3861 | 4219 | 4576 | 4934 | 5291 | 5647 | 6004 |
| 22 | 6360 | 6716 | 7071 | 7426 | 7781 | 8136 | 8490 | 8845 | 9198 | 9552 |
| 23 | 9905 | *0258 | *0611 | *0963 | *1315 | *1667 | *2018 | *2370 | *2721 | *3071 |
| 24 | 09 3422 | 3772 | 4122 | 4471 | 4820 | 5169 | 5518 | 5866 | 6215 | 6562 |
| 25 | 6910 | 7257 | 7604 | 7951 | 8298 | 8644 | 8990 | 9335 | 9681 | *0026 |
| 26 | 10 0371 | 0715 | 1059 | 1403 | 1747 | 2091 | 2434 | 2777 | 3119 | 3462 |
| 27 | 3804 | 4146 | 4487 | 4828 | 5169 | 5510 | 5851 | 6191 | 6531 | 6871 |
| 28 | 7210 | 7549 | 7888 | 8227 | 8565 | 8903 | 9241 | 9579 | 9916 | *0253 |
| 29 | 11 0590 | 0926 | 1263 | 1599 | 1934 | 2270 | 2605 | 2940 | 3275 | 3609 |
| 130 | 11 3943 | 4277 | 4611 | 4944 | 5278 | 5611 | 5943 | 6276 | 6608 | 6940 |

| Proportional Parts | 333 | 332 |
|---|---|---|
| 1 | 33.3 | 33.2 |
| 2 | 66.6 | 66.4 |
| 3 | 99.9 | 99.6 |
| 4 | 133.2 | 132.8 |
| 5 | 166.5 | 166.0 |
| 6 | 199.8 | 199.2 |
| 7 | 233.1 | 232.4 |
| 8 | 266.4 | 265.6 |
| 9 | 299.7 | 298.8 |

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| Proportional Parts | | | | Proportional Parts | | | Proportional Parts | | | Proportional Parts | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 331 | 330 | 329 | 328 | 327 | 326 | 325 | 324 | 323 | 322 | 321 | 320 |
| 1 | 33.1 | 33.0 | 32.9 | 32.8 | 32.7 | 32.6 | 32.5 | 32.4 | 32.3 | 32.2 | 32.1 | 32.0 |
| 2 | 66.2 | 66.0 | 65.8 | 65.6 | 65.4 | 65.2 | 65.0 | 64.8 | 64.6 | 64.4 | 64.2 | 64.0 |
| 3 | 99.3 | 99.0 | 98.7 | 98.4 | 98.1 | 97.8 | 97.5 | 97.2 | 96.9 | 96.6 | 96.3 | 96.0 |
| 4 | 132.4 | 132.0 | 131.6 | 131.2 | 130.8 | 130.4 | 130.0 | 129.6 | 129.2 | 128.8 | 128.4 | 128.0 |
| 5 | 165.5 | 165.0 | 164.5 | 164.0 | 163.5 | 163.0 | 162.5 | 162.0 | 161.5 | 161.0 | 160.5 | 160.0 |
| 6 | 198.6 | 198.0 | 197.4 | 196.8 | 196.2 | 195.6 | 195.0 | 194.4 | 193.8 | 193.2 | 192.6 | 192.0 |
| 7 | 231.7 | 231.0 | 230.3 | 229.6 | 228.9 | 228.2 | 227.5 | 226.8 | 226.1 | 225.4 | 224.7 | 224.0 |
| 8 | 264.8 | 264.0 | 263.2 | 262.4 | 261.6 | 260.8 | 260.0 | 259.2 | 258.4 | 257.6 | 256.8 | 256.0 |
| 9 | 297.9 | 297.0 | 296.1 | 295.2 | 294.3 | 293.4 | 292.5 | 291.6 | 290.7 | 289.8 | 288.9 | 288.0 |
| | 319 | 318 | 317 | 316 | 315 | 314 | 313 | 312 | 311 | 310 | 309 | 308 |
| 1 | 31.9 | 31.8 | 31.7 | 31.6 | 31.5 | 31.4 | 31.3 | 31.2 | 31.1 | 31.0 | 30.9 | 30.8 |
| 2 | 63.8 | 63.6 | 63.4 | 63.2 | 63.0 | 62.8 | 62.6 | 62.4 | 62.2 | 62.0 | 61.8 | 61.6 |
| 3 | 95.7 | 95.4 | 95.1 | 94.8 | 94.5 | 94.2 | 93.9 | 93.6 | 93.3 | 93.0 | 92.7 | 92.4 |
| 4 | 127.6 | 127.2 | 126.8 | 126.4 | 126.0 | 125.6 | 125.2 | 124.8 | 124.4 | 124.0 | 123.6 | 123.2 |
| 5 | 159.5 | 159.0 | 158.5 | 158.0 | 157.5 | 157.0 | 156.5 | 156.0 | 155.5 | 155.0 | 154.5 | 154.0 |
| 6 | 191.4 | 190.8 | 190.2 | 189.6 | 189.0 | 188.4 | 187.8 | 187.2 | 186.6 | 186.0 | 185.4 | 184.8 |
| 7 | 223.3 | 222.6 | 221.9 | 221.2 | 220.5 | 219.8 | 219.1 | 218.4 | 217.7 | 217.0 | 216.3 | 215.6 |
| 8 | 255.2 | 254.4 | 253.6 | 252.8 | 252.0 | 251.2 | 250.4 | 249.6 | 248.8 | 248.0 | 247.2 | 246.4 |
| 9 | 287.1 | 286.2 | 285.3 | 284.4 | 283.5 | 282.6 | 281.7 | 280.8 | 279.9 | 279.0 | 278.1 | 277.2 |
| | 307 | 306 | 305 | 304 | 303 | 302 | 301 | 300 | 299 | 298 | 297 | 296 |
| 1 | 30.7 | 30.6 | 30.5 | 30.4 | 30.3 | 30.2 | 30.1 | 30.0 | 29.9 | 29.8 | 29.7 | 29.6 |
| 2 | 61.4 | 61.2 | 61.0 | 60.8 | 60.6 | 60.4 | 60.2 | 60.0 | 59.8 | 59.6 | 59.4 | 59.2 |
| 3 | 92.1 | 91.8 | 91.5 | 91.2 | 90.9 | 90.6 | 90.3 | 90.0 | 89.7 | 89.4 | 89.1 | 88.8 |
| 4 | 122.8 | 122.4 | 122.0 | 121.6 | 121.2 | 120.8 | 120.4 | 120.0 | 119.6 | 119.2 | 118.8 | 118.4 |
| 5 | 153.5 | 153.0 | 152.5 | 152.0 | 151.5 | 151.0 | 150.5 | 150.0 | 149.5 | 149.0 | 148.5 | 148.0 |
| 6 | 184.2 | 183.6 | 183.0 | 182.4 | 181.8 | 181.2 | 180.6 | 180.0 | 179.4 | 178.8 | 178.2 | 177.6 |
| 7 | 214.9 | 214.2 | 213.5 | 212.8 | 212.1 | 211.4 | 210.7 | 210.0 | 209.3 | 208.6 | 207.9 | 207.2 |
| 8 | 245.6 | 244.8 | 244.0 | 243.2 | 242.4 | 241.6 | 240.8 | 240.0 | 239.2 | 238.4 | 237.6 | 236.8 |
| 9 | 276.3 | 275.4 | 274.5 | 273.6 | 272.7 | 271.8 | 270.9 | 270.0 | 269.1 | 268.2 | 267.3 | 266.4 |
| | | | | | | | | | | 295 | 294 | 293 |
| 1 | | | | | | | | | | 29.5 | 29.4 | 29.3 |
| 2 | | | | | | | | | | 59.0 | 58.8 | 58.6 |
| 3 | | | | | | | | | | 88.5 | 88.2 | 87.9 |
| 4 | | | | | | | | | | 118.0 | 117.6 | 117.2 |
| 5 | | | | | | | | | | 147.5 | 147.0 | 146.5 |
| 6 | | | | | | | | | | 177.0 | 176.4 | 175.8 |
| 7 | | | | | | | | | | 206.5 | 205.8 | 205.1 |
| 8 | | | | | | | | | | 236.0 | 235.2 | 234.4 |
| 9 | | | | | | | | | | 265.5 | 264.6 | 263.7 |
| | | | | | | | | | | 292 | 291 | 290 |
| 1 | | | | | | | | | | 29.2 | 29.1 | 29.0 |
| 2 | | | | | | | | | | 58.4 | 58.2 | 58.0 |
| 3 | | | | | | | | | | 87.6 | 87.3 | 87.0 |
| 4 | | | | | | | | | | 116.8 | 116.4 | 116.0 |
| 5 | | | | | | | | | | 146.0 | 145.5 | 145.0 |
| 6 | | | | | | | | | | 175.2 | 174.6 | 174.0 |
| 7 | | | | | | | | | | 204.4 | 203.7 | 203.0 |
| 8 | | | | | | | | | | 233.6 | 232.8 | 232.0 |
| 9 | | | | | | | | | | 262.8 | 261.9 | 261.0 |
| | | | | | | | | | | 289 | 288 | 287 |
| 1 | | | | | | | | | | 28.9 | 28.8 | 28.7 |
| 2 | | | | | | | | | | 57.8 | 57.6 | 57.4 |
| 3 | | | | | | | | | | 86.7 | 86.4 | 86.1 |
| 4 | | | | | | | | | | 115.6 | 115.2 | 114.8 |
| 5 | | | | | | | | | | 144.5 | 144.0 | 143.5 |
| 6 | | | | | | | | | | 173.4 | 172.8 | 172.2 |
| 7 | | | | | | | | | | 202.3 | 201.6 | 200.9 |
| 8 | | | | | | | | | | 231.2 | 230.4 | 229.6 |
| 9 | | | | | | | | | | 260.1 | 259.2 | 258.3 |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 130 | 11 3943 | 4277 | 4611 | 4944 | 5278 | 5611 | 5943 | 6276 | 6608 | 6940 |
| 31 | 7271 | 7603 | 7934 | 8265 | 8595 | 8926 | 9256 | 9586 | 9915 | *0245 |
| 32 | 12 0574 | 0903 | 1231 | 1560 | 1888 | 2216 | 2544 | 2871 | 3198 | 3525 |
| 33 | 3852 | 4178 | 4504 | 4830 | 5156 | 5481 | 5806 | 6131 | 6456 | 6781 |
| 34 | 7105 | 7429 | 7753 | 8076 | 8399 | 8722 | 9045 | 9368 | 9690 | *0012 |
| 35 | 13 0334 | 0655 | 0977 | 1298 | 1619 | 1939 | 2260 | 2580 | 2900 | 3219 |
| 36 | 3539 | 3858 | 4177 | 4496 | 4814 | 5133 | 5451 | 5769 | 6086 | 6403 |
| 37 | 6721 | 7037 | 7354 | 7671 | 7987 | 8303 | 8618 | 8934 | 9249 | 9564 |
| 38 | 9879 | *0194 | *0508 | *0822 | *1136 | *1450 | *1763 | *2076 | *2389 | *2702 |
| 39 | 14 3015 | 3327 | 3639 | 3951 | 4263 | 4574 | 4885 | 5196 | 5507 | 5818 |
| 140 | 14 6128 | 6438 | 6748 | 7058 | 7367 | 7676 | 7985 | 8294 | 8603 | 8911 |
| 41 | 9219 | 9527 | 9835 | *0142 | *0449 | *0756 | *1063 | *1370 | *1676 | *1982 |
| 42 | 15 2288 | 2594 | 2900 | 3205 | 3510 | 3815 | 4120 | 4424 | 4728 | 5032 |
| 43 | 5336 | 5640 | 5943 | 6246 | 6549 | 6852 | 7154 | 7457 | 7759 | 8061 |
| 44 | 8362 | 8664 | 8965 | 9266 | 9567 | 9868 | *0168 | *0469 | *0769 | *1068 |
| 45 | 16 1368 | 1667 | 1967 | 2266 | 2564 | 2863 | 3161 | 3460 | 3758 | 4055 |
| 46 | 4353 | 4650 | 4947 | 5244 | 5541 | 5838 | 6134 | 6430 | 6726 | 7022 |
| 47 | 7317 | 7613 | 7908 | 8203 | 8497 | 8792 | 9086 | 9380 | 9674 | 9968 |
| 48 | 17 0262 | 0555 | 0848 | 1141 | 1434 | 1726 | 2019 | 2311 | 2603 | 2895 |
| 49 | 3186 | 3478 | 3769 | 4060 | 4351 | 4641 | 4932 | 5222 | 5512 | 5802 |
| 150 | 6091 | 6381 | 6670 | 6959 | 7248 | 7536 | 7825 | 8113 | 8401 | 8689 |

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## TABLE I

| Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 290 | 289 | 288 | | 287 | 286 | 285 | | 284 | 283 | 282 | | 281 | 280 | 279 |
| 1 | 29.0 | 28.9 | 28.8 | 1 | 28.7 | 28.6 | 28.5 | 1 | 28.4 | 28.3 | 28.2 | 1 | 28.1 | 28.0 | 27.9 |
| 2 | 58.0 | 57.8 | 57.6 | 2 | 57.4 | 57.2 | 57.0 | 2 | 56.8 | 56.6 | 56.4 | 2 | 56.2 | 56.0 | 55.8 |
| 3 | 87.0 | 86.7 | 86.4 | 3 | 86.1 | 85.8 | 85.5 | 3 | 85.2 | 84.9 | 84.6 | 3 | 84.3 | 84.0 | 83.7 |
| 4 | 116.0 | 115.6 | 115.2 | 4 | 114.8 | 114.4 | 114.0 | 4 | 113.6 | 113.2 | 112.8 | 4 | 112.4 | 112.0 | 111.6 |
| 5 | 145.0 | 144.5 | 144.0 | 5 | 143.5 | 143.0 | 142.5 | 5 | 142.0 | 141.5 | 141.0 | 5 | 140.5 | 140.0 | 139.5 |
| 6 | 174.0 | 173.4 | 172.8 | 6 | 172.2 | 171.6 | 171.0 | 6 | 170.4 | 169.8 | 169.2 | 6 | 168.6 | 168.0 | 167.4 |
| 7 | 203.0 | 202.3 | 201.6 | 7 | 200.9 | 200.2 | 199.5 | 7 | 198.8 | 198.1 | 197.4 | 7 | 196.7 | 196.0 | 195.3 |
| 8 | 232.0 | 231.2 | 230.4 | 8 | 229.6 | 228.8 | 228.0 | 8 | 227.2 | 226.4 | 225.6 | 8 | 224.8 | 224.0 | 223.2 |
| 9 | 261.0 | 260.1 | 259.2 | 9 | 258.3 | 257.4 | 256.5 | 9 | 255.6 | 254.7 | 253.8 | 9 | 252.9 | 252.0 | 251.1 |
| | 278 | 277 | 276 | | 275 | 274 | 273 | | 272 | 271 | 270 | | 269 | 268 | 267 |
| 1 | 27.8 | 27.7 | 27.6 | 1 | 27.5 | 27.4 | 27.3 | 1 | 27.2 | 27.1 | 27.0 | 1 | 26.9 | 26.8 | 26.7 |
| 2 | 55.6 | 55.4 | 55.2 | 2 | 55.0 | 54.8 | 54.6 | 2 | 54.4 | 54.2 | 54.0 | 2 | 53.8 | 53.6 | 53.4 |
| 3 | 83.4 | 83.1 | 82.8 | 3 | 82.5 | 82.2 | 81.9 | 3 | 81.6 | 81.3 | 81.0 | 3 | 80.7 | 80.4 | 80.1 |
| 4 | 111.2 | 110.8 | 110.4 | 4 | 110.0 | 109.6 | 109.2 | 4 | 108.8 | 108.4 | 108.0 | 4 | 107.6 | 107.2 | 106.8 |
| 5 | 139.0 | 138.5 | 138.0 | 5 | 137.5 | 137.0 | 136.5 | 5 | 136.0 | 135.5 | 135.0 | 5 | 134.5 | 134.0 | 133.5 |
| 6 | 166.8 | 166.2 | 165.6 | 6 | 165.0 | 164.4 | 163.8 | 6 | 163.2 | 162.6 | 162.0 | 6 | 161.4 | 160.8 | 160.2 |
| 7 | 194.6 | 193.9 | 193.2 | 7 | 192.5 | 191.8 | 191.1 | 7 | 190.4 | 189.7 | 189.0 | 7 | 188.3 | 187.6 | 186.9 |
| 8 | 222.4 | 221.6 | 220.8 | 8 | 220.0 | 219.2 | 218.4 | 8 | 217.6 | 216.8 | 216.0 | 8 | 215.2 | 214.4 | 213.6 |
| 9 | 250.2 | 249.3 | 248.4 | 9 | 247.5 | 246.6 | 245.7 | 9 | 244.8 | 243.9 | 243.0 | 9 | 242.1 | 241.2 | 240.3 |
| | 266 | 265 | 264 | | 263 | 262 | 261 | | 260 | 259 | 258 | | 257 | 256 | 255 |
| 1 | 26.6 | 26.5 | 26.4 | 1 | 26.3 | 26.2 | 26.1 | 1 | 26.0 | 25.9 | 25.8 | 1 | 25.7 | 25.6 | 25.5 |
| 2 | 53.2 | 53.0 | 52.8 | 2 | 52.6 | 52.4 | 52.2 | 2 | 52.0 | 51.8 | 51.6 | 2 | 51.4 | 51.2 | 51.0 |
| 3 | 79.8 | 79.5 | 79.2 | 3 | 78.9 | 78.6 | 78.3 | 3 | 78.0 | 77.7 | 77.4 | 3 | 77.1 | 76.8 | 76.5 |
| 4 | 106.4 | 106.0 | 105.6 | 4 | 105.2 | 104.8 | 104.4 | 4 | 104.0 | 103.6 | 103.2 | 4 | 102.8 | 102.4 | 102.0 |
| 5 | 133.0 | 132.5 | 132.0 | 5 | 131.5 | 131.0 | 130.5 | 5 | 130.0 | 129.5 | 129.0 | 5 | 128.5 | 128.0 | 127.5 |
| 6 | 159.6 | 159.0 | 158.4 | 6 | 157.8 | 157.2 | 156.6 | 6 | 156.0 | 155.4 | 154.8 | 6 | 154.2 | 153.6 | 153.0 |
| 7 | 186.2 | 185.5 | 184.8 | 7 | 184.1 | 183.4 | 182.7 | 7 | 182.0 | 181.3 | 180.6 | 7 | 179.9 | 179.2 | 178.5 |
| 8 | 212.8 | 212.0 | 211.2 | 8 | 210.4 | 209.6 | 208.8 | 8 | 208.0 | 207.2 | 206.4 | 8 | 205.6 | 204.8 | 204.0 |
| 9 | 239.4 | 238.5 | 237.6 | 9 | 236.7 | 235.8 | 234.9 | 9 | 234.0 | 233.1 | 232.2 | 9 | 231.3 | 230.4 | 229.5 |

| | 254 |
|---|---|
| 1 | 25.4 |
| 2 | 50.8 |
| 3 | 76.2 |
| 4 | 101.6 |
| 5 | 127.0 |
| 6 | 152.4 |
| 7 | 177.8 |
| 8 | 203.2 |
| 9 | 228.6 |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 150 | 17 6091 | 6381 | 6670 | 6959 | 7248 | 7536 | 7825 | 8113 | 8401 | 8689 |
| 51 | 8977 | 9264 | 9552 | 9839 | *0126 | *0413 | *0699 | *0986 | *1272 | *1558 |
| 52 | 18 1844 | 2129 | 2415 | 2700 | 2985 | 3270 | 3555 | 3839 | 4123 | 4407 |
| 53 | 4691 | 4975 | 5259 | 5542 | 5825 | 6108 | 6391 | 6674 | 6956 | 7239 |
| 54 | 7521 | 7803 | 8084 | 8366 | 8647 | 8928 | 9209 | 9490 | 9771 | *0051 |
| 55 | 19 0332 | 0612 | 0892 | 1171 | 1451 | 1730 | 2010 | 2289 | 2567 | 2846 |
| 56 | 3125 | 3403 | 3681 | 3959 | 4237 | 4514 | 4792 | 5069 | 5346 | 5623 |
| 57 | 5900 | 6176 | 6453 | 6729 | 7005 | 7281 | 7556 | 7832 | 8107 | 8382 |
| 58 | 8657 | 8932 | 9206 | 9481 | 9755 | *0029 | *0303 | *0577 | *0850 | *1124 |
| 59 | 20 1397 | 1670 | 1943 | 2216 | 2488 | 2761 | 3033 | 3305 | 3577 | 3848 |
| 160 | 20 4120 | 4391 | 4663 | 4934 | 5204 | 5475 | 5746 | 6016 | 6286 | 6556 |
| 61 | 6826 | 7096 | 7365 | 7634 | 7904 | 8173 | 8441 | 8710 | 8979 | 9247 |
| 62 | 9515 | 9783 | *0051 | *0319 | *0586 | *0853 | *1121 | *1388 | *1654 | *1921 |
| 63 | 21 2188 | 2454 | 2720 | 2986 | 3252 | 3518 | 3783 | 4049 | 4314 | 4579 |
| 64 | 4844 | 5109 | 5373 | 5638 | 5902 | 6166 | 6430 | 6694 | 6957 | 7221 |
| 65 | 7484 | 7747 | 8010 | 8273 | 8536 | 8798 | 9060 | 9323 | 9585 | 9846 |
| 66 | 22 0108 | 0370 | 0631 | 0892 | 1153 | 1414 | 1675 | 1936 | 2196 | 2456 |
| 67 | 2716 | 2976 | 3236 | 3496 | 3755 | 4015 | 4274 | 4533 | 4792 | 5051 |
| 68 | 5309 | 5568 | 5826 | 6084 | 6342 | 6600 | 6858 | 7115 | 7372 | 7630 |
| 69 | 7887 | 8144 | 8400 | 8657 | 8913 | 9170 | 9426 | 9682 | 9938 | *0193 |
| 170 | 23 0449 | 0704 | 0960 | 1215 | 1470 | 1724 | 1979 | 2234 | 2488 | 2742 |

170–200 TABLE I

| Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 255 | 254 | 253 | | 252 | 251 | 250 | | 249 | 248 | 247 | | 246 | 245 | 244 |
| 1 | 25.5 | 25.4 | 25.3 | 1 | 25.2 | 25.1 | 25.0 | 1 | 24.9 | 24.8 | 24.7 | 1 | 24.6 | 24.5 | 24.4 |
| 2 | 51.0 | 50.8 | 50.6 | 2 | 50.4 | 50.2 | 50.0 | 2 | 49.8 | 49.6 | 49.4 | 2 | 49.2 | 49.0 | 48.8 |
| 3 | 76.5 | 76.2 | 75.9 | 3 | 75.6 | 75.3 | 75.0 | 3 | 74.7 | 74.4 | 74.1 | 3 | 73.8 | 73.5 | 73.2 |
| 4 | 102.0 | 101.6 | 101.2 | 4 | 100.8 | 100.4 | 100.0 | 4 | 99.6 | 99.2 | 98.8 | 4 | 98.4 | 98.0 | 97.6 |
| 5 | 127.5 | 127.0 | 126.5 | 5 | 126.0 | 125.5 | 125.0 | 5 | 124.5 | 124.0 | 123.5 | 5 | 123.0 | 122.5 | 122.0 |
| 6 | 153.0 | 152.4 | 151.8 | 6 | 151.2 | 150.6 | 150.0 | 6 | 149.4 | 148.8 | 148.2 | 6 | 147.6 | 147.0 | 146.4 |
| 7 | 178.5 | 177.8 | 177.1 | 7 | 176.4 | 175.7 | 175.0 | 7 | 174.3 | 173.6 | 172.9 | 7 | 172.2 | 171.5 | 170.8 |
| 8 | 204.0 | 203.2 | 202.4 | 8 | 201.6 | 200.8 | 200.0 | 8 | 199.2 | 198.4 | 197.6 | 8 | 196.8 | 196.0 | 195.2 |
| 9 | 229.5 | 228.6 | 227.7 | 9 | 226.8 | 225.9 | 225.0 | 9 | 224.1 | 223.2 | 222.3 | 9 | 221.4 | 220.5 | 219.6 |
| | 243 | 242 | 241 | | 240 | 239 | 238 | | 237 | 236 | 235 | | 234 | 233 | 232 |
| 1 | 24.3 | 24.2 | 24.1 | 1 | 24.0 | 23.9 | 23.8 | 1 | 23.7 | 23.6 | 23.5 | 1 | 23.4 | 23.3 | 23.2 |
| 2 | 48.6 | 48.4 | 48.2 | 2 | 48.0 | 47.8 | 47.6 | 2 | 47.4 | 47.2 | 47.0 | 2 | 46.8 | 46.6 | 46.4 |
| 3 | 72.9 | 72.6 | 72.3 | 3 | 72.0 | 71.7 | 71.4 | 3 | 71.1 | 70.8 | 70.5 | 3 | 70.2 | 69.9 | 69.6 |
| 4 | 97.2 | 96.8 | 96.4 | 4 | 96.0 | 95.6 | 95.2 | 4 | 94.8 | 94.4 | 94.0 | 4 | 93.6 | 93.2 | 92.8 |
| 5 | 121.5 | 121.0 | 120.5 | 5 | 120.0 | 119.5 | 119.0 | 5 | 118.5 | 118.0 | 117.5 | 5 | 117.0 | 116.5 | 116.0 |
| 6 | 145.8 | 145.2 | 144.6 | 6 | 144.0 | 143.4 | 142.8 | 6 | 142.2 | 141.6 | 141.0 | 6 | 140.4 | 139.8 | 139.2 |
| 7 | 170.1 | 169.4 | 168.7 | 7 | 168.0 | 167.3 | 166.6 | 7 | 165.9 | 165.2 | 164.5 | 7 | 163.8 | 163.1 | 162.4 |
| 8 | 194.4 | 193.6 | 192.8 | 8 | 192.0 | 191.2 | 190.4 | 8 | 189.6 | 188.8 | 188.0 | 8 | 187.2 | 186.4 | 185.6 |
| 9 | 218.7 | 217.8 | 216.9 | 9 | 216.0 | 215.1 | 214.2 | 9 | 213.3 | 212.4 | 211.5 | 9 | 210.6 | 209.7 | 208.8 |
| | | | | | | | | | | | | | 231 | 230 | 229 |
| | | | | | | | | | | | | 1 | 23.1 | 23.0 | 22.9 |
| | | | | | | | | | | | | 2 | 46.2 | 46.0 | 45.8 |
| | | | | | | | | | | | | 3 | 69.3 | 69.0 | 68.7 |
| | | | | | | | | | | | | 4 | 92.4 | 92.0 | 91.6 |
| | | | | | | | | | | | | 5 | 115.5 | 115.0 | 114.5 |
| | | | | | | | | | | | | 6 | 138.6 | 138.0 | 137.4 |
| | | | | | | | | | | | | 7 | 161.7 | 161.0 | 160.3 |
| | | | | | | | | | | | | 8 | 184.8 | 184.0 | 183.2 |
| | | | | | | | | | | | | 9 | 207.9 | 207.0 | 206.1 |
| | | | | | | | | | | | | | 228 | 227 | 226 |
| | | | | | | | | | | | | 1 | 22.8 | 22.7 | 22.6 |
| | | | | | | | | | | | | 2 | 45.6 | 45.4 | 45.2 |
| | | | | | | | | | | | | 3 | 68.4 | 68.1 | 67.8 |
| | | | | | | | | | | | | 4 | 91.2 | 90.8 | 90.4 |
| | | | | | | | | | | | | 5 | 114.0 | 113.5 | 113.0 |
| | | | | | | | | | | | | 6 | 136.8 | 136.2 | 135.6 |
| | | | | | | | | | | | | 7 | 159.6 | 158.9 | 158.2 |
| | | | | | | | | | | | | 8 | 182.4 | 181.6 | 180.8 |
| | | | | | | | | | | | | 9 | 205.2 | 204.3 | 203.4 |
| | | | | | | | | | | | | | 225 | 224 | 223 |
| | | | | | | | | | | | | 1 | 22.5 | 22.4 | 22.3 |
| | | | | | | | | | | | | 2 | 45.0 | 44.8 | 44.6 |
| | | | | | | | | | | | | 3 | 67.5 | 67.2 | 66.9 |
| | | | | | | | | | | | | 4 | 90.0 | 89.6 | 89.2 |
| | | | | | | | | | | | | 5 | 112.5 | 112.0 | 111.5 |
| | | | | | | | | | | | | 6 | 135.0 | 134.4 | 133.8 |
| | | | | | | | | | | | | 7 | 157.5 | 156.8 | 156.1 |
| | | | | | | | | | | | | 8 | 180.0 | 179.2 | 178.4 |
| | | | | | | | | | | | | 9 | 202.5 | 201.6 | 200.7 |
| | | | | | | | | | | | | | 222 | 221 | 220 |
| | | | | | | | | | | | | 1 | 22.2 | 22.1 | 22.0 |
| | | | | | | | | | | | | 2 | 44.4 | 44.2 | 44.0 |
| | | | | | | | | | | | | 3 | 66.6 | 66.3 | 66.0 |
| | | | | | | | | | | | | 4 | 88.8 | 88.4 | 88.0 |
| | | | | | | | | | | | | 5 | 111.0 | 110.5 | 110.0 |
| | | | | | | | | | | | | 6 | 133.2 | 132.6 | 132.0 |
| | | | | | | | | | | | | 7 | 155.4 | 154.7 | 154.0 |
| | | | | | | | | | | | | 8 | 177.6 | 176.8 | 176.0 |
| | | | | | | | | | | | | 9 | 199.8 | 198.9 | 198.0 |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 170 | 23 0449 | 0704 | 0960 | 1215 | 1470 | 1724 | 1979 | 2234 | 2488 | 2742 |
| 71 | 2996 | 3250 | 3504 | 3757 | 4011 | 4264 | 4517 | 4770 | 5023 | 5276 |
| 72 | 5528 | 5781 | 6033 | 6285 | 6537 | 6789 | 7041 | 7292 | 7544 | 7795 |
| 73 | 8046 | 8297 | 8548 | 8799 | 9049 | 9299 | 9550 | 9800 | *0050 | *0300 |
| 74 | 24 0549 | 0799 | 1048 | 1297 | 1546 | 1795 | 2044 | 2293 | 2541 | 2790 |
| 75 | 3038 | 3286 | 3534 | 3782 | 4030 | 4277 | 4525 | 4772 | 5019 | 5266 |
| 76 | 5513 | 5759 | 6006 | 6252 | 6499 | 6745 | 6991 | 7237 | 7482 | 7728 |
| 77 | 7973 | 8219 | 8464 | 8709 | 8954 | 9198 | 9443 | 9687 | 9932 | *0176 |
| 78 | 25 0420 | 0664 | 0908 | 1151 | 1395 | 1638 | 1881 | 2125 | 2368 | 2610 |
| 79 | 2853 | 3096 | 3338 | 3580 | 3822 | 4064 | 4306 | 4548 | 4790 | 5031 |
| 180 | 25 5273 | 5514 | 5755 | 5996 | 6237 | 6477 | 6718 | 6958 | 7198 | 7439 |
| 81 | 7679 | 7918 | 8158 | 8398 | 8637 | 8877 | 9116 | 9355 | 9594 | 9833 |
| 82 | 26 0071 | 0310 | 0548 | 0787 | 1025 | 1263 | 1501 | 1739 | 1976 | 2214 |
| 83 | 2451 | 2688 | 2925 | 3162 | 3399 | 3636 | 3873 | 4109 | 4346 | 4582 |
| 84 | 4818 | 5054 | 5290 | 5525 | 5761 | 5996 | 6232 | 6467 | 6702 | 6937 |
| 85 | 7172 | 7406 | 7641 | 7875 | 8110 | 8344 | 8578 | 8812 | 9046 | 9279 |
| 86 | 9513 | 9746 | 9980 | *0213 | *0446 | *0679 | *0912 | *1144 | *1377 | *1609 |
| 87 | 27 1842 | 2074 | 2306 | 2538 | 2770 | 3001 | 3233 | 3464 | 3696 | 3927 |
| 88 | 4158 | 4389 | 4620 | 4850 | 5081 | 5311 | 5542 | 5772 | 6002 | 6232 |
| 89 | 6462 | 6692 | 6921 | 7151 | 7380 | 7609 | 7838 | 8067 | 8296 | 8525 |
| 190 | 27 8754 | 8982 | 9211 | 9439 | 9667 | 9895 | *0123 | *0351 | *0578 | *0806 |
| 91 | 28 1033 | 1261 | 1488 | 1715 | 1942 | 2169 | 2396 | 2622 | 2849 | 3075 |
| 92 | 3301 | 3527 | 3753 | 3979 | 4205 | 4431 | 4656 | 4882 | 5107 | 5332 |
| 93 | 5557 | 5782 | 6007 | 6232 | 6456 | 6681 | 6905 | 7130 | 7354 | 7578 |
| 94 | 7802 | 8026 | 8249 | 8473 | 8696 | 8920 | 9143 | 9366 | 9589 | 9812 |
| 95 | 29 0035 | 0257 | 0480 | 0702 | 0925 | 1147 | 1369 | 1591 | 1813 | 2034 |
| 96 | 2256 | 2478 | 2699 | 2920 | 3141 | 3363 | 3584 | 3804 | 4025 | 4246 |
| 97 | 4466 | 4687 | 4907 | 5127 | 5347 | 5567 | 5787 | 6007 | 6226 | 6446 |
| 98 | 6665 | 6884 | 7104 | 7323 | 7542 | 7761 | 7979 | 8198 | 8416 | 8635 |
| 99 | 8853 | 9071 | 9289 | 9507 | 9725 | 9943 | *0161 | *0378 | *0595 | *0813 |
| 200 | 30 1030 | 1247 | 1464 | 1681 | 1898 | 2114 | 2331 | 2547 | 2764 | 2980 |

200–230

## TABLE I

| Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 219 | 218 | 217 | | 216 | 215 | 214 | | 213 | 212 | 211 | | 210 | 209 | 208 |
| 1 | 21.9 | 21.8 | 21.7 | 1 | 21.6 | 21.5 | 21.4 | 1 | 21.3 | 21.2 | 21.1 | 1 | 21.0 | 20.9 | 20.8 |
| 2 | 43.8 | 43.6 | 43.4 | 2 | 43.2 | 43.0 | 42.8 | 2 | 42.6 | 42.4 | 42.2 | 2 | 42.0 | 41.8 | 41.6 |
| 3 | 65.7 | 65.4 | 65.1 | 3 | 64.8 | 64.5 | 64.2 | 3 | 63.9 | 63.6 | 63.3 | 3 | 63.0 | 62.7 | 62.4 |
| 4 | 87.6 | 87.2 | 86.8 | 4 | 86.4 | 86.0 | 85.6 | 4 | 85.2 | 84.8 | 84.4 | 4 | 84.0 | 83.6 | 83.2 |
| 5 | 109.5 | 109.0 | 108.5 | 5 | 108.0 | 107.5 | 107.0 | 5 | 106.5 | 106.0 | 105.5 | 5 | 105.0 | 104.5 | 104.0 |
| 6 | 131.4 | 130.8 | 130.2 | 6 | 129.6 | 129.0 | 128.4 | 6 | 127.8 | 127.2 | 126.6 | 6 | 126.0 | 125.4 | 124.8 |
| 7 | 153.3 | 152.6 | 151.9 | 7 | 151.2 | 150.5 | 149.8 | 7 | 149.1 | 148.4 | 147.7 | 7 | 147.0 | 146.3 | 145.6 |
| 8 | 175.2 | 174.4 | 173.6 | 8 | 172.8 | 172.0 | 171.2 | 8 | 170.4 | 169.6 | 168.8 | 8 | 168.0 | 167.2 | 166.4 |
| 9 | 197.1 | 196.2 | 195.3 | 9 | 194.4 | 193.5 | 192.6 | 9 | 191.7 | 190.8 | 189.9 | 9 | 189.0 | 188.1 | 187.2 |
| | 207 | 206 | 205 | | 204 | 203 | 202 | | 201 | 200 | 199 | | 198 | 197 | 196 |
| 1 | 20.7 | 20.6 | 20.5 | 1 | 20.4 | 20.3 | 20.2 | 1 | 20.1 | 20.0 | 19.9 | 1 | 19.8 | 19.7 | 19.6 |
| 2 | 41.4 | 41.2 | 41.0 | 2 | 40.8 | 40.6 | 40.4 | 2 | 40.2 | 40.0 | 39.8 | 2 | 39.6 | 39.4 | 39.2 |
| 3 | 62.1 | 61.8 | 61.5 | 3 | 61.2 | 60.9 | 60.6 | 3 | 60.3 | 60.0 | 59.7 | 3 | 59.4 | 59.1 | 58.8 |
| 4 | 82.8 | 82.4 | 82.0 | 4 | 81.6 | 81.2 | 80.8 | 4 | 80.4 | 80.0 | 79.6 | 4 | 79.2 | 78.8 | 78.4 |
| 5 | 103.5 | 103.0 | 102.5 | 5 | 102.0 | 101.5 | 101.0 | 5 | 100.5 | 100.0 | 99.5 | 5 | 99.0 | 98.5 | 98.0 |
| 6 | 124.2 | 123.6 | 123.0 | 6 | 122.4 | 121.8 | 121.2 | 6 | 120.6 | 120.0 | 119.4 | 6 | 118.8 | 118.2 | 117.6 |
| 7 | 144.9 | 144.2 | 143.5 | 7 | 142.8 | 142.1 | 141.4 | 7 | 140.7 | 140.0 | 139.3 | 7 | 138.6 | 137.9 | 137.2 |
| 8 | 165.6 | 164.8 | 164.0 | 8 | 163.2 | 162.4 | 161.6 | 8 | 160.8 | 160.0 | 159.2 | 8 | 158.4 | 157.6 | 156.8 |
| 9 | 186.3 | 185.4 | 184.5 | 9 | 183.6 | 182.7 | 181.8 | 9 | 180.9 | 180.0 | 179.1 | 9 | 178.2 | 177.3 | 176.4 |

| | 195 | 194 | 193 |
|---|---|---|---|
| 1 | 19.5 | 19.4 | 19.3 |
| 2 | 39.0 | 38.8 | 38.6 |
| 3 | 58.5 | 58.2 | 57.9 |
| 4 | 78.0 | 77.6 | 77.2 |
| 5 | 97.5 | 97.0 | 96.5 |
| 6 | 117.0 | 116.4 | 115.8 |
| 7 | 136.5 | 135.8 | 135.1 |
| 8 | 156.0 | 155.2 | 154.4 |
| 9 | 175.5 | 174.6 | 173.7 |
| | 192 | 191 | 190 |
| 1 | 19.2 | 19.1 | 19.0 |
| 2 | 38.4 | 38.2 | 38.0 |
| 3 | 57.6 | 57.3 | 57.0 |
| 4 | 76.8 | 76.4 | 76.0 |
| 5 | 96.0 | 95.5 | 95.0 |
| 6 | 115.2 | 114.6 | 114.0 |
| 7 | 134.4 | 133.7 | 133.0 |
| 8 | 153.6 | 152.8 | 152.0 |
| 9 | 172.8 | 171.9 | 171.0 |
| | 189 | | |
| 1 | 18.9 | | |
| 2 | 37.8 | | |
| 3 | 56.7 | | |
| 4 | 75.6 | | |
| 5 | 94.5 | | |
| 6 | 113.4 | | |
| 7 | 132.3 | | |
| 8 | 151.2 | | |
| 9 | 170.1 | | |
| | 188 | | |
| 1 | 18.8 | | |
| 2 | 37.6 | | |
| 3 | 56.4 | | |
| 4 | 75.2 | | |
| 5 | 94.0 | | |
| 6 | 112.8 | | |
| 7 | 131.6 | | |
| 8 | 150.4 | | |
| 9 | 169.2 | | |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 200 | 30 1030 | 1247 | 1464 | 1681 | 1898 | 2114 | 2331 | 2547 | 2764 | 2980 |
| 01 | 3196 | 3412 | 3628 | 3844 | 4059 | 4275 | 4491 | 4706 | 4921 | 5136 |
| 02 | 5351 | 5566 | 5781 | 5996 | 6211 | 6425 | 6639 | 6854 | 7068 | 7282 |
| 03 | 7496 | 7710 | 7924 | 8137 | 8351 | 8564 | 8778 | 8991 | 9204 | 9417 |
| 04 | 9630 | 9843 | *0056 | *0268 | *0481 | *0693 | *0906 | *1118 | *1330 | *1542 |
| 05 | 31 1754 | 1966 | 2177 | 2389 | 2600 | 2812 | 3023 | 3234 | 3445 | 3656 |
| 06 | 3867 | 4078 | 4289 | 4499 | 4710 | 4920 | 5130 | 5340 | 5551 | 5760 |
| 07 | 5970 | 6180 | 6390 | 6599 | 6809 | 7018 | 7227 | 7436 | 7646 | 7854 |
| 08 | 8063 | 8272 | 8481 | 8689 | 8898 | 9106 | 9314 | 9522 | 9730 | 9938 |
| 09 | 32 0146 | 0354 | 0562 | 0769 | 0977 | 1184 | 1391 | 1598 | 1805 | 2012 |
| 210 | 32 2219 | 2426 | 2633 | 2839 | 3046 | 3252 | 3458 | 3665 | 3871 | 4077 |
| 11 | 4282 | 4488 | 4694 | 4899 | 5105 | 5310 | 5516 | 5721 | 5926 | 6131 |
| 12 | 6336 | 6541 | 6745 | 6950 | 7155 | 7359 | 7563 | 7767 | 7972 | 8176 |
| 13 | 8380 | 8583 | 8787 | 8991 | 9194 | 9398 | 9601 | 9805 | *0008 | *0211 |
| 14 | 33 0414 | 0617 | 0819 | 1022 | 1225 | 1427 | 1630 | 1832 | 2034 | 2236 |
| 15 | 2438 | 2640 | 2842 | 3044 | 3246 | 3447 | 3649 | 3850 | 4051 | 4253 |
| 16 | 4454 | 4655 | 4856 | 5057 | 5257 | 5458 | 5658 | 5859 | 6059 | 6260 |
| 17 | 6460 | 6660 | 6860 | 7060 | 7260 | 7459 | 7659 | 7858 | 8058 | 8257 |
| 18 | 8456 | 8656 | 8855 | 9054 | 9253 | 9451 | 9650 | 9849 | *0047 | *0246 |
| 19 | 34 0444 | 0642 | 0841 | 1039 | 1237 | 1435 | 1632 | 1830 | 2028 | 2225 |
| 220 | 34 2423 | 2620 | 2817 | 3014 | 3212 | 3409 | 3606 | 3802 | 3999 | 4196 |
| 21 | 4392 | 4589 | 4785 | 4981 | 5178 | 5374 | 5570 | 5766 | 5962 | 6157 |
| 22 | 6353 | 6549 | 6744 | 6939 | 7135 | 7330 | 7525 | 7720 | 7915 | 8110 |
| 23 | 8305 | 8500 | 8694 | 8889 | 9083 | 9278 | 9472 | 9666 | 9860 | *0054 |
| 24 | 35 0248 | 0442 | 0636 | 0829 | 1023 | 1216 | 1410 | 1603 | 1796 | 1989 |
| 25 | 2183 | 2375 | 2568 | 2761 | 2954 | 3147 | 3339 | 3532 | 3724 | 3916 |
| 26 | 4108 | 4301 | 4493 | 4685 | 4876 | 5068 | 5260 | 5452 | 5643 | 5834 |
| 27 | 6026 | 6217 | 6408 | 6599 | 6790 | 6981 | 7172 | 7363 | 7554 | 7744 |
| 28 | 7935 | 8125 | 8316 | 8506 | 8696 | 8886 | 9076 | 9266 | 9456 | 9646 |
| 29 | 9835 | *0025 | *0215 | *0404 | *0593 | *0783 | *0972 | *1161 | *1350 | *1539 |
| 230 | 36 1728 | 1917 | 2105 | 2294 | 2482 | 2671 | 2859 | 3048 | 3236 | 3424 |

**230–270** **TABLE I**

| Proportional Parts | | | | Proportional Parts | | | Proportional Parts | | | | Proportional Parts | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 189 | 188 | 187 | | 186 | 185 | 184 | | 183 | 182 | 181 | | 180 | 179 | 178 |
| 1 | 18.9 | 18.8 | 18.7 | 1 | 18.6 | 18.5 | 18.4 | 1 | 18.3 | 18.2 | 18.1 | 1 | 18.0 | 17.9 | 17.8 |
| 2 | 37.8 | 37.6 | 37.4 | 2 | 37.2 | 37.0 | 36.8 | 2 | 36.6 | 36.4 | 36.2 | 2 | 36.0 | 35.8 | 35.6 |
| 3 | 56.7 | 56.4 | 56.1 | 3 | 55.8 | 55.5 | 55.2 | 3 | 54.9 | 54.6 | 54.3 | 3 | 54.0 | 53.7 | 53.4 |
| 4 | 75.6 | 75.2 | 74.8 | 4 | 74.4 | 74.0 | 73.6 | 4 | 73.2 | 72.8 | 72.4 | 4 | 72.0 | 71.6 | 71.2 |
| 5 | 94.5 | 94.0 | 93.5 | 5 | 93.0 | 92.5 | 92.0 | 5 | 91.5 | 91.0 | 90.5 | 5 | 90.0 | 89.5 | 89.0 |
| 6 | 113.4 | 112.8 | 112.2 | 6 | 111.6 | 111.0 | 110.4 | 6 | 109.8 | 109.2 | 108.6 | 6 | 108.0 | 107.4 | 106.8 |
| 7 | 132.3 | 131.6 | 130.9 | 7 | 130.2 | 129.5 | 128.8 | 7 | 128.1 | 127.4 | 126.7 | 7 | 126.0 | 125.3 | 124.6 |
| 8 | 151.2 | 150.4 | 149.6 | 8 | 148.8 | 148.0 | 147.2 | 8 | 146.4 | 145.6 | 144.8 | 8 | 144.0 | 143.2 | 142.4 |
| 9 | 170.1 | 169.2 | 168.3 | 9 | 167.4 | 166.5 | 165.6 | 9 | 164.7 | 163.8 | 162.9 | 9 | 162.0 | 161.1 | 160.2 |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | | 177 | 176 | 175 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 230 | 36 1728 | 1917 | 2105 | 2294 | 2482 | 2671 | 2859 | 3048 | 3236 | 3424 | 1 | 17.7 | 17.6 | 17.5 |
| 31 | 3612 | 3800 | 3988 | 4176 | 4363 | 4551 | 4739 | 4926 | 5113 | 5301 | 2 | 35.4 | 35.2 | 35.0 |
| 32 | 5488 | 5675 | 5862 | 6049 | 6236 | 6423 | 6610 | 6796 | 6983 | 7169 | 3 | 53.1 | 52.8 | 52.5 |
| 33 | 7356 | 7542 | 7729 | 7915 | 8101 | 8287 | 8473 | 8659 | 8845 | 9030 | 4 | 70.8 | 70.4 | 70.0 |
| 34 | 9216 | 9401 | 9587 | 9772 | 9958 | *0143 | *0328 | *0513 | *0698 | *0883 | 5 | 88.5 | 88.0 | 87.5 |
| | | | | | | | | | | | 6 | 106.2 | 105.6 | 105.0 |
| 35 | 37 1068 | 1253 | 1437 | 1622 | 1806 | 1991 | 2175 | 2360 | 2544 | 2728 | 7 | 123.9 | 123.2 | 122.5 |
| 36 | 2912 | 3096 | 3280 | 3464 | 3647 | 3831 | 4015 | 4198 | 4382 | 4565 | 8 | 141.6 | 140.8 | 140.0 |
| 37 | 4748 | 4932 | 5115 | 5298 | 5481 | 5664 | 5846 | 6029 | 6212 | 6394 | 9 | 159.3 | 158.4 | 157.5 |
| 38 | 6577 | 6759 | 6942 | 7124 | 7306 | 7488 | 7670 | 7852 | 8034 | 8216 | | | | |
| 39 | 8398 | 8580 | 8761 | 8943 | 9124 | 9306 | 9487 | 9668 | 9849 | *0030 | | 174 | 173 | 172 |
| | | | | | | | | | | | 1 | 17.4 | 17.3 | 17.2 |
| 240 | 38 0211 | 0392 | 0573 | 0754 | 0934 | 1115 | 1296 | 1476 | 1656 | 1837 | 2 | 34.8 | 34.6 | 34.4 |
| 41 | 2017 | 2197 | 2377 | 2557 | 2737 | 2917 | 3097 | 3277 | 3456 | 3636 | 3 | 52.2 | 51.9 | 51.6 |
| 42 | 3815 | 3995 | 4174 | 4353 | 4533 | 4712 | 4891 | 5070 | 5249 | 5428 | 4 | 69.6 | 69.2 | 68.8 |
| 43 | 5606 | 5785 | 5964 | 6142 | 6321 | 6499 | 6677 | 6856 | 7034 | 7212 | 5 | 87.0 | 86.5 | 86.0 |
| 44 | 7390 | 7568 | 7746 | 7923 | 8101 | 8279 | 8456 | 8634 | 8811 | 8989 | 5 | 104.4 | 103.8 | 103.2 |
| | | | | | | | | | | | 7 | 121.8 | 121.1 | 120.4 |
| 45 | 9166 | 9343 | 9520 | 9698 | 9875 | *0051 | *0228 | *0405 | *0582 | *0759 | 8 | 139.2 | 138.4 | 137.6 |
| 46 | 39 0935 | 1112 | 1288 | 1464 | 1641 | 1817 | 1993 | 2169 | 2345 | 2521 | 9 | 156.6 | 155.7 | 154.8 |
| 47 | 2697 | 2873 | 3048 | 3224 | 3400 | 3575 | 3751 | 3926 | 4101 | 4277 | | | | |
| 48 | 4452 | 4627 | 4802 | 4977 | 5152 | 5326 | 5501 | 5676 | 5850 | 6025 | | 171 | 170 | 169 |
| 49 | 6199 | 6374 | 6548 | 6722 | 6896 | 7071 | 7245 | 7419 | 7592 | 7766 | 1 | 17.1 | 17.0 | 16.9 |
| | | | | | | | | | | | 2 | 34.2 | 34.0 | 33.8 |
| 250 | 39 7940 | 8114 | 8287 | 8461 | 8634 | 8808 | 8981 | 9154 | 9328 | 9501 | 3 | 51.3 | 51.0 | 50.7 |
| 51 | 9674 | 9847 | *0020 | *0192 | *0365 | *0538 | *0711 | *0883 | *1056 | *1228 | 4 | 68.4 | 68.0 | 67.6 |
| 52 | 40 1401 | 1573 | 1745 | 1917 | 2089 | 2261 | 2433 | 2605 | 2777 | 2949 | 5 | 85.5 | 85.0 | 84.5 |
| 53 | 3121 | 3292 | 3464 | 3635 | 3807 | 3978 | 4149 | 4320 | 4492 | 4663 | 6 | 102.6 | 102.0 | 101.4 |
| 54 | 4834 | 5005 | 5176 | 5346 | 5517 | 5688 | 5858 | 6029 | 6199 | 6370 | 7 | 119.7 | 119.0 | 118.3 |
| | | | | | | | | | | | 8 | 136.8 | 136.0 | 135.2 |
| 55 | 6540 | 6710 | 6881 | 7051 | 7221 | 7391 | 7561 | 7731 | 7901 | 8070 | 9 | 153.9 | 153.0 | 152.1 |
| 56 | 8240 | 8410 | 8579 | 8749 | 8918 | 9087 | 9257 | 9426 | 9595 | 9764 | | | | |
| 57 | 9933 | *0102 | *0271 | *0440 | *0609 | *0777 | *0946 | *1114 | *1283 | *1451 | | 165 | 167 | 166 |
| 58 | 41 1620 | 1788 | 1956 | 2124 | 2293 | 2461 | 2629 | 2796 | 2964 | 3132 | 1 | 16.8 | 16.7 | 16.6 |
| 59 | 3300 | 3467 | 3635 | 3803 | 3970 | 4137 | 4305 | 4472 | 4639 | 4806 | 2 | 33.6 | 33.4 | 33.2 |
| | | | | | | | | | | | 3 | 50.4 | 50.1 | 49.8 |
| 260 | 41 4973 | 5140 | 5307 | 5474 | 5641 | 5808 | 5974 | 6141 | 6308 | 6474 | 4 | 67.2 | 66.8 | 66.4 |
| 61 | 6641 | 6807 | 6973 | 7139 | 7306 | 7472 | 7638 | 7804 | 7970 | 8135 | 5 | 84.0 | 83.5 | 83.0 |
| 62 | 8301 | 8467 | 8633 | 8798 | 8964 | 9129 | 9295 | 9460 | 9625 | 9791 | 6 | 100.8 | 100.2 | 99.6 |
| 63 | 9956 | *0121 | *0286 | *0451 | *0616 | *0781 | *0945 | *1110 | *1275 | *1439 | 7 | 117.6 | 116.9 | 116.2 |
| 64 | 42 1604 | 1768 | 1933 | 2097 | 2261 | 2426 | 2590 | 2754 | 2918 | 3082 | 8 | 134.4 | 133.6 | 132.8 |
| | | | | | | | | | | | 9 | 151.2 | 150.3 | 149.4 |
| 65 | 3246 | 3410 | 3574 | 3737 | 3901 | 4065 | 4228 | 4392 | 4555 | 4718 | | 165 | 164 | 163 |
| 66 | 4882 | 5045 | 5208 | 5371 | 5534 | 5697 | 5860 | 6023 | 6186 | 6349 | 1 | 16.5 | 16.4 | 16.3 |
| 67 | 6511 | 6674 | 6836 | 6999 | 7161 | 7324 | 7486 | 7648 | 7811 | 7973 | 2 | 33.0 | 32.8 | 32.6 |
| 68 | 8135 | 8297 | 8459 | 8621 | 8783 | 8944 | 9106 | 9268 | 9429 | 9591 | 3 | 49.5 | 49.2 | 48.9 |
| 69 | 9752 | 9914 | *0075 | *0236 | *0398 | *0559 | *0720 | *0881 | *1042 | *1203 | 4 | 66.0 | 65.6 | 65.2 |
| | | | | | | | | | | | 5 | 82.5 | 82.0 | 81.5 |
| 270 | 43 1364 | 1525 | 1685 | 1846 | 2007 | 2167 | 2328 | 2488 | 2649 | 2809 | 6 | 99.0 | 98.4 | 97.8 |
| | | | | | | | | | | | 7 | 115.5 | 114.8 | 114.1 |
| | | | | | | | | | | | 8 | 132.0 | 131.2 | 130.4 |
| | | | | | | | | | | | 9 | 148.5 | 147.6 | 146.7 |

270–310 **TABLE I**

| Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 162 | 161 | 160 | | 159 | 158 | 157 | | 156 | 155 | 154 | | 153 | 152 | 151 |
| 1 | 16.2 | 16.1 | 16.0 | 1 | 15.9 | 15.8 | 15.7 | 1 | 15.6 | 15.5 | 15.4 | 1 | 15.3 | 15.2 | 15.1 |
| 2 | 32.4 | 32.2 | 32.0 | 2 | 31.8 | 31.6 | 31.4 | 2 | 31.2 | 31.0 | 30.8 | 2 | 30.6 | 30.4 | 30.2 |
| 3 | 48.6 | 48.3 | 48.0 | 3 | 47.7 | 47.4 | 47.1 | 3 | 46.8 | 46.5 | 46.2 | 3 | 45.9 | 45.6 | 45.3 |
| 4 | 64.8 | 64.4 | 64.0 | 4 | 63.6 | 63.2 | 62.8 | 4 | 62.4 | 62.0 | 61.6 | 4 | 61.2 | 60.8 | 60.4 |
| 5 | 81.0 | 80.5 | 80.0 | 5 | 79.5 | 79.0 | 78.5 | 5 | 78.0 | 77.5 | 77.0 | 5 | 76.5 | 76.0 | 75.5 |
| 6 | 97.2 | 96.6 | 96.0 | 6 | 95.4 | 94.8 | 94.2 | 6 | 93.6 | 93.0 | 92.4 | 6 | 91.8 | 91.2 | 90.6 |
| 7 | 113.4 | 112.7 | 112.0 | 7 | 111.3 | 110.6 | 109.9 | 7 | 109.2 | 108.5 | 107.8 | 7 | 107.1 | 106.4 | 105.7 |
| 8 | 129.6 | 128.8 | 128.0 | 8 | 127.2 | 126.4 | 125.6 | 8 | 124.8 | 124.0 | 123.2 | 8 | 122.4 | 121.6 | 120.8 |
| 9 | 145.8 | 144.9 | 144.0 | 9 | 143.1 | 142.2 | 141.3 | 9 | 140.4 | 139.5 | 138.6 | 9 | 137.7 | 136.8 | 135.9 |

| Proportional Parts | | | |
|---|---|---|---|
| | 150 | 149 | 148 |
| 1 | 15.0 | 14.9 | 14.8 |
| 2 | 30.0 | 29.8 | 29.6 |
| 3 | 45.0 | 44.7 | 44.4 |
| 4 | 60.0 | 59.6 | 59.2 |
| 5 | 75.0 | 74.5 | 74.0 |
| 6 | 90.0 | 89.4 | 88.8 |
| 7 | 105.0 | 104.3 | 103.6 |
| 8 | 120.0 | 119.2 | 118.4 |
| 9 | 135.0 | 134.1 | 133.2 |

| Proportional Parts | | | |
|---|---|---|---|
| | 147 | 146 | 145 |
| 1 | 14.7 | 14.6 | 14.5 |
| 2 | 29.4 | 29.2 | 29.0 |
| 3 | 44.1 | 43.8 | 43.5 |
| 4 | 58.8 | 58.4 | 58.0 |
| 5 | 73.5 | 73.0 | 72.5 |
| 6 | 88.2 | 87.6 | 87.0 |
| 7 | 102.9 | 102.2 | 101.5 |
| 8 | 117.6 | 116.8 | 116.0 |
| 9 | 132.3 | 131.4 | 130.5 |

| Proportional Parts | | |
|---|---|---|
| | 144 | 143 |
| 1 | 14.4 | 14.3 |
| 2 | 28.8 | 28.6 |
| 3 | 43.2 | 42.9 |
| 4 | 57.6 | 57.2 |
| 5 | 72.0 | 71.5 |
| 6 | 86.4 | 85.8 |
| 7 | 100.8 | 100.1 |
| 8 | 115.2 | 114.4 |
| 9 | 129.6 | 128.7 |

| Proportional Parts | | |
|---|---|---|
| | 142 | 141 |
| 1 | 14.2 | 14.1 |
| 2 | 28.4 | 28.2 |
| 3 | 42.6 | 42.3 |
| 4 | 56.8 | 56.4 |
| 5 | 71.0 | 70.5 |
| 6 | 85.2 | 84.6 |
| 7 | 99.4 | 98.7 |
| 8 | 113.6 | 112.8 |
| 9 | 127.8 | 126.9 |

| Proportional Parts | |
|---|---|
| | 140 |
| 1 | 14.0 |
| 2 | 28.0 |
| 3 | 42.0 |
| 4 | 56.0 |
| 5 | 70.0 |
| 6 | 84.0 |
| 7 | 98.0 |
| 8 | 112.0 |
| 9 | 126.0 |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 270 | 43 1364 | 1525 | 1685 | 1846 | 2007 | 2167 | 2328 | 2488 | 2649 | 2809 |
| 71 | 2969 | 3130 | 3290 | 3450 | 3610 | 3770 | 3930 | 4090 | 4249 | 4409 |
| 72 | 4569 | 4729 | 4888 | 5048 | 5207 | 5367 | 5526 | 5685 | 5844 | 6004 |
| 73 | 6163 | 6322 | 6481 | 6640 | 6799 | 6957 | 7116 | 7275 | 7433 | 7592 |
| 74 | 7751 | 7909 | 8067 | 8226 | 8384 | 8542 | 8701 | 8859 | 9017 | 9175 |
| 75 | 9333 | 9491 | 9648 | 9806 | 9964 | *0122 | *0279 | *0437 | *0594 | *0752 |
| 76 | 44 0909 | 1066 | 1224 | 1381 | 1538 | 1695 | 1852 | 2009 | 2166 | 2323 |
| 77 | 2480 | 2637 | 2793 | 2950 | 3106 | 3263 | 3419 | 3576 | 3732 | 3889 |
| 78 | 4045 | 4201 | 4357 | 4513 | 4669 | 4825 | 4981 | 5137 | 5293 | 5449 |
| 79 | 5604 | 5760 | 5915 | 6071 | 6226 | 6382 | 6537 | 6692 | 6848 | 7003 |
| 280 | 44 7158 | 7313 | 7468 | 7623 | 7778 | 7933 | 8088 | 8242 | 8397 | 8552 |
| 81 | 8706 | 8861 | 9015 | 9170 | 9324 | 9478 | 9633 | 9787 | 9941 | *0095 |
| 82 | 45 0249 | 0403 | 0557 | 0711 | 0865 | 1018 | 1172 | 1326 | 1479 | 1633 |
| 83 | 1786 | 1940 | 2093 | 2247 | 2400 | 2553 | 2706 | 2859 | 3012 | 3165 |
| 84 | 3318 | 3471 | 3624 | 3777 | 3930 | 4082 | 4235 | 4387 | 4540 | 4692 |
| 85 | 4845 | 4997 | 5150 | 5302 | 5454 | 5606 | 5758 | 5910 | 6062 | 6214 |
| 86 | 6366 | 6518 | 6670 | 6821 | 6973 | 7125 | 7276 | 7428 | 7579 | 7731 |
| 87 | 7882 | 8033 | 8184 | 8336 | 8487 | 8638 | 8789 | 8940 | 9091 | 9242 |
| 88 | 9392 | 9543 | 9694 | 9845 | 9995 | *0146 | *0296 | *0447 | *0597 | *0748 |
| 89 | 46 0898 | 1048 | 1198 | 1348 | 1499 | 1649 | 1799 | 1948 | 2098 | 2248 |
| 290 | 46 2398 | 2548 | 2697 | 2847 | 2997 | 3146 | 3296 | 3445 | 3594 | 3744 |
| 91 | 3893 | 4042 | 4191 | 4340 | 4490 | 4639 | 4788 | 4936 | 5085 | 5234 |
| 92 | 5383 | 5532 | 5680 | 5829 | 5977 | 6126 | 6274 | 6423 | 6571 | 6719 |
| 93 | 6868 | 7016 | 7164 | 7312 | 7460 | 7608 | 7756 | 7904 | 8052 | 8200 |
| 94 | 8347 | 8495 | 8643 | 8790 | 8938 | 9085 | 9233 | 9380 | 9527 | 9675 |
| 95 | 9822 | 9969 | *0116 | *0263 | *0410 | *0557 | *0704 | *0851 | *0998 | *1145 |
| 96 | 47 1292 | 1438 | 1585 | 1732 | 1878 | 2025 | 2171 | 2318 | 2464 | 2610 |
| 97 | 2756 | 2903 | 3049 | 3195 | 3341 | 3487 | 3633 | 3779 | 3925 | 4071 |
| 98 | 4216 | 4362 | 4508 | 4653 | 4799 | 4944 | 5090 | 5235 | 5381 | 5526 |
| 99 | 5671 | 5816 | 5962 | 6107 | 6252 | 6397 | 6542 | 6687 | 6832 | 6976 |
| 300 | 47 7121 | 7266 | 7411 | 7555 | 7700 | 7844 | 7989 | 8133 | 8278 | 8422 |
| 01 | 8566 | 8711 | 8855 | 8999 | 9143 | 9287 | 9431 | 9575 | 9719 | 9863 |
| 02 | 48 0007 | 0151 | 0294 | 0438 | 0582 | 0725 | 0869 | 1012 | 1156 | 1299 |
| 03 | 1443 | 1586 | 1729 | 1872 | 2016 | 2159 | 2302 | 2445 | 2588 | 2731 |
| 04 | 2874 | 3016 | 3159 | 3302 | 3445 | 3587 | 3730 | 3872 | 4015 | 4157 |
| 05 | 4300 | 4442 | 4585 | 4727 | 4869 | 5011 | 5153 | 5295 | 5437 | 5579 |
| 06 | 5721 | 5863 | 6005 | 6147 | 6289 | 6430 | 6572 | 6714 | 6855 | 6997 |
| 07 | 7138 | 7280 | 7421 | 7563 | 7704 | 7845 | 7986 | 8127 | 8269 | 8410 |
| 08 | 8551 | 8692 | 8833 | 8974 | 9114 | 9255 | 9396 | 9537 | 9677 | 9818 |
| 09 | 9958 | *0099 | *0239 | *0380 | *0520 | *0661 | *0801 | *0941 | *1081 | *1222 |
| 310 | 49 1362 | 1502 | 1642 | 1782 | 1922 | 2062 | 2201 | 2341 | 2481 | 2621 |

**310–350** **TABLE I**

| Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | | Proportional Parts | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 140 | 139 | 138 | | 137 | 136 | 135 | | 134 | 133 | 132 | | 131 | 130 | 129 |
| 1 | 14.0 | 13.9 | 13.8 | 1 | 13.7 | 13.6 | 13.5 | 1 | 13.4 | 13.3 | 13.2 | 1 | 13.1 | 13.0 | 12.9 |
| 2 | 28.0 | 27.8 | 27.6 | 2 | 27.4 | 27.2 | 27.0 | 2 | 26.8 | 26.6 | 26.4 | 2 | 26.2 | 26.0 | 25.8 |
| 3 | 42.0 | 41.7 | 41.4 | 3 | 41.1 | 40.8 | 40.5 | 3 | 40.2 | 39.9 | 39.6 | 3 | 39.3 | 39.0 | 38.7 |
| 4 | 56.0 | 55.6 | 55.2 | 4 | 54.8 | 54.4 | 54.0 | 4 | 53.6 | 53.2 | 52.8 | 4 | 52.4 | 52.0 | 51.6 |
| 5 | 70.0 | 69.5 | 69.0 | 5 | 68.5 | 68.0 | 67.5 | 5 | 67.0 | 66.5 | 66.0 | 5 | 65.5 | 65.0 | 64.5 |
| 6 | 84.0 | 83.4 | 82.8 | 6 | 82.2 | 81.6 | 81.0 | 6 | 80.4 | 79.8 | 79.2 | 6 | 78.6 | 78.0 | 77.4 |
| 7 | 98.0 | 97.3 | 96.6 | 7 | 95.9 | 95.2 | 94.5 | 7 | 93.8 | 93.1 | 92.4 | 7 | 91.7 | 91.0 | 90.3 |
| 8 | 112.0 | 111.2 | 110.4 | 8 | 109.6 | 108.8 | 108.0 | 8 | 107.2 | 106.4 | 105.6 | 8 | 104.8 | 104.0 | 103.2 |
| 9 | 126.0 | 125.1 | 124.2 | 9 | 123.3 | 122.4 | 121.5 | 9 | 120.6 | 119.7 | 118.8 | 9 | 117.9 | 117.0 | 116.1 |

## LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 310 | 49 1362 | 1502 | 1642 | 1782 | 1922 | 2062 | 2201 | 2341 | 2481 | 2621 |
| 11 | 2760 | 2900 | 3040 | 3179 | 3319 | 3458 | 3597 | 3737 | 3876 | 4015 |
| 12 | 4155 | 4294 | 4433 | 4572 | 4711 | 4850 | 4989 | 5128 | 5267 | 5406 |
| 13 | 5544 | 5683 | 5822 | 5960 | 6099 | 6238 | 6376 | 6515 | 6653 | 6791 |
| 14 | 6930 | 7068 | 7206 | 7344 | 7483 | 7621 | 7759 | 7897 | 8035 | 8173 |
| 15 | 8311 | 8448 | 8586 | 8724 | 8862 | 8999 | 9137 | 9275 | 9412 | 9550 |
| 16 | 9687 | 9824 | 9962 | *0099 | *0236 | *0374 | *0511 | *0648 | *0785 | *0922 |
| 17 | 50 1059 | 1196 | 1333 | 1470 | 1607 | 1744 | 1880 | 2017 | 2154 | 2291 |
| 18 | 2427 | 2564 | 2700 | 2837 | 2973 | 3109 | 3246 | 3382 | 3518 | 3655 |
| 19 | 3791 | 3927 | 4063 | 4199 | 4335 | 4471 | 4607 | 4743 | 4878 | 5014 |
| 320 | 50 5150 | 5286 | 5421 | 5557 | 5693 | 5828 | 5964 | 6099 | 6234 | 6370 |
| 21 | 6505 | 6640 | 6776 | 6911 | 7046 | 7181 | 7316 | 7451 | 7586 | 7721 |
| 22 | 7856 | 7991 | 8126 | 8260 | 8395 | 8530 | 8664 | 8799 | 8934 | 9068 |
| 23 | 9203 | 9337 | 9471 | 9606 | 9740 | 9874 | *0009 | *0143 | *0277 | *0411 |
| 24 | 51 0545 | 0679 | 0813 | 0947 | 1081 | 1215 | 1349 | 1482 | 1616 | 1750 |
| 25 | 1883 | 2017 | 2151 | 2284 | 2418 | 2551 | 2684 | 2818 | 2951 | 3084 |
| 26 | 3218 | 3351 | 3484 | 3617 | 3750 | 3883 | 4016 | 4149 | 4282 | 4415 |
| 27 | 4548 | 4681 | 4813 | 4946 | 5079 | 5211 | 5344 | 5476 | 5609 | 5741 |
| 28 | 5874 | 6006 | 6139 | 6271 | 6403 | 6535 | 6668 | 6800 | 6932 | 7064 |
| 29 | 7196 | 7328 | 7460 | 7592 | 7724 | 7855 | 7987 | 8119 | 8251 | 8382 |
| 330 | 51 8514 | 8646 | 8777 | 8909 | 9040 | 9171 | 9303 | 9434 | 9566 | 9697 |
| 31 | 9828 | 9959 | *0090 | *0221 | *0353 | *0484 | *0615 | *0745 | *0876 | *1007 |
| 32 | 52 1138 | 1269 | 1400 | 1530 | 1661 | 1792 | 1922 | 2053 | 2183 | 2314 |
| 33 | 2444 | 2575 | 2705 | 2835 | 2966 | 3096 | 3226 | 3356 | 3486 | 3616 |
| 34 | 3746 | 3876 | 4006 | 4136 | 4266 | 4396 | 4526 | 4656 | 4785 | 4915 |
| 35 | 5045 | 5174 | 5304 | 5434 | 5563 | 5693 | 5822 | 5951 | 6081 | 6210 |
| 36 | 6339 | 6469 | 6598 | 6727 | 6856 | 6985 | 7114 | 7243 | 7372 | 7501 |
| 37 | 7630 | 7759 | 7888 | 8016 | 8145 | 8274 | 8402 | 8531 | 8660 | 8788 |
| 38 | 8917 | 9045 | 9174 | 9302 | 9430 | 9559 | 9687 | 9815 | 9943 | *0072 |
| 39 | 53 0200 | 0328 | 0456 | 0584 | 0712 | 0840 | 0968 | 1096 | 1223 | 1351 |
| 340 | 53 1479 | 1607 | 1734 | 1862 | 1990 | 2117 | 2245 | 2372 | 2500 | 2627 |
| 41 | 2754 | 2882 | 3009 | 3136 | 3264 | 3391 | 3518 | 3645 | 3772 | 3899 |
| 42 | 4026 | 4153 | 4280 | 4407 | 4534 | 4661 | 4787 | 4914 | 5041 | 5167 |
| 43 | 5294 | 5421 | 5547 | 5674 | 5800 | 5927 | 6053 | 6180 | 6306 | 6432 |
| 44 | 6558 | 6685 | 6811 | 6937 | 7063 | 7189 | 7315 | 7441 | 7567 | 7693 |
| 45 | 7819 | 7945 | 8071 | 8197 | 8322 | 8448 | 8574 | 8699 | 8825 | 8951 |
| 46 | 9076 | 9202 | 9327 | 9452 | 9578 | 9703 | 9829 | 9954 | *0079 | *0204 |
| 47 | 54 0329 | 0455 | 0580 | 0705 | 0830 | 0955 | 1080 | 1205 | 1330 | 1454 |
| 48 | 1579 | 1704 | 1829 | 1953 | 2078 | 2203 | 2327 | 2452 | 2576 | 2701 |
| 49 | 2825 | 2950 | 3074 | 3199 | 3323 | 3447 | 3571 | 3696 | 3820 | 3944 |
| 350 | 54 4068 | 4192 | 4316 | 4440 | 4564 | 4688 | 4812 | 4937 | 5060 | 5183 |

| | 128 | 127 |
|---|---|---|
| 1 | 12.8 | 12.7 |
| 2 | 25.6 | 25.4 |
| 3 | 38.4 | 38.1 |
| 4 | 51.2 | 50 8 |
| 5 | 64.0 | 63.5 |
| 6 | 76.8 | 76.2 |
| 7 | 89.6 | 88.9 |
| 8 | 102.4 | 101.6 |
| 9 | 115.2 | 114.3 |

| | 126 | 125 |
|---|---|---|
| 1 | 12.6 | 12.5 |
| 2 | 25.2 | 25.0 |
| 3 | 37.8 | 37.5 |
| 4 | 50.4 | 50.0 |
| 5 | 63.0 | 62.5 |
| 6 | 75.6 | 75.0 |
| 7 | 88.2 | 87.5 |
| 8 | 100.8 | 100.0 |
| 9 | 113.4 | 112.5 |

| | 124 |
|---|---|
| 1 | 12.4 |
| 2 | 24.8 |
| 3 | 37.2 |
| 4 | 49.6 |
| 5 | 62.0 |
| 6 | 74.4 |
| 7 | 86.8 |
| 8 | 99.2 |
| 9 | 111.6 |

| | 123 |
|---|---|
| 1 | 12.3 |
| 2 | 24.6 |
| 3 | 36.9 |
| 4 | 49.2 |
| 5 | 61.5 |
| 6 | 73.8 |
| 7 | 86.1 |
| 8 | 98.4 |
| 9 | 110.7 |

350–400 **TABLE I**

| LOGARITHMS | | | | | | | | | | | PROPORTIONAL PARTS | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | | 124 | 123 | 122 |
| 350 | 54 4068 | 4192 | 4316 | 4440 | 4564 | 4688 | 4812 | 4936 | 5060 | 5183 | 1 | 12.4 | 12.3 | 12.2 |
| 51 | 5307 | 5431 | 5555 | 5678 | 5802 | 5925 | 6049 | 6172 | 6296 | 6419 | 2 | 24.8 | 24.6 | 24.4 |
| 52 | 6543 | 6666 | 6789 | 6913 | 7036 | 7159 | 7282 | 7405 | 7529 | 7652 | 3 | 37.2 | 36.9 | 36.6 |
| 53 | 7775 | 7898 | 8021 | 8144 | 8267 | 8389 | 8512 | 8635 | 8758 | 8881 | 4 | 49.6 | 49.2 | 48.8 |
| 54 | 9003 | 9126 | 9249 | 9371 | 9494 | 9616 | 9739 | 9861 | 9984 | *0106 | 5 | 62.0 | 61.5 | 61.0 |
| | | | | | | | | | | | 6 | 74.4 | 73.8 | 73.2 |
| 55 | 55 0228 | 0351 | 0473 | 0595 | 0717 | 0840 | 0962 | 1084 | 1206 | 1328 | 7 | 86.8 | 86.1 | 85.4 |
| 56 | 1450 | 1572 | 1694 | 1816 | 1938 | 2060 | 2181 | 2303 | 2425 | 2547 | 8 | 99.2 | 98.4 | 97.6 |
| 57 | 2668 | 2790 | 2911 | 3033 | 3155 | 3276 | 3398 | 3519 | 3640 | 3762 | 9 | 111.6 | 110.7 | 109.8 |
| 58 | 3883 | 4004 | 4126 | 4247 | 4368 | 4489 | 4610 | 4731 | 4852 | 4973 | | 121 | 120 | 119 |
| 59 | 5094 | 5215 | 5336 | 5457 | 5578 | 5699 | 5820 | 5940 | 6061 | 6182 | | | | |
| | | | | | | | | | | | 1 | 12.1 | 12.0 | 11.9 |
| 360 | 55 6303 | 6423 | 6544 | 6664 | 6785 | 6905 | 7026 | 7146 | 7267 | 7387 | 2 | 24.2 | 24.0 | 23.8 |
| 61 | 7507 | 7627 | 7748 | 7868 | 7988 | 8108 | 8228 | 8349 | 8469 | 8589 | 3 | 36.3 | 36.0 | 35.7 |
| 62 | 8709 | 8829 | 8949 | 9068 | 9188 | 9308 | 9428 | 9548 | 9667 | 9787 | 4 | 48.4 | 48.0 | 47.6 |
| 63 | 9907 | *0026 | *0146 | *0265 | *0385 | *0504 | *0624 | *0743 | *0863 | *0982 | 5 | 60.5 | 60.0 | 59.5 |
| 64 | 56 1101 | 1221 | 1340 | 1459 | 1578 | 1698 | 1817 | 1936 | 2055 | 2174 | 6 | 72.6 | 72.0 | 71.4 |
| | | | | | | | | | | | 7 | 84.7 | 84.0 | 83.3 |
| 65 | 2293 | 2412 | 2531 | 2650 | 2769 | 2887 | 3006 | 3125 | 3244 | 3362 | 8 | 96.8 | 96.0 | 95.2 |
| 66 | 3481 | 3600 | 3718 | 3837 | 3955 | 4074 | 4192 | 4311 | 4429 | 4548 | 9 | 108.9 | 108.0 | 107.1 |
| 67 | 4666 | 4784 | 4903 | 5021 | 5139 | 5257 | 5376 | 5494 | 5612 | 5730 | | 118 | 117 | 116 |
| 68 | 5848 | 5966 | 6084 | 6202 | 6320 | 6437 | 6555 | 6673 | 6791 | 6909 | | | | |
| 69 | 7026 | 7144 | 7262 | 7379 | 7497 | 7614 | 7732 | 7849 | 7967 | 8084 | 1 | 11.8 | 11.7 | 11.6 |
| | | | | | | | | | | | 2 | 23.6 | 23.4 | 23.2 |
| 370 | 56 8202 | 8319 | 8436 | 8554 | 8671 | 8788 | 8905 | 9023 | 9140 | 9257 | 3 | 35.4 | 35.1 | 34.8 |
| 71 | 9374 | 9491 | 9608 | 9725 | 9842 | 9959 | *0076 | *0193 | *0309 | *0426 | 4 | 47.2 | 46.8 | 46.4 |
| 72 | 57 0543 | 0660 | 0776 | 0893 | 1010 | 1126 | 1243 | 1359 | 1476 | 1592 | 5 | 59.0 | 58.5 | 58.0 |
| 73 | 1709 | 1825 | 1942 | 2058 | 2174 | 2291 | 2407 | 2523 | 2639 | 2755 | 6 | 70.8 | 70.2 | 69.6 |
| 74 | 2872 | 2988 | 3104 | 3220 | 3336 | 3452 | 3568 | 3684 | 3800 | 3915 | 7 | 82.6 | 81.9 | 81.2 |
| | | | | | | | | | | | 8 | 94.4 | 93.6 | 92.8 |
| 75 | 4031 | 4147 | 4263 | 4379 | 4494 | 4610 | 4726 | 4841 | 4957 | 5072 | 9 | 106.2 | 105.3 | 104.4 |
| 76 | 5188 | 5303 | 5419 | 5534 | 5650 | 5765 | 5880 | 5996 | 6111 | 6226 | | 115 | 114 | 113 |
| 77 | 6341 | 6457 | 6572 | 6687 | 6802 | 6917 | 7032 | 7147 | 7262 | 7377 | | | | |
| 78 | 7492 | 7607 | 7722 | 7836 | 7951 | 8066 | 8181 | 8295 | 8410 | 8525 | 1 | 11.5 | 11.4 | 11.3 |
| 79 | 8639 | 8754 | 8868 | 8983 | 9097 | 9212 | 9326 | 9441 | 9555 | 9669 | 2 | 23.0 | 22.8 | 22.6 |
| | | | | | | | | | | | 3 | 34.5 | 34.2 | 33.9 |
| 380 | 57 9784 | 9898 | *0012 | *0126 | *0241 | *0355 | *0469 | *0583 | *0697 | *0811 | 4 | 46.0 | 45.6 | 45.2 |
| 81 | 58 0925 | 1039 | 1153 | 1267 | 1381 | 1495 | 1608 | 1722 | 1836 | 1950 | 5 | 57.5 | 57.0 | 56.5 |
| 82 | 2063 | 2177 | 2291 | 2404 | 2518 | 2631 | 2745 | 2858 | 2972 | 3085 | 6 | 69.0 | 68.4 | 67.8 |
| 83 | 3199 | 3312 | 3426 | 3539 | 3652 | 3765 | 3879 | 3992 | 4105 | 4218 | 7 | 80.5 | 79.8 | 79.1 |
| 84 | 4331 | 4444 | 4557 | 4670 | 4783 | 4896 | 5009 | 5122 | 5235 | 5348 | 8 | 92.0 | 91.2 | 90.4 |
| | | | | | | | | | | | 9 | 103.5 | 102.6 | 101.7 |
| 85 | 5461 | 5574 | 5686 | 5799 | 5912 | 6024 | 6137 | 6250 | 6362 | 6475 | | 112 | 111 | 110 |
| 86 | 6587 | 6700 | 6812 | 6925 | 7037 | 7149 | 7262 | 7374 | 7486 | 7599 | | | | |
| 87 | 7711 | 7823 | 7935 | 8047 | 8160 | 8272 | 8384 | 8496 | 8608 | 8720 | 1 | 11.2 | 11.1 | 11.0 |
| 88 | 8832 | 8944 | 9056 | 9167 | 9279 | 9391 | 9503 | 9615 | 9726 | 9838 | 2 | 22.4 | 22.2 | 22.0 |
| 89 | 9950 | *0061 | *0173 | *0284 | *0396 | *0507 | *0619 | *0730 | *0842 | *0953 | 3 | 33.6 | 33.3 | 33.0 |
| | | | | | | | | | | | 4 | 44.8 | 44.4 | 44.0 |
| 390 | 59 1065 | 1176 | 1287 | 1399 | 1510 | 1621 | 1732 | 1843 | 1955 | 2066 | 5 | 56.0 | 55.5 | 55.0 |
| 91 | 2177 | 2288 | 2399 | 2510 | 2621 | 2732 | 2843 | 2954 | 3064 | 3175 | 6 | 67.2 | 66.6 | 66.0 |
| 92 | 3286 | 3397 | 3508 | 3618 | 3729 | 3840 | 3950 | 4061 | 4171 | 4282 | 7 | 78.4 | 77.7 | 77.0 |
| 93 | 4393 | 4503 | 4614 | 4724 | 4834 | 4945 | 5055 | 5165 | 5276 | 5386 | 8 | 89.6 | 88.8 | 88.0 |
| 94 | 5496 | 5606 | 5717 | 5827 | 5937 | 6047 | 6157 | 6267 | 6377 | 6487 | 9 | 100.8 | 99.9 | 99.0 |
| | | | | | | | | | | | | 109 | 108 | |
| 95 | 6597 | 6707 | 6817 | 6927 | 7037 | 7146 | 7256 | 7366 | 7476 | 7586 | 1 | 10.9 | 10.8 | |
| 96 | 7695 | 7805 | 7914 | 8024 | 8134 | 8243 | 8353 | 8462 | 8572 | 8681 | 2 | 21.8 | 21.6 | |
| 97 | 8791 | 8900 | 9009 | 9119 | 9228 | 9337 | 9446 | 9556 | 9665 | 9774 | 3 | 32.7 | 32.4 | |
| 98 | 9883 | 9992 | *0101 | *0210 | *0319 | *0428 | *0537 | *0646 | *0755 | *0864 | 4 | 43.6 | 43.2 | |
| 99 | 60 0973 | 1082 | 1191 | 1299 | 1408 | 1517 | 1625 | 1734 | 1843 | 1951 | 5 | 54.5 | 54.0 | |
| | | | | | | | | | | | 6 | 65.4 | 64.8 | |
| 400 | 60 2060 | 2169 | 2277 | 2386 | 2494 | 2603 | 2711 | 2819 | 2928 | 3036 | 7 | 76.3 | 75.6 | |
| | | | | | | | | | | | 8 | 87.2 | 86.4 | |
| | | | | | | | | | | | 9 | 98.1 | 97.2 | |

400–450 **TABLE I**

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 400 | 60 2060 | 2169 | 2277 | 2386 | 2494 | 2603 | 2711 | 2819 | 2928 | 3036 |
| 01 | 3144 | 3253 | 3361 | 3469 | 3577 | 3686 | 3794 | 3902 | 4010 | 4118 |
| 02 | 4226 | 4334 | 4442 | 4550 | 4658 | 4766 | 4874 | 4982 | 5089 | 5197 |
| 03 | 5305 | 5413 | 5521 | 5628 | 5736 | 5844 | 5951 | 6059 | 6166 | 6274 |
| 04 | 6381 | 6489 | 6596 | 6704 | 6811 | 6919 | 7026 | 7133 | 7241 | 7348 |
| 05 | 7455 | 7562 | 7669 | 7777 | 7884 | 7991 | 8098 | 8205 | 8312 | 8419 |
| 06 | 8526 | 8633 | 8740 | 8847 | 8954 | 9061 | 9167 | 9274 | 9381 | 9488 |
| 07 | 9594 | 9701 | 9808 | 9914 | *0021 | *0128 | *0234 | *0341 | *0447 | *0554 |
| 08 | 61 0660 | 0767 | 0873 | 0979 | 1086 | 1192 | 1298 | 1405 | 1511 | 1617 |
| 09 | 1723 | 1829 | 1936 | 2042 | 2148 | 2254 | 2360 | 2466 | 2572 | 2678 |
| 410 | 61 2784 | 2890 | 2996 | 3102 | 3207 | 3313 | 3419 | 3525 | 3630 | 3736 |
| 11 | 3842 | 3947 | 4053 | 4159 | 4264 | 4370 | 4475 | 4581 | 4686 | 4792 |
| 12 | 4897 | 5003 | 5108 | 5213 | 5319 | 5424 | 5529 | 5634 | 5740 | 5845 |
| 13 | 5950 | 6055 | 6160 | 6265 | 6370 | 6476 | 6581 | 6686 | 6790 | 6895 |
| 14 | 7000 | 7105 | 7210 | 7315 | 7420 | 7525 | 7629 | 7734 | 7839 | 7943 |
| 15 | 8048 | 8153 | 8257 | 8362 | 8466 | 8571 | 8676 | 8780 | 8884 | 8989 |
| 16 | 9093 | 9198 | 9302 | 9406 | 9511 | 9615 | 9719 | 9824 | 9928 | *0032 |
| 17 | 62 0136 | 0240 | 0344 | 0448 | 0552 | 0656 | 0760 | 0864 | 0968 | 1072 |
| 18 | 1176 | 1280 | 1384 | 1488 | 1592 | 1695 | 1799 | 1903 | 2007 | 2110 |
| 19 | 2214 | 2318 | 2421 | 2525 | 2628 | 2732 | 2835 | 2939 | 3042 | 3146 |
| 420 | 62 3249 | 3353 | 3456 | 3559 | 3663 | 3766 | 3869 | 3973 | 4076 | 4179 |
| 21 | 4282 | 4385 | 4488 | 4591 | 4695 | 4798 | 4901 | 5004 | 5107 | 5210 |
| 22 | 5312 | 5415 | 5518 | 5621 | 5724 | 5827 | 5929 | 6032 | 6135 | 6238 |
| 23 | 6340 | 6443 | 6546 | 6648 | 6751 | 6853 | 6956 | 7058 | 7161 | 7263 |
| 24 | 7366 | 7468 | 7571 | 7673 | 7775 | 7878 | 7980 | 8082 | 8185 | 8287 |
| 25 | 8389 | 8491 | 8593 | 8695 | 8797 | 8900 | 9002 | 9104 | 9206 | 9308 |
| 26 | 9410 | 9512 | 9613 | 9715 | 9817 | 9919 | *0021 | *0123 | *0224 | *0326 |
| 27 | 63 0428 | 0530 | 0631 | 0733 | 0835 | 0936 | 1038 | 1139 | 1241 | 1342 |
| 28 | 1444 | 1545 | 1647 | 1748 | 1849 | 1951 | 2052 | 2153 | 2255 | 2356 |
| 29 | 2457 | 2559 | 2660 | 2761 | 2862 | 2963 | 3064 | 3165 | 3266 | 3367 |
| 430 | 63 3468 | 3569 | 3670 | 3771 | 3872 | 3973 | 4074 | 4175 | 4276 | 4376 |
| 31 | 4477 | 4578 | 4679 | 4779 | 4880 | 4981 | 5081 | 5182 | 5283 | 5383 |
| 32 | 5484 | 5584 | 5685 | 5785 | 5886 | 5986 | 6087 | 6187 | 6287 | 6388 |
| 33 | 6488 | 6588 | 6688 | 6789 | 6889 | 6989 | 7089 | 7189 | 7290 | 7390 |
| 34 | 7490 | 7590 | 7690 | 7790 | 7890 | 7990 | 8090 | 8190 | 8290 | 8389 |
| 35 | 8489 | 8589 | 8689 | 8789 | 8888 | 8988 | 9088 | 9188 | 9287 | 9387 |
| 36 | 9486 | 9586 | 9686 | 9785 | 9885 | 9984 | *0084 | *0183 | *0283 | *0382 |
| 37 | 64 0481 | 0581 | 0680 | 0779 | 0879 | 0978 | 1077 | 1177 | 1276 | 1375 |
| 38 | 1474 | 1573 | 1672 | 1771 | 1871 | 1970 | 2069 | 2168 | 2267 | 2366 |
| 39 | 2465 | 2563 | 2662 | 2761 | 2860 | 2959 | 3058 | 3156 | 3255 | 3354 |
| 440 | 64 3453 | 3551 | 3650 | 3749 | 3847 | 3946 | 4044 | 4143 | 4242 | 4340 |
| 41 | 4439 | 4537 | 4636 | 4734 | 4832 | 4931 | 5029 | 5127 | 5226 | 5324 |
| 42 | 5422 | 5521 | 5619 | 5717 | 5815 | 5913 | 6011 | 6110 | 6208 | 6306 |
| 43 | 6404 | 6502 | 6600 | 6698 | 6796 | 6894 | 6992 | 7089 | 7187 | 7285 |
| 44 | 7383 | 7481 | 7579 | 7676 | 7774 | 7872 | 7969 | 8067 | 8165 | 8262 |
| 45 | 8360 | 8458 | 8555 | 8653 | 8750 | 8848 | 8945 | 9043 | 9140 | 9237 |
| 46 | 9335 | 9432 | 9530 | 9627 | 9724 | 9821 | 9919 | *0016 | *0113 | *0210 |
| 47 | 65 0308 | 0405 | 0502 | 0599 | 0696 | 0793 | 0890 | 0987 | 1084 | 1181 |
| 48 | 1278 | 1375 | 1472 | 1569 | 1666 | 1762 | 1859 | 1956 | 2053 | 2150 |
| 49 | 2246 | 2343 | 2440 | 2536 | 2633 | 2730 | 2826 | 2923 | 3019 | 3116 |
| 450 | 653213 | 3309 | 3405 | 3502 | 3598 | 3695 | 3791 | 3888 | 3984 | 4080 |

PROPORTIONAL PARTS

| | 109 | 108 | 107 |
|---|---|---|---|
| 1 | 10.9 | 10.8 | 10.7 |
| 2 | 21.8 | 21.6 | 21.4 |
| 3 | 32.7 | 32.4 | 32.1 |
| 4 | 43.6 | 43.2 | 42.8 |
| 5 | 54.5 | 54.0 | 53.5 |
| 6 | 65.4 | 64.8 | 64.2 |
| 7 | 76.3 | 75.6 | 74.9 |
| 8 | 87.2 | 86.4 | 85.6 |
| 9 | 98.1 | 97.2 | 96.3 |

| | 106 | 105 | 104 |
|---|---|---|---|
| 1 | 10.6 | 10.5 | 10.4 |
| 2 | 21.2 | 21.0 | 20.8 |
| 3 | 31.8 | 31.5 | 31.2 |
| 4 | 42.4 | 42.0 | 41.6 |
| 5 | 53.0 | 52.5 | 52.0 |
| 6 | 63.6 | 63.0 | 62.4 |
| 7 | 74.2 | 73.5 | 72.8 |
| 8 | 84.8 | 84.0 | 83.2 |
| 9 | 95.4 | 94.5 | 93.6 |

| | 103 | 102 | 101 |
|---|---|---|---|
| 1 | 10.3 | 10.2 | 10.1 |
| 2 | 20.6 | 20.4 | 20.2 |
| 3 | 30.9 | 30.6 | 30.3 |
| 4 | 41.2 | 40.8 | 40.4 |
| 5 | 51.5 | 51.0 | 50.5 |
| 6 | 61.8 | 61.2 | 60.6 |
| 7 | 72.1 | 71.4 | 70.7 |
| 8 | 82.4 | 81.6 | 80.8 |
| 9 | 92.7 | 91.8 | 90.9 |

| | 100 | 99 |
|---|---|---|
| 1 | 10.0 | 9.9 |
| 2 | 20.0 | 19.8 |
| 3 | 30.0 | 29.7 |
| 4 | 40.0 | 39.6 |
| 5 | 50.0 | 49.5 |
| 6 | 60.0 | 59.4 |
| 7 | 70.0 | 69.3 |
| 8 | 80.0 | 79.2 |
| 9 | 90.0 | 89.1 |

| | 98 |
|---|---|
| 1 | 9.8 |
| 2 | 19.6 |
| 3 | 29.4 |
| 4 | 39.2 |
| 5 | 49.0 |
| 6 | 58.8 |
| 7 | 68.6 |
| 8 | 78.4 |
| 9 | 88.2 |

| | 97 |
|---|---|
| 1 | 9.7 |
| 2 | 19.4 |
| 3 | 29.1 |
| 4 | 38.8 |
| 5 | 48.5 |
| 6 | 58.2 |
| 7 | 67.9 |
| 8 | 77.6 |
| 9 | 87.3 |

450–500

## TABLE I

| N | LOGARITHMS 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 450 | 65 3213 | 3309 | 3405 | 3502 | 3598 | 3695 | 3791 | 3888 | 3984 | 4080 |
| 51 | 4177 | 4273 | 4369 | 4465 | 4562 | 4658 | 4754 | 4850 | 4946 | 5042 |
| 52 | 5138 | 5235 | 5331 | 5427 | 5523 | 5619 | 5715 | 5810 | 5906 | 6002 |
| 53 | 6098 | 6194 | 6290 | 6386 | 6482 | 6577 | 6673 | 6769 | 6864 | 6960 |
| 54 | 7056 | 7152 | 7247 | 7343 | 7438 | 7534 | 7629 | 7725 | 7820 | 7916 |
| 55 | 8011 | 8107 | 8202 | 8298 | 8393 | 8488 | 8584 | 8679 | 8774 | 8870 |
| 56 | 8965 | 9060 | 9155 | 9250 | 9346 | 9441 | 9536 | 9631 | 9726 | 9821 |
| 57 | 9916 | *0011 | *0106 | *0201 | *0296 | *0391 | *0486 | *0581 | *0676 | *0771 |
| 58 | 66 0865 | 0960 | 1055 | 1150 | 1245 | 1339 | 1434 | 1529 | 1623 | 1718 |
| 59 | 1813 | 1907 | 2002 | 2096 | 2191 | 2286 | 2380 | 2475 | 2569 | 2663 |
| 460 | 66 2758 | 2852 | 2947 | 3041 | 3135 | 3230 | 3324 | 3418 | 3512 | 3607 |
| 61 | 3701 | 3795 | 3889 | 3983 | 4078 | 4172 | 4266 | 4360 | 4454 | 4548 |
| 62 | 4642 | 4736 | 4830 | 4924 | 5018 | 5112 | 5206 | 5299 | 5393 | 5487 |
| 63 | 5581 | 5675 | 5769 | 5862 | 5956 | 6050 | 6143 | 6237 | 6331 | 6424 |
| 64 | 6518 | 6612 | 6705 | 6799 | 6892 | 6986 | 7079 | 7173 | 7266 | 7360 |
| 65 | 7453 | 7546 | 7640 | 7733 | 7826 | 7920 | 8013 | 8106 | 8199 | 8293 |
| 66 | 8386 | 8479 | 8572 | 8665 | 8759 | 8852 | 8945 | 9038 | 9131 | 9224 |
| 67 | 9317 | 9410 | 9503 | 9596 | 9689 | 9782 | 9875 | 9967 | *0060 | *0153 |
| 68 | 67 0246 | 0339 | 0431 | 0524 | 0617 | 0710 | 0802 | 0895 | 0988 | 1080 |
| 69 | 1173 | 1265 | 1358 | 1451 | 1543 | 1636 | 1728 | 1821 | 1913 | 2005 |
| 470 | 67 2098 | 2190 | 2283 | 2375 | 2467 | 2560 | 2652 | 2744 | 2836 | 2929 |
| 71 | 3021 | 3113 | 3205 | 3297 | 3390 | 3482 | 3574 | 3666 | 3758 | 3850 |
| 72 | 3942 | 4034 | 4126 | 4218 | 4310 | 4402 | 4494 | 4586 | 4677 | 4769 |
| 73 | 4861 | 4953 | 5045 | 5137 | 5228 | 5320 | 5412 | 5503 | 5595 | 5687 |
| 74 | 5778 | 5870 | 5962 | 6053 | 6145 | 6236 | 6328 | 6419 | 6511 | 6602 |
| 75 | 6694 | 6785 | 6876 | 6968 | 7059 | 7151 | 7242 | 7333 | 7424 | 7516 |
| 76 | 7607 | 7698 | 7789 | 7881 | 7972 | 8063 | 8154 | 8245 | 8336 | 8427 |
| 77 | 8518 | 8609 | 8700 | 8791 | 8882 | 8973 | 9064 | 9155 | 9246 | 9337 |
| 78 | 9428 | 9519 | 9610 | 9700 | 9791 | 9882 | 9973 | *0063 | *0154 | *0245 |
| 79 | 68 0336 | 0426 | 0517 | 0607 | 0698 | 0789 | 0879 | 0970 | 1060 | 1151 |
| 480 | 68 1241 | 1332 | 1422 | 1513 | 1603 | 1693 | 1784 | 1874 | 1964 | 2055 |
| 81 | 2145 | 2235 | 2326 | 2416 | 2506 | 2596 | 2686 | 2777 | 2867 | 2957 |
| 82 | 3047 | 3137 | 3227 | 3317 | 3407 | 3497 | 3587 | 3677 | 3767 | 3857 |
| 83 | 3947 | 4037 | 4127 | 4217 | 4307 | 4396 | 4486 | 4576 | 4666 | 4756 |
| 84 | 4845 | 4935 | 5025 | 5114 | 5204 | 5294 | 5383 | 5473 | 5563 | 5652 |
| 85 | 5742 | 5831 | 5921 | 6010 | 6100 | 6189 | 6279 | 6368 | 6458 | 6547 |
| 86 | 6636 | 6726 | 6815 | 6904 | 6994 | 7083 | 7172 | 7261 | 7351 | 7440 |
| 87 | 7529 | 7618 | 7707 | 7796 | 7886 | 7975 | 8064 | 8153 | 8242 | 8331 |
| 88 | 8420 | 8509 | 8598 | 8687 | 8776 | 8865 | 8953 | 9042 | 9131 | 9220 |
| 89 | 9309 | 9398 | 9486 | 9575 | 9664 | 9753 | 9841 | 9930 | *0019 | *0107 |
| 490 | 69 0196 | 0285 | 0373 | 0462 | 0550 | 0639 | 0728 | 0816 | 0905 | 0993 |
| 91 | 1081 | 1170 | 1258 | 1347 | 1435 | 1524 | 1612 | 1700 | 1789 | 1877 |
| 92 | 1965 | 2053 | 2142 | 2230 | 2318 | 2406 | 2494 | 2583 | 2671 | 2759 |
| 93 | 2847 | 2935 | 3023 | 3111 | 3199 | 3287 | 3375 | 3463 | 3551 | 3639 |
| 94 | 3727 | 3815 | 3903 | 3991 | 4078 | 4166 | 4254 | 4342 | 4430 | 4517 |
| 95 | 4605 | 4693 | 4781 | 4868 | 4956 | 5044 | 5131 | 5219 | 5307 | 5394 |
| 96 | 5482 | 5569 | 5657 | 5744 | 5832 | 5919 | 6007 | 6094 | 6182 | 6269 |
| 97 | 6356 | 6444 | 6531 | 6618 | 6706 | 6793 | 6880 | 6968 | 7055 | 7142 |
| 98 | 7229 | 7317 | 7404 | 7491 | 7578 | 7665 | 7752 | 7839 | 7926 | 8014 |
| 99 | 8101 | 8188 | 8275 | 8362 | 8449 | 8535 | 8622 | 8709 | 8796 | 8883 |
| 500 | 69 8970 | 9057 | 9144 | 9231 | 9317 | 9404 | 9491 | 9578 | 9664 | 9751 |

| PROPORTIONAL PARTS | | | |
|---|---|---|---|
| | 96 | 95 | 94 |
| 1 | 9.6 | 9.5 | 9.4 |
| 2 | 19.2 | 19.0 | 18.8 |
| 3 | 28.8 | 28.5 | 28.2 |
| 4 | 38.4 | 38.0 | 37.6 |
| 5 | 48.0 | 47.5 | 47.0 |
| 6 | 57.6 | 57.0 | 56.4 |
| 7 | 67.2 | 66.5 | 65.8 |
| 8 | 76.8 | 76.0 | 75.2 |
| 9 | 86.4 | 85.5 | 84.6 |

| | 93 | 92 |
|---|---|---|
| 1 | 9.3 | 9.2 |
| 2 | 18.6 | 18.4 |
| 3 | 27.9 | 27.6 |
| 4 | 37.2 | 36.8 |
| 5 | 46.5 | 46.0 |
| 6 | 55.8 | 55.2 |
| 7 | 65.1 | 64.4 |
| 8 | 74.4 | 73.6 |
| 9 | 83.7 | 82.8 |

| | 91 | 90 |
|---|---|---|
| 1 | 9.1 | 9.0 |
| 2 | 18.2 | 18.0 |
| 3 | 27.3 | 27.0 |
| 4 | 36.4 | 36.0 |
| 5 | 45.5 | 45.0 |
| 6 | 54.6 | 54.0 |
| 7 | 63.7 | 63.0 |
| 8 | 72.8 | 72.0 |
| 9 | 81.9 | 81.0 |

| | 89 |
|---|---|
| 1 | 8.9 |
| 2 | 17.8 |
| 3 | 26.7 |
| 4 | 35.6 |
| 5 | 44.5 |
| 6 | 53.4 |
| 7 | 62.3 |
| 8 | 71.2 |
| 9 | 80.1 |

| | 88 |
|---|---|
| 1 | 8.8 |
| 2 | 17.6 |
| 3 | 26.4 |
| 4 | 35.2 |
| 5 | 44.0 |
| 6 | 52.8 |
| 7 | 61.6 |
| 8 | 70.4 |
| 9 | 79.2 |

| | 87 |
|---|---|
| 1 | 8.7 |
| 2 | 17.4 |
| 3 | 26.1 |
| 4 | 34.8 |
| 5 | 43.5 |
| 6 | 52.2 |
| 7 | 60.9 |
| 8 | 69.6 |
| 9 | 78.3 |

500–550 TABLE I

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 500 | 69 8370 | 8057 | 9144 | 9231 | 9317 | 9404 | 9491 | 9578 | 9664 | 9751 |
| 01 | 9838 | 9924 | *0011 | *0098 | *0184 | *0271 | *0358 | *0444 | *0531 | *0617 |
| 02 | 70 0704 | 0790 | 0877 | 0963 | 1050 | 1136 | 1222 | 1309 | 1395 | 1482 |
| 03 | 1568 | 1654 | 1741 | 1827 | 1913 | 1999 | 2086 | 2172 | 2258 | 2344 |
| 04 | 2431 | 2517 | 2603 | 2689 | 2775 | 2861 | 2947 | 3033 | 3119 | 3205 |
| 05 | 3291 | 3377 | 3463 | 3549 | 3635 | 3721 | 3807 | 3893 | 3979 | 4065 |
| 06 | 4151 | 4236 | 4322 | 4408 | 4494 | 4579 | 4665 | 4751 | 4837 | 4922 |
| 07 | 5008 | 5094 | 5179 | 5265 | 5350 | 5436 | 5522 | 5607 | 5693 | 5778 |
| 08 | 5864 | 5949 | 6035 | 6120 | 6206 | 6291 | 6376 | 6462 | 6547 | 6632 |
| 09 | 6718 | 6803 | 6888 | 6974 | 7059 | 7144 | 7229 | 7315 | 7400 | 7485 |
| 510 | 70 7570 | 7655 | 7740 | 7826 | 7911 | 7996 | 8081 | 8166 | 8251 | 8336 |
| 11 | 8421 | 8506 | 8591 | 8676 | 8761 | 8846 | 8931 | 9015 | 9100 | 9185 |
| 12 | 9270 | 9355 | 9440 | 9524 | 9609 | 9694 | 9779 | 9863 | 9948 | *0033 |
| 13 | 71 0117 | 0202 | 0287 | 0371 | 0456 | 0540 | 0625 | 0710 | 0794 | 0879 |
| 14 | 0963 | 1048 | 1132 | 1217 | 130 | 1385 | 1470 | 1554 | 1639 | 1723 |
| 15 | 1807 | 1892 | 1976 | 2060 | 2144 | 2229 | 2313 | 2397 | 2481 | 2566 |
| 16 | 2650 | 2734 | 2818 | 2902 | 2986 | 3070 | 3154 | 3238 | 3323 | 3407 |
| 17 | 3491 | 3575 | 3659 | 3742 | 3826 | 3910 | 3994 | 4078 | 4162 | 4246 |
| 18 | 4330 | 4414 | 4497 | 4581 | 4665 | 4749 | 4833 | 4916 | 5000 | 5084 |
| 19 | 5167 | 5251 | 5335 | 5418 | 5502 | 5586 | 5669 | 5753 | 5836 | 5920 |
| 520 | 71 6003 | 6087 | 6170 | 6254 | 6337 | 6421 | 6504 | 6588 | 6671 | 6754 |
| 21 | 6838 | 6921 | 7004 | 7088 | 7171 | 7254 | 7338 | 7421 | 7504 | 7587 |
| 22 | 7671 | 7754 | 7837 | 7920 | 8003 | 8086 | 8169 | 8253 | 8336 | 8419 |
| 23 | 8502 | 8585 | 8668 | 8751 | 8834 | 8917 | 9000 | 9083 | 9165 | 9248 |
| 24 | 9331 | 9414 | 9497 | 9580 | 9663 | 9745 | 9828 | 9911 | 9994 | *0077 |
| 25 | 72 0159 | 0242 | 0325 | 0407 | 0490 | 0573 | 0655 | 0738 | 0821 | 0903 |
| 26 | 0986 | 1068 | 1151 | 1233 | 1316 | 1398 | 1481 | 1563 | 1646 | 1728 |
| 27 | 1811 | 1893 | 1975 | 2058 | 2140 | 2222 | 2305 | 2387 | 2469 | 2552 |
| 28 | 2634 | 2716 | 2798 | 2881 | 2963 | 3045 | 3127 | 3209 | 3291 | 3374 |
| 29 | 3456 | 3538 | 3620 | 3702 | 3784 | 3866 | 3948 | 4030 | 4112 | 4194 |
| 530 | 72 4276 | 4358 | 4440 | 4522 | 4604 | 4685 | 4767 | 4849 | 4931 | 5013 |
| 31 | 5095 | 5176 | 5258 | 5340 | 5422 | 5503 | 5585 | 5667 | 5748 | 5830 |
| 32 | 5912 | 5993 | 6075 | 6156 | 6238 | 6320 | 6401 | 6483 | 6564 | 6646 |
| 33 | 6727 | 6809 | 6890 | 6972 | 7053 | 7134 | 7216 | 7297 | 7379 | 7460 |
| 34 | 7541 | 7623 | 7704 | 7785 | 7866 | 7948 | 8029 | 8110 | 8191 | 8273 |
| 35 | 8354 | 8435 | 8516 | 8597 | 8678 | 8759 | 8841 | 8922 | 9003 | 9084 |
| 36 | 9165 | 9246 | 9327 | 9408 | 9489 | 9570 | 9651 | 9732 | 9813 | 9893 |
| 37 | 9974 | *0055 | *0136 | *0217 | *0298 | *0378 | *0459 | *0540 | *0621 | *0702 |
| 38 | 73 0782 | 0863 | 0944 | 1024 | 1105 | 1186 | 1266 | 1347 | 1428 | 1508 |
| 39 | 1589 | 1669 | 1750 | 1830 | 1911 | 1991 | 2072 | 2152 | 2233 | 2313 |
| 540 | 73 2394 | 2474 | 2555 | 2635 | 2715 | 2796 | 2876 | 2956 | 3037 | 3117 |
| 41 | 3197 | 3278 | 3358 | 3438 | 3518 | 3598 | 3679 | 3759 | 3839 | 3919 |
| 42 | 3999 | 4079 | 4160 | 4240 | 4320 | 4400 | 4480 | 4560 | 4640 | 4720 |
| 43 | 4800 | 4880 | 4960 | 5040 | 5120 | 5200 | 5279 | 5359 | 5439 | 5519 |
| 44 | 5599 | 5679 | 5759 | 5838 | 5918 | 5998 | 6078 | 6157 | 6237 | 6317 |
| 45 | 6397 | 6476 | 6556 | 6635 | 6715 | 6795 | 6874 | 6954 | 7034 | 7113 |
| 46 | 7193 | 7272 | 7352 | 7431 | 7511 | 7590 | 7670 | 7749 | 7829 | 7908 |
| 47 | 7987 | 8067 | 8146 | 8225 | 8305 | 8384 | 8463 | 8543 | 8622 | 8701 |
| 48 | 8781 | 8860 | 8939 | 9018 | 9097 | 9177 | 9256 | 9335 | 9414 | 9493 |
| 49 | 9572 | 9651 | 9731 | 9810 | 9889 | 9968 | *0047 | *0126 | *0205 | *0284 |
| 550 | 74 0363 | 0442 | 0521 | 0600 | 0678 | 0757 | 0836 | 0915 | 0994 | 1073 |

PROPORTIONAL PARTS

| | 87 | 86 | 85 |
|---|---|---|---|
| 1 | 8.7 | 8.6 | 8.5 |
| 2 | 17.4 | 17.2 | 17.0 |
| 3 | 26.1 | 25.8 | 25.5 |
| 4 | 34.8 | 34.4 | 34.0 |
| 5 | 43.5 | 43.0 | 42.5 |
| 6 | 52.2 | 51.6 | 51.0 |
| 7 | 60.9 | 60.2 | 59.5 |
| 8 | 69.6 | 68.8 | 68.0 |
| 9 | 78.3 | 77.4 | 76.5 |

| | 84 | 83 |
|---|---|---|
| 1 | 8.4 | 8.3 |
| 2 | 16.8 | 16.6 |
| 3 | 25.2 | 24.9 |
| 4 | 33.6 | 33.2 |
| 5 | 42.0 | 41.5 |
| 6 | 50.4 | 49.8 |
| 7 | 58.8 | 58.1 |
| 8 | 67.2 | 66.4 |
| 9 | 75.6 | 74.7 |

| | 82 |
|---|---|
| 1 | 8.2 |
| 2 | 16.4 |
| 3 | 24.6 |
| 4 | 32.8 |
| 5 | 41.0 |
| 6 | 49.2 |
| 7 | 57.4 |
| 8 | 65.6 |
| 9 | 73.8 |

| | 81 |
|---|---|
| 1 | 8.1 |
| 2 | 16.2 |
| 3 | 24.3 |
| 4 | 32.4 |
| 5 | 40.5 |
| 6 | 48.6 |
| 7 | 56.7 |
| 8 | 64.8 |
| 9 | 72.9 |

| | 80 |
|---|---|
| 1 | 8.0 |
| 2 | 16.0 |
| 3 | 24.0 |
| 4 | 32.0 |
| 5 | 40.0 |
| 6 | 48.0 |
| 7 | 56.0 |
| 8 | 64.0 |
| 9 | 72.0 |

| | 79 |
|---|---|
| 1 | 7.9 |
| 2 | 15.8 |
| 3 | 23.7 |
| 4 | 31.6 |
| 5 | 39.5 |
| 6 | 47.4 |
| 7 | 55.3 |
| 8 | 63.2 |
| 9 | 71.1 |

550–600 TABLE I

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 550 | 74 0363 | 0442 | 0521 | 0600 | 0678 | 0757 | 0836 | 0915 | 0994 | 1073 |
| 51 | 1152 | 1230 | 1309 | 1388 | 1467 | 1546 | 1624 | 1703 | 1782 | 1860 |
| 52 | 1939 | 2018 | 2096 | 2175 | 2254 | 2332 | 2411 | 2489 | 2568 | 2647 |
| 53 | 2725 | 2804 | 2882 | 2961 | 3039 | 3118 | 3196 | 3275 | 3353 | 3431 |
| 54 | 3510 | 3588 | 3667 | 3745 | 3823 | 3902 | 3980 | 4058 | 4136 | 4215 |
| 55 | 4293 | 4371 | 4449 | 4528 | *4606 | 4684 | 4762 | 4840 | 4919 | 4997 |
| 56 | 5075 | 5153 | 5231 | 5309 | 5387 | 5465 | 5543 | 5621 | 5699 | 5777 |
| 57 | 5855 | 5933 | 6011 | 6089 | 6167 | 6245 | 6323 | 6401 | 6479 | 6556 |
| 58 | 6634 | 6712 | 6790 | 6868 | 6945 | 7023 | 7101 | 7179 | 7256 | 7334 |
| 59 | 7412 | 7489 | 7567 | 7645 | 7722 | 7800 | 7878 | 7955 | 8033 | 8110 |
| 560 | 74 8188 | 8266 | 8343 | 8421 | 8498 | 8576 | 8653 | 8731 | 8808 | 8885 |
| 61 | 8963 | 9040 | 9118 | 9195 | 9272 | 9350 | 9427 | 9504 | 9582 | 9659 |
| 62 | 9736 | 9814 | 9891 | 9968 | *0045 | *0123 | *0200 | *0277 | *0354 | *0431 |
| 63 | 75 0508 | 0586 | 0663 | 0740 | 0817 | 0894 | 0971 | 1048 | 1125 | 1202 |
| 64 | 1279 | 1356 | 1433 | 1510 | 1587 | 1664 | 1741 | 1818 | 1895 | 1972 |
| 65 | 2048 | 2125 | 2202 | 2279 | 2356 | 2433 | 2509 | 2586 | 2663 | 2740 |
| 66 | 2816 | 2893 | 2970 | 3047 | 3123 | 3200 | 3277 | 3353 | 3430 | 3506 |
| 67 | 3583 | 3660 | 3736 | 3813 | 3889 | 3966 | 4042 | 4119 | 4195 | 4272 |
| 68 | 4348 | 4425 | 4501 | 4578 | 4654 | 4730 | 4807 | 4883 | 4960 | 5036 |
| 69 | 5112 | 5189 | 5265 | 5341 | 5417 | 5494 | 5570 | 5646 | 5722 | 5799 |
| 570 | 75 5875 | 5951 | 6027 | 6103 | 6180 | 6256 | 6332 | 6408 | 6484 | 6560 |
| 71 | 6636 | 6712 | 6788 | 6864 | 6940 | 7016 | 7092 | 7168 | 7244 | 7320 |
| 72 | 7396 | 7472 | 7548 | 7624 | 7700 | 7775 | 7851 | 7927 | 8003 | 8079 |
| 73 | 8155 | 8230 | 8306 | 8382 | 8458 | 8533 | 8609 | 8685 | 8761 | 8836 |
| 74 | 8912 | 8988 | 9063 | 9139 | 9214 | 9290 | 9366 | 9441 | 9517 | 9592 |
| 75 | 9668 | 9743 | 9819 | 9894 | 9970 | *0045 | *0121 | *0196 | *0272 | *0347 |
| 76 | 76 0422 | 0498 | 0573 | 0649 | 0724 | 0799 | 0875 | 0950 | 1025 | 1101 |
| 77 | 1176 | 1251 | 1326 | 1402 | 1477 | 1552 | 1627 | 1702 | 1778 | 1853 |
| 78 | 1928 | 2003 | 2078 | 2153 | 2228 | 2303 | 2378 | 2453 | 2529 | 2604 |
| 79 | 2679 | 2754 | 2829 | 2904 | 2978 | 3053 | 3128 | 3203 | 3278 | 3353 |
| 580 | 76 3428 | 3503 | 3578 | 3653 | 3727 | 3802 | 3877 | 3952 | 4027 | 4101 |
| 81 | 4176 | 4251 | 4326 | 4400 | 4475 | 4550 | 4624 | 4699 | 4774 | 4848 |
| 82 | 4923 | 4998 | 5072 | 5147 | 5221 | 5296 | 5370 | 5445 | 5520 | 5594 |
| 83 | 5669 | 5743 | 5818 | 5892 | 5966 | 6041 | 6115 | 6190 | 6264 | 6338 |
| 84 | 6413 | 6487 | 6562 | 6636 | 6710 | 6785 | 6859 | 6933 | 7007 | 7082 |
| 85 | 7156 | 7230 | 7304 | 7379 | 7453 | 7527 | 7601 | 7675 | 7749 | 7823 |
| 86 | 7898 | 7972 | 8046 | 8120 | 8194 | 8268 | 8342 | 8416 | 8490 | 8564 |
| 87 | 8638 | 8712 | 8786 | 8860 | 8934 | 9008 | 9082 | 9156 | 9230 | 9303 |
| 88 | 9377 | 9451 | 9525 | 9599 | 9673 | 9746 | 9820 | 9894 | 9968 | *0042 |
| 89 | 77 0115 | 0189 | 0263 | 0336 | 0410 | 0484 | 0557 | 0631 | 0705 | 0778 |
| 590 | 77 0852 | 0926 | 0999 | 1073 | 1146 | 1220 | 1293 | 1367 | 1440 | 1514 |
| 91 | 1587 | 1661 | 1734 | 1808 | 1881 | 1955 | 2028 | 2102 | 2175 | 2248 |
| 92 | 2322 | 2395 | 2468 | 2542 | 2615 | 2688 | 2762 | 2835 | 2908 | 2981 |
| 93 | 3055 | 3128 | 3201 | 3274 | 3348 | 3421 | 3494 | 3567 | 3640 | 3713 |
| 94 | 3786 | 3860 | 3933 | 4006 | 4079 | 4152 | 4225 | 4298 | 4371 | 4444 |
| 95 | 4517 | 4590 | 4663 | 4736 | 4809 | 4882 | 4955 | 5028 | 5100 | 5173 |
| 96 | 5246 | 5319 | 5392 | 5465 | 5538 | 5610 | 5683 | 5756 | 5829 | 5902 |
| 97 | 5974 | 6047 | 6120 | 6193 | 6265 | 6338 | 6411 | 6483 | 6556 | 6629 |
| 98 | 6701 | 6774 | 6846 | 6919 | 6992 | 7064 | 7137 | 7209 | 7282 | 7354 |
| 99 | 7427 | 7499 | 7572 | 7644 | 7717 | 7789 | 7862 | 7934 | 8006 | 8079 |
| 600 | 77 8151 | 8224 | 8296 | 8368 | 8441 | 8513 | 8585 | 8658 | 8730 | 8802 |

| Proportional Parts | | |
|---|---|---|
| | 79 | 78 |
| 1 | 7.9 | 7.8 |
| 2 | 15.8 | 15.6 |
| 3 | 23.7 | 23.4 |
| 4 | 31.6 | 31.2 |
| 5 | 39.5 | 39.0 |
| 6 | 47.4 | 46.8 |
| 7 | 55.3 | 54.6 |
| 8 | 63.2 | 62.4 |
| 9 | 71.1 | 70.2 |

| | 77 | 76 |
|---|---|---|
| 1 | 7.7 | 7.6 |
| 2 | 15.4 | 15.2 |
| 3 | 23.1 | 22.8 |
| 4 | 30.8 | 30.4 |
| 5 | 38.5 | 38.0 |
| 6 | 46.2 | 45.6 |
| 7 | 53.9 | 53.2 |
| 8 | 61.6 | 60.8 |
| 9 | 69.3 | 68.4 |

| | 75 |
|---|---|
| 1 | 7.5 |
| 2 | 15.0 |
| 3 | 22.5 |
| 4 | 30.0 |
| 5 | 37.5 |
| 6 | 45.0 |
| 7 | 52.5 |
| 8 | 60.0 |
| 9 | 67.5 |

| | 74 |
|---|---|
| 1 | 7.4 |
| 2 | 14.8 |
| 3 | 22.2 |
| 4 | 29.6 |
| 5 | 37.0 |
| 6 | 44.4 |
| 7 | 51.8 |
| 8 | 59.2 |
| 9 | 66.6 |

| | 73 |
|---|---|
| 1 | 7.3 |
| 2 | 14.6 |
| 3 | 21.9 |
| 4 | 29.2 |
| 5 | 36.5 |
| 6 | 43.8 |
| 7 | 51.1 |
| 8 | 58.4 |
| 9 | 65.7 |

| | 72 |
|---|---|
| 1 | 7.2 |
| 2 | 14.4 |
| 3 | 21.6 |
| 4 | 28.8 |
| 5 | 36.0 |
| 6 | 43.2 |
| 7 | 50.4 |
| 8 | 57.6 |
| 9 | 64.8 |

600–650 TABLE I

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 600 | 77 8151 | 8224 | 8296 | 8368 | 8441 | 8513 | 8585 | 8658 | 8730 | 8802 |
| 01 | 8874 | 8947 | 9019 | 9091 | 9163 | 9236 | 9308 | 9380 | 9452 | 9524 |
| 02 | 9596 | 9669 | 9741 | 9813 | 9885 | 9957 | *0029 | *0101 | *0173 | *0245 |
| 03 | 78 0317 | 0389 | 0461 | 0533 | 0605 | 0677 | 0749 | 0821 | 0893 | 0965 |
| 04 | 1037 | 1109 | 1181 | 1253 | 1324 | 1396 | 1468 | 1540 | 1612 | 1684 |
| 05 | 1755 | 1827 | 1899 | 1971 | 2042 | 2114 | 2186 | 2258 | 2329 | 2401 |
| 06 | 2473 | 2544 | 2616 | 2688 | 2759 | 2831 | 2902 | 2974 | 3046 | 3117 |
| 07 | 3189 | 3260 | 3332 | 3403 | 3475 | 3546 | 3618 | 3689 | 3761 | 3832 |
| 08 | 3904 | 3975 | 4046 | 4118 | 4189 | 4261 | 4332 | 4403 | 4475 | 4546 |
| 09 | 4617 | 4689 | 4760 | 4831 | 4902 | 4974 | 5045 | 5116 | 5187 | 5259 |
| 610 | 78 5330 | 5401 | 5472 | 5543 | 5615 | 5686 | 5757 | 5828 | 5899 | 5970 |
| 11 | 6041 | 6112 | 6183 | 6254 | 6325 | 6396 | 6467 | 6538 | 6609 | 6680 |
| 12 | 6751 | 6822 | 6893 | 6964 | 7035 | 7106 | 7177 | 7248 | 7319 | 7390 |
| 13 | 7460 | 7531 | 7602 | 7673 | 7744 | 7815 | 7885 | 7956 | 8027 | 8098 |
| 14 | 8168 | 8239 | 8310 | 8381 | 8451 | 8522 | 8593 | 8663 | 8734 | 8804 |
| 15 | 8875 | 8946 | 9016 | 9087 | 9157 | 9228 | 9299 | 9369 | 9440 | 9510 |
| 16 | 9581 | 9651 | 9722 | 9792 | 9863 | 9933 | *0004 | *0074 | *0144 | *0215 |
| 17 | 79 0285 | 0356 | 0426 | 0496 | 0567 | 0637 | 0707 | 0778 | 0848 | 0918 |
| 18 | 0988 | 1059 | 1129 | 1199 | 1269 | 1340 | 1410 | 1480 | 1550 | 1620 |
| 19 | 1691 | 1761 | 1831 | 1901 | 1971 | 2041 | 2111 | 2181 | 2252 | 2322 |
| 620 | 79 2392 | 2462 | 2532 | 2602 | 2672 | 2742 | 2812 | 2882 | 2952 | 3022 |
| 21 | 3092 | 3162 | 3231 | 3301 | 3371 | 3441 | 3511 | 3581 | 3651 | 3721 |
| 22 | 3790 | 3860 | 3930 | 4000 | 4070 | 4139 | 4209 | 4279 | 4349 | 4418 |
| 23 | 4488 | 4558 | 4627 | 4697 | 4767 | 4836 | 4906 | 4976 | 5045 | 5115 |
| 24 | 5185 | 5254 | 5324 | 5393 | 5463 | 5532 | 5602 | 5672 | 5741 | 5811 |
| 25 | 5880 | 5949 | 6019 | 6088 | 6158 | 6227 | 6297 | 6366 | 6436 | 6505 |
| 26 | 6574 | 6644 | 6713 | 6782 | 6852 | 6921 | 6990 | 7060 | 7129 | 7198 |
| 27 | 7268 | 7337 | 7406 | 7475 | 7545 | 7614 | 7683 | 7752 | 7821 | 7890 |
| 28 | 7960 | 8029 | 8098 | 8167 | 8236 | 8305 | 8374 | 8443 | 8513 | 8582 |
| 29 | 8651 | 8720 | 8789 | 8858 | 8927 | 8996 | 9065 | 9134 | 9203 | 9272 |
| 630 | 79 9341 | 9409 | 9478 | 9547 | 9616 | 9685 | 9754 | 9823 | 9892 | 9961 |
| 31 | 80 0029 | 0098 | 0167 | 0236 | 0305 | 0373 | 0442 | 0511 | 0580 | 0648 |
| 32 | 0717 | 0786 | 0854 | 0923 | 0992 | 1061 | 1129 | 1198 | 1266 | 1335 |
| 33 | 1404 | 1472 | 1541 | 1609 | 1678 | 1747 | 1815 | 1884 | 1952 | 2021 |
| 34 | 2089 | 2158 | 2226 | 2295 | 2363 | 2432 | 2500 | 2568 | 2637 | 2705 |
| 35 | 2774 | 2842 | 2910 | 2979 | 3047 | 3116 | 3184 | 3252 | 3321 | 3389 |
| 36 | 3457 | 3525 | 3594 | 3662 | 3730 | 3798 | 3867 | 3935 | 4003 | 4071 |
| 37 | 4139 | 4208 | 4276 | 4344 | 4412 | 4480 | 4548 | 4616 | 4685 | 4753 |
| 38 | 4821 | 4889 | 4957 | 5025 | 5093 | 5161 | 5229 | 5297 | 5365 | 5433 |
| 39 | 5501 | 5569 | 5637 | 5705 | 5773 | 5841 | 5908 | 5976 | 6044 | 6112 |
| 640 | 80 6180 | 6248 | 6316 | 6384 | 6451 | 6519 | 6587 | 6655 | 6723 | 6790 |
| 41 | 6858 | 6926 | 6994 | 7061 | 7129 | 7197 | 7264 | 7332 | 7400 | 7467 |
| 42 | 7535 | 7603 | 7670 | 7738 | 7806 | 7873 | 7941 | 8008 | 8076 | 8143 |
| 43 | 8211 | 8279 | 8346 | 8414 | 8481 | 8549 | 8616 | 8684 | 8751 | 8818 |
| 44 | 8886 | 8953 | 9021 | 9088 | 9156 | 9223 | 9290 | 9358 | 9425 | 9492 |
| 45 | 9560 | 9627 | 9694 | 9762 | 9829 | 9896 | 9964 | *0031 | *0098 | *0165 |
| 46 | 81 0233 | 0300 | 0367 | 0434 | 0501 | 0569 | 0636 | 0703 | 0770 | 0837 |
| 47 | 0904 | 0971 | 1039 | 1106 | 1173 | 1240 | 1307 | 1374 | 1441 | 1508 |
| 48 | 1575 | 1642 | 1709 | 1776 | 1843 | 1910 | 1977 | 2044 | 2111 | 2178 |
| 49 | 2245 | 2312 | 2379 | 2445 | 2512 | 2579 | 2646 | 2713 | 2780 | 2847 |
| 650 | 81 2913 | 2980 | 3047 | 3114 | 3181 | 3247 | 3314 | 3381 | 3448 | 3514 |

PROPORTIONAL PARTS

| | 73 | 72 |
|---|---|---|
| 1 | 7.3 | 7.2 |
| 2 | 14.6 | 14.4 |
| 3 | 21.9 | 21.6 |
| 4 | 29.2 | 28.8 |
| 5 | 36.5 | 36.0 |
| 6 | 43.8 | 43.2 |
| 7 | 51.1 | 50.4 |
| 8 | 58.4 | 57.6 |
| 9 | 65.7 | 64.8 |

| | 71 | 70 |
|---|---|---|
| 1 | 7.1 | 7.0 |
| 2 | 14.2 | 14.0 |
| 3 | 21.3 | 21.0 |
| 4 | 28.4 | 28.0 |
| 5 | 35.5 | 35.0 |
| 6 | 42.6 | 42.0 |
| 7 | 49.7 | 49.0 |
| 8 | 56.8 | 56.0 |
| 9 | 63.9 | 63.0 |

| | 69 |
|---|---|
| 1 | 6.9 |
| 2 | 13.8 |
| 3 | 20.7 |
| 4 | 27.6 |
| 5 | 34.5 |
| 6 | 41.4 |
| 7 | 48.3 |
| 8 | 55.2 |
| 9 | 62.1 |

| | 68 |
|---|---|
| 1 | 6.8 |
| 2 | 13.6 |
| 3 | 20.4 |
| 4 | 27.2 |
| 5 | 34.0 |
| 6 | 40.8 |
| 7 | 47.6 |
| 8 | 54.4 |
| 9 | 61.2 |

| | 67 |
|---|---|
| 1 | 6.7 |
| 2 | 13.4 |
| 3 | 20.1 |
| 4 | 26.8 |
| 5 | 33.5 |
| 6 | 40.2 |
| 7 | 46.9 |
| 8 | 53.6 |
| 9 | 60.3 |

| | 66 |
|---|---|
| 1 | 6.6 |
| 2 | 13.2 |
| 3 | 19.8 |
| 4 | 26.4 |
| 5 | 33.0 |
| 6 | 39.6 |
| 7 | 46.2 |
| 8 | 52.8 |
| 9 | 59.4 |

650–700 **TABLE I**

| N | LOGARITHMS 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Proportional Parts |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 650 | 81 2913 | 2980 | 3047 | 3114 | 3181 | 3247 | 3314 | 3381 | 3448 | 3514 | 67 |
| 51 | 3581 | 3648 | 3714 | 3781 | 3848 | 3914 | 3981 | 4048 | 4114 | 4181 | 1 6.7 |
| 52 | 4248 | 4314 | 4381 | 4447 | 4514 | 4581 | 4647 | 4714 | 4780 | 4847 | 2 13.4 |
| 53 | 4913 | 4980 | 5046 | 5113 | 5179 | 5246 | 5312 | 5378 | 5445 | 5511 | 3 20.1 |
| 54 | 5578 | 5644 | 5711 | 5777 | 5843 | 5910 | 5976 | 6042 | 6109 | 6175 | 4 26.8 |
| | | | | | | | | | | | 5 33.5 |
| 55 | 6241 | 6308 | 6374 | 6440 | 6506 | 6573 | 6639 | 6705 | 6771 | 6838 | 6 40.2 |
| 56 | 6904 | 6970 | 7036 | 7102 | 7169 | 7235 | 7301 | 7367 | 7433 | 7499 | 7 46.9 |
| 57 | 7565 | 7631 | 7698 | 7764 | 7830 | 7896 | 7962 | 8028 | 8094 | 8160 | 8 53.6 |
| 58 | 8226 | 8292 | 8358 | 8424 | 8490 | 8556 | 8622 | 8688 | 8754 | 8820 | 9 60.3 |
| 59 | 8885 | 8951 | 9017 | 9083 | 9149 | 9215 | 9281 | 9346 | 9412 | 9478 | |
| | | | | | | | | | | | 66 |
| 660 | 81 9544 | 9610 | 9676 | 9741 | 9807 | 9873 | 9939 | *0004 | *0070 | *0136 | 1 6.6 |
| 61 | 82 0201 | 0267 | 0333 | 0399 | 0464 | 0530 | 0595 | 0661 | 0727 | 0792 | 2 13.2 |
| 62 | 0858 | 0924 | 0989 | 1055 | 1120 | 1186 | 1251 | 1317 | 1382 | 1448 | 3 19.8 |
| 63 | 1514 | 1579 | 1645 | 1710 | 1775 | 1841 | 1906 | 1972 | 1037 | 2103 | 4 26.4 |
| 64 | 2168 | 2233 | 2299 | 2364 | 2430 | 2495 | 2560 | 2626 | 2691 | 2756 | 5 33.0 |
| | | | | | | | | | | | 6 39.6 |
| 65 | 2822 | 2887 | 2952 | 3018 | 3083 | 3148 | 3213 | 3279 | 3344 | 3409 | 7 46.2 |
| 66 | 3474 | 3539 | 3605 | 3670 | 3735 | 3800 | 3865 | 3930 | 3996 | 4061 | 8 52.8 |
| 67 | 4126 | 4191 | 4256 | 4321 | 4386 | 4451 | 4516 | 4581 | 4646 | 4711 | 9 59.4 |
| 68 | 4776 | 4841 | 4906 | 4971 | 5036 | 5101 | 5166 | 5231 | 5296 | 5361 | |
| 69 | 5426 | 5491 | 5556 | 5621 | 5686 | 5751 | 5815 | 5880 | 5945 | 6010 | 65 |
| | | | | | | | | | | | 1 6.5 |
| 670 | 82 6075 | 6140 | 6204 | 6269 | 6334 | 6399 | 6464 | 6528 | 6593 | 6658 | 2 13.0 |
| 71 | 6723 | 6787 | 6852 | 6917 | 6981 | 7046 | 7111 | 7175 | 7240 | 7305 | 3 19.5 |
| 72 | 7369 | 7434 | 7499 | 7563 | 7628 | 7692 | 7757 | 7821 | 7886 | 7951 | 4 26.0 |
| 73 | 8015 | 8080 | 8144 | 8209 | 8273 | 8338 | 8402 | 8467 | 8531 | 8595 | 5 32.5 |
| 74 | 8660 | 8724 | 8789 | 8853 | 8918 | 8982 | 9046 | 9111 | 9175 | 9239 | 6 39.0 |
| | | | | | | | | | | | 7 45.5 |
| 75 | 9304 | 9368 | 9432 | 9497 | 9561 | 9625 | 9690 | 9754 | 9818 | 9882 | 8 52.0 |
| 76 | 9947 | *0011 | *0075 | *0139 | *0204 | *0268 | *0332 | *0396 | *0460 | *0525 | 9 58.5 |
| 77 | 83 0589 | 0653 | 0717 | 0781 | 0845 | 0909 | 0973 | 1037 | 1102 | 1166 | |
| 78 | 1230 | 1294 | 1358 | 1422 | 1486 | 1550 | 1614 | 1678 | 1742 | 1806 | 64 |
| 79 | 1870 | 1934 | 1998 | 2062 | 2126 | 2189 | 2253 | 2317 | 2381 | 2445 | 1 6.4 |
| | | | | | | | | | | | 2 12.8 |
| 680 | 83 2509 | 2573 | 2637 | 2700 | 2764 | 2828 | 2892 | 2956 | 3020 | 3083 | 3 19.2 |
| 81 | 3147 | 3211 | 3275 | 3338 | 3402 | 3466 | 3530 | 3593 | 3657 | 3721 | 4 25.6 |
| 82 | 3784 | 3848 | 3912 | 3975 | 4039 | 4103 | 4166 | 4230 | 4294 | 4357 | 5 32.0 |
| 83 | 4421 | 4484 | 4548 | 4611 | 4675 | 4739 | 4802 | 4866 | 4929 | 4993 | 6 38.4 |
| 84 | 5056 | 5120 | 5183 | 5247 | 5310 | 5373 | 5437 | 5500 | 5564 | 5627 | 7 44.8 |
| | | | | | | | | | | | 8 51.2 |
| 85 | 5691 | 5754 | 5817 | 5881 | 5944 | 6007 | 6071 | 6134 | 6197 | 6261 | 9 57.6 |
| 86 | 6324 | 6387 | 6451 | 6514 | 6577 | 6641 | 6704 | 6767 | 6830 | 6894 | |
| 87 | 6957 | 7020 | 7083 | 7146 | 7210 | 7273 | 7336 | 7399 | 7462 | 7525 | 63 |
| 88 | 7588 | 7652 | 7715 | 7778 | 7841 | 7904 | 7967 | 8030 | 8093 | 8156 | 1 6.3 |
| 89 | 8219 | 8282 | 8345 | 8408 | 8471 | 8534 | 8597 | 8660 | 8723 | 8786 | 2 12.6 |
| | | | | | | | | | | | 3 18.9 |
| 690 | 83 8849 | 8912 | 8975 | 9038 | 9101 | 9164 | 9227 | 9289 | 9352 | 9415 | 4 25.2 |
| 91 | 9478 | 9541 | 9604 | 9667 | 9729 | 9792 | 9855 | 9918 | 9981 | *0043 | 5 31.5 |
| 92 | 84 0106 | 0169 | 0232 | 0294 | 0357 | 0420 | 0482 | 0545 | 0608 | 0671 | 6 37.8 |
| 93 | 0733 | 0796 | 0859 | 0921 | 0984 | 1046 | 1109 | 1172 | 1234 | 1297 | 7 44.1 |
| 94 | 1359 | 1422 | 1485 | 1547 | 1610 | 1672 | 1735 | 1797 | 1860 | 1922 | 8 50.4 |
| | | | | | | | | | | | 9 56.7 |
| 95 | 1985 | 2047 | 2110 | 2172 | 2235 | 2297 | 2360 | 2422 | 2484 | 2547 | 62 |
| 96 | 2609 | 2672 | 2734 | 2796 | 2859 | 2921 | 2983 | 3046 | 3108 | 3170 | 1 6.2 |
| 97 | 3233 | 3295 | 3357 | 3420 | 3482 | 3544 | 3606 | 3669 | 3731 | 3793 | 2 12.4 |
| 98 | 3855 | 3918 | 3980 | 4042 | 4104 | 4166 | 4229 | 4291 | 4353 | 4415 | 3 18.6 |
| 99 | 4477 | 4539 | 4601 | 4664 | 4726 | 4788 | 4850 | 4912 | 4974 | 5036 | 4 24.8 |
| | | | | | | | | | | | 5 31.0 |
| 700 | 84 5098 | 5160 | 5222 | 5284 | 5346 | 5408 | 5470 | 5532 | 5594 | 5656 | 6 37.2 |
| | | | | | | | | | | | 7 43.4 |
| | | | | | | | | | | | 8 49.6 |
| | | | | | | | | | | | 9 55.8 |

DEPARTMENT OF MATHEMATICS
CASE SCHOOL OF APPLIED SCIENCE

700–750 **TABLE I**

| N | LOGARITHMS 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Proportional Parts | 62 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 700 | 84 5098 | 5160 | 5222 | 5284 | 5346 | 5408 | 5470 | 5532 | 5594 | 5656 | 1 | 6.2 |
| 01 | 5718 | 5780 | 5842 | 5904 | 5966 | 6028 | 6090 | 6151 | 6213 | 6275 | 2 | 12.4 |
| 02 | 6337 | 6399 | 6461 | 6523 | 6585 | 6646 | 6708 | 6770 | 6832 | 6894 | 3 | 16.8 |
| 03 | 6955 | 7017 | 7079 | 7141 | 7202 | 7264 | 7326 | 7388 | 7449 | 7511 | 4 | 24.8 |
| 04 | 7573 | 7634 | 7696 | 7758 | 7819 | 7881 | 7943 | 8004 | 8066 | 8128 | 5 | 31.0 |
| | | | | | | | | | | | 6 | 37.2 |
| | | | | | | | | | | | 7 | 43.4 |
| 05 | 8189 | 8251 | 8312 | 8374 | 8435 | 8497 | 8559 | 8620 | 8682 | 8743 | 8 | 49.6 |
| 06 | 8805 | 8866 | 8928 | 8989 | 9051 | 9112 | 9174 | 9235 | 9297 | 9358 | 9 | 55.8 |
| 07 | 9419 | 9481 | 9542 | 9604 | 9665 | 9726 | 9788 | 9849 | 9911 | 9972 | | |
| 08 | 85 0033 | 0095 | 0156 | 0217 | 0279 | 0340 | 0401 | 0462 | 0524 | 0585 | | 61 |
| 09 | 0646 | 0707 | 0769 | 0830 | 0891 | 0952 | 1014 | 1075 | 1136 | 1197 | | |
| | | | | | | | | | | | 1 | 6.1 |
| 710 | 85 1258 | 1320 | 1381 | 1442 | 1503 | 1564 | 1625 | 1686 | 1747 | 1809 | 2 | 12.2 |
| 11 | 1870 | 1931 | 1992 | 2053 | 2114 | 2175 | 2236 | 2297 | 2358 | 2419 | 3 | 18.3 |
| 12 | 2480 | 2541 | 2602 | 2663 | 2724 | 2785 | 2846 | 2907 | 2968 | 3029 | 4 | 24.4 |
| 13 | 3090 | 3150 | 3211 | 3272 | 3333 | 3394 | 3455 | 3516 | 3577 | 3637 | 5 | 30.5 |
| 14 | 3698 | 3759 | 3820 | 3881 | 3941 | 4002 | 4063 | 4124 | 4185 | 4245 | 6 | 36.6 |
| | | | | | | | | | | | 7 | 42.7 |
| | | | | | | | | | | | 8 | 48.8 |
| 15 | 4306 | 4367 | 4428 | 4488 | 4549 | 4610 | 4670 | 4731 | 4792 | 4852 | 9 | 54.9 |
| 16 | 4913 | 4974 | 5034 | 5095 | 5156 | 5216 | 5277 | 5337 | 5398 | 5459 | | |
| 17 | 5519 | 5580 | 5640 | 5701 | 5761 | 5822 | 5882 | 5943 | 6003 | 6064 | | 60 |
| 18 | 6124 | 6185 | 6245 | 6306 | 6366 | 6427 | 6487 | 6548 | 6608 | 6668 | | |
| 19 | 6729 | 6789 | 6850 | 6910 | 6970 | 7031 | 7091 | 7152 | 7212 | 7272 | 1 | 6.0 |
| | | | | | | | | | | | 2 | 12.0 |
| | | | | | | | | | | | 3 | 18.0 |
| 720 | 85 7332 | 7393 | 7453 | 7513 | 7574 | 7634 | 7694 | 7755 | 7815 | 7875 | 4 | 24.0 |
| 21 | 7935 | 7995 | 8056 | 8116 | 8176 | 8236 | 8297 | 8357 | 8417 | 8477 | 5 | 30.0 |
| 22 | 8537 | 8597 | 8657 | 8718 | 8778 | 8838 | 8898 | 8958 | 9018 | 9078 | 6 | 36.0 |
| 23 | 9138 | 9198 | 9258 | 9318 | 9379 | 9439 | 9499 | 9559 | 9619 | 9679 | 7 | 42.0 |
| 24 | 9739 | 9799 | 9859 | 9918 | 9978 | *0038 | *0098 | *0158 | *0218 | *0278 | 8 | 48.0 |
| | | | | | | | | | | | 9 | 54.0 |
| 25 | 86 0338 | 0398 | 0458 | 0518 | 0578 | 0637 | 0697 | 0757 | 0817 | 0877 | | 59 |
| 26 | 0937 | 0996 | 1056 | 1116 | 1176 | 1236 | 1295 | 1355 | 1415 | 1475 | | |
| 27 | 1534 | 1594 | 1654 | 1714 | 1773 | 1833 | 1893 | 1952 | 2012 | 2072 | 1 | 5.9 |
| 28 | 2131 | 2191 | 2251 | 2310 | 2370 | 2430 | 2489 | 2549 | 2608 | 2668 | 2 | 11.8 |
| 29 | 2728 | 2787 | 2847 | 2906 | 2966 | 3025 | 3085 | 3144 | 3204 | 3263 | 3 | 17.7 |
| | | | | | | | | | | | 4 | 23.6 |
| 730 | 86 3323 | 3382 | 3442 | 3501 | 3561 | 3620 | 3680 | 3739 | 3799 | 3858 | 5 | 29.5 |
| 31 | 3917 | 3977 | 4036 | 4096 | 4155 | 4214 | 4274 | 4333 | 4392 | 4452 | 6 | 35.4 |
| 32 | 4511 | 4570 | 4630 | 4689 | 4748 | 4808 | 4867 | 4926 | 4985 | 5045 | 7 | 41.3 |
| 33 | 5104 | 5163 | 5222 | 5282 | 5341 | 5400 | 5459 | 5519 | 5578 | 5637 | 8 | 47.2 |
| 34 | 5696 | 5755 | 5814 | 5874 | 5933 | 5992 | 6051 | 6110 | 6169 | 6228 | 9 | 53.1 |
| | | | | | | | | | | | | 58 |
| 35 | 6287 | 6346 | 6405 | 6465 | 6524 | 6583 | 6642 | 6701 | 6760 | 681[illegible] | 1 | 5.8 |
| 36 | 6878 | 6937 | 6996 | 7055 | 7114 | 7173 | 7232 | 7291 | 7350 | 740[illegible] | 2 | 11.6 |
| 37 | 7467 | 7526 | 7585 | 7644 | 7703 | 7762 | 7821 | 7880 | 7939 | 7998 | 3 | 17.4 |
| 38 | 8056 | 8115 | 8174 | 8233 | 8292 | 8350 | 8409 | 8468 | 8527 | 8586 | 4 | 23.2 |
| 39 | 8644 | 8703 | 8762 | 8821 | 8879 | 8938 | 8997 | 9056 | 9114 | 9173 | 5 | 29.0 |
| | | | | | | | | | | | 6 | 34.8 |
| 740 | 86 9232 | 9290 | 9349 | 9408 | 9466 | 9525 | 9584 | 9642 | 9701 | 9760 | 7 | 40.6 |
| 41 | 9818 | 9877 | 9935 | 9994 | *0053 | *0111 | *0170 | *0228 | *0287 | *0345 | 8 | 46.4 |
| 42 | 87 0404 | 0462 | 0521 | 0579 | 0638 | 0696 | 0755 | 0813 | 0872 | 0930 | 9 | 52.2 |
| 43 | 0989 | 1047 | 1106 | 1164 | 1223 | 1281 | 1339 | 1398 | 1456 | 1515 | | |
| 44 | 1573 | 1631 | 1690 | 1748 | 1806 | 1865 | 1923 | 1981 | 2040 | 2098 | | 57 |
| | | | | | | | | | | | 1 | 5.7 |
| 45 | 2156 | 2215 | 2273 | 2331 | 2389 | 2448 | 2506 | 2564 | 2622 | 2681 | 2 | 11.4 |
| 46 | 2739 | 2797 | 2855 | 2913 | 2972 | 3030 | 3088 | 3146 | 3204 | 3262 | 3 | 17.1 |
| 47 | 3321 | 3379 | 3437 | 3495 | 3553 | 3611 | 3669 | 3727 | 3785 | 3844 | 4 | 22.8 |
| 48 | 3902 | 3960 | 4018 | 4076 | 4134 | 4192 | 4250 | 4308 | 4366 | 4424 | 5 | 28.5 |
| 49 | 4482 | 4540 | 4598 | 4656 | 4714 | 4772 | 4830 | 4888 | 4945 | 5003 | 6 | 34.2 |
| | | | | | | | | | | | 7 | 39.9 |
| 750 | 87 5061 | 5119 | 5177 | 5235 | 5293 | 5351 | 5409 | 5466 | 5524 | 5582 | 8 | 45.6 |
| | | | | | | | | | | | 9 | 51.3 |

750–800

## TABLE I

| N | LOGARITHMS 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | PROPORTIONAL PARTS | 55 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 750 | 87 5061 | 5119 | 5177 | 5235 | 5293 | 5351 | 5409 | 5466 | 5524 | 5582 | 1 | 5.8 |
| 51 | 5640 | 5698 | 5756 | 5813 | 5871 | 5929 | 5987 | 6045 | 6102 | 6160 | 2 | 11.6 |
| 52 | 6218 | 6276 | 6333 | 6391 | 6449 | 6507 | 6564 | 6622 | 6680 | 6737 | 3 | 17.4 |
| 53 | 6795 | 6853 | 6910 | 6968 | 7026 | 7083 | 7141 | 7199 | 7256 | 7314 | 4 | 23.2 |
| 54 | 7371 | 7429 | 7487 | 7544 | 7602 | 7659 | 7717 | 7774 | 7832 | 7889 | 5 | 29.0 |
| | | | | | | | | | | | 6 | 34.8 |
| 55 | 7947 | 8004 | 8062 | 8119 | 8177 | 8234 | 8292 | 8349 | 8407 | 8464 | 7 | 40.6 |
| 56 | 8522 | 8579 | 8637 | 8694 | 8752 | 8809 | 8866 | 8924 | 8981 | 9039 | 8 | 46.4 |
| 57 | 9096 | 9153 | 9211 | 9268 | 9325 | 9383 | 9440 | 9497 | 9555 | 9612 | 9 | 52.2 |
| 58 | 9669 | 9726 | 9784 | 9841 | 9898 | 9956 | *0013 | *0070 | *0127 | *0185 | | |
| 59 | 88 0242 | 0299 | 0356 | 0413 | 0471 | 0528 | 0585 | 0642 | 0699 | 0756 | | 57 |
| 760 | 88 0814 | 0871 | 0928 | 0985 | 1042 | 1099 | 1156 | 1213 | 1271 | 1328 | | |
| 61 | 1385 | 1442 | 1499 | 1556 | 1613 | 1670 | 1727 | 1784 | 1841 | 1898 | | |
| 62 | 1955 | 2012 | 2069 | 2126 | 2183 | 2240 | 2297 | 2354 | 2411 | 2468 | 1 | 5.7 |
| 63 | 2525 | 2581 | 2638 | 2695 | 2752 | 2809 | 2866 | 2923 | 2980 | 3037 | 2 | 11.4 |
| 64 | 3093 | 3150 | 3207 | 3264 | 3321 | 3377 | 3434 | 3491 | 3548 | 3605 | 3 | 17.1 |
| | | | | | | | | | | | 4 | 22.8 |
| 65 | 3661 | 3718 | 3775 | 3832 | 3888 | 3945 | 4002 | 4059 | 4115 | 4172 | 5 | 28.5 |
| 66 | 4229 | 4285 | 4342 | 4399 | 4455 | 4512 | 4569 | 4625 | 4682 | 4739 | 6 | 34.2 |
| 67 | 4795 | 4852 | 4909 | 4965 | 5022 | 5078 | 5135 | 5192 | 5248 | 5305 | 7 | 39.9 |
| 68 | 5361 | 5418 | 5474 | 5531 | 5587 | 5644 | 5700 | 5757 | 5813 | 5870 | 8 | 45.6 |
| 69 | 5926 | 5983 | 6039 | 6096 | 6152 | 6209 | 6265 | 6321 | 6378 | 6434 | 9 | 51·3 |
| 770 | 88 6491 | 6547 | 6604 | 6660 | 6716 | 6773 | 6829 | 6885 | 6942 | 6998 | | |
| 71 | 7054 | 7111 | 7167 | 7223 | 7280 | 7336 | 7392 | 7449 | 7505 | 7561 | | 56 |
| 72 | 7617 | 7674 | 7730 | 7786 | 7842 | 7898 | 7955 | 8011 | 8067 | 8123 | | |
| 73 | 8179 | 8236 | 8292 | 8348 | 8404 | 8460 | 8516 | 8573 | 8629 | 8685 | | |
| 74 | 8741 | 8797 | 8853 | 8909 | 8965 | 9021 | 9077 | 9134 | 9190 | 9246 | 1 | 5.6 |
| | | | | | | | | | | | 2 | 11.2 |
| 75 | 9302 | 9358 | 9414 | 9470 | 9526 | 9582 | 9638 | 9694 | 9750 | 9806 | 3 | 16.8 |
| 76 | 9862 | 9918 | 9974 | *0030 | *0086 | *0141 | *0197 | *0253 | *0309 | *0365 | 4 | 22.4 |
| 77 | 89 0421 | 0477 | 0533 | 0589 | 0645 | 0700 | 0756 | 0812 | 0868 | 0924 | 5 | 28.0 |
| 78 | 0980 | 1035 | 1091 | 1147 | 1203 | 1259 | 1314 | 1370 | 1426 | 1482 | 6 | 33.6 |
| 79 | 1537 | 1593 | 1649 | 1705 | 1760 | 1816 | 1872 | 1928 | 1983 | 2039 | 7 | 39.2 |
| | | | | | | | | | | | 8 | 44.8 |
| 780 | 89 2095 | 2150 | 2206 | 2262 | 2317 | 2373 | 2429 | 2484 | 2540 | 2595 | 9 | 50.4 |
| 81 | 2651 | 2707 | 2762 | 2818 | 2873 | 2929 | 2985 | 3040 | 3096 | 3151 | | |
| 82 | 3207 | 3262 | 3318 | 3373 | 3429 | 3484 | 3540 | 3595 | 3651 | 3706 | | |
| 83 | 3762 | 3817 | 3873 | 3928 | 3984 | 4039 | 4094 | 4150 | 4205 | 4261 | | 55 |
| 84 | 4316 | 4371 | 4427 | 4482 | 4538 | 4593 | 4648 | 4704 | 4759 | 4814 | | |
| 85 | 4870 | 4925 | 4980 | 5036 | 5091 | 5146 | 5201 | 5257 | 5312 | 5367 | | |
| 86 | 5423 | 5478 | 5533 | 5588 | 5644 | 5699 | 5754 | 5809 | 5864 | 5920 | 1 | 5.5 |
| 87 | 5975 | 6030 | 6085 | 6140 | 6195 | 6251 | 6306 | 6361 | 6416 | 6471 | 2 | 11.0 |
| 88 | 6526 | 6581 | 6636 | 6692 | 6747 | 6802 | 6857 | 6912 | 6967 | 7022 | 3 | 16.5 |
| 89 | 7077 | 7132 | 7187 | 7242 | 7297 | 7352 | 7407 | 7462 | 7517 | 7572 | 4 | 22.0 |
| | | | | | | | | | | | 5 | 27.5 |
| 790 | 89 7627 | 7682 | 7737 | 7792 | 7847 | 7902 | 7957 | 8012 | 8067 | 8122 | 6 | 33.0 |
| 91 | 8176 | 8231 | 8286 | 8341 | 8396 | 8451 | 8506 | 8561 | 8615 | 8670 | 7 | 38.5 |
| 92 | 8725 | 8780 | 8835 | 8890 | 8944 | 8999 | 9054 | 9109 | 9164 | 9218 | 8 | 44.0 |
| 93 | 9273 | 9328 | 9383 | 9437 | 9492 | 9547 | 9602 | 9656 | 9711 | 9766 | 9 | 49.5 |
| 94 | 9821 | 9875 | 9930 | 9985 | *0039 | *0094 | *0149 | *0203 | *0258 | *0312 | | |
| 95 | 90 0367 | 0422 | 0476 | 0531 | 0586 | 0640 | 0695 | 0749 | 0804 | 0859 | | |
| 96 | 0913 | 0968 | 1022 | 1077 | 1131 | 1186 | 1240 | 1295 | 1349 | 1404 | | |
| 97 | 1458 | 1513 | 1567 | 1622 | 1676 | 1731 | 1785 | 1840 | 1894 | 1948 | | |
| 98 | 2003 | 2057 | 2112 | 2166 | 2221 | 2275 | 2329 | 2384 | 2438 | 2492 | | |
| 99 | 2547 | 2601 | 2655 | 2710 | 2764 | 2818 | 2873 | 2927 | 2981 | 3036 | | |
| 800 | 90 3090 | 3144 | 3199 | 3253 | 3307 | 3361 | 3416 | 3470 | 3524 | 3578 | | |

800–850 TABLE I

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 800 | 90 3090 | 3144 | 3199 | 3253 | 3307 | 3361 | 3416 | 3470 | 3524 | 3578 |
| 01 | 3633 | 3687 | 3741 | 3795 | 3849 | 3904 | 3958 | 4012 | 4066 | 4120 |
| 02 | 4174 | 4229 | 4283 | 4337 | 4391 | 4445 | 4499 | 4553 | 4607 | 4661 |
| 03 | 4716 | 4770 | 4824 | 4878 | 4932 | 4986 | 5040 | 5094 | 5148 | 5202 |
| 04 | 5256 | 5310 | 5364 | 5418 | 5472 | 5526 | 5580 | 5634 | 5688 | 5742 |
| 05 | 5796 | 5850 | 5904 | 5958 | 6012 | 6066 | 6119 | 6173 | 6227 | 6281 |
| 06 | 6335 | 6389 | 6443 | 6497 | 6551 | 6604 | 6658 | 6712 | 6766 | 6820 |
| 07 | 6874 | 6927 | 6981 | 7035 | 7089 | 7143 | 7196 | 7250 | 7304 | 7358 |
| 08 | 7411 | 7465 | 7519 | 7573 | 7626 | 7680 | 7734 | 7787 | 7841 | 7895 |
| 09 | 7949 | 8002 | 8056 | 8110 | 8163 | 8217 | 8270 | 8324 | 8378 | 8431 |
| 810 | 90 8485 | 8539 | 8592 | 8646 | 8699 | 8753 | 8807 | 8860 | 8914 | 8967 |
| 11 | 9021 | 9074 | 9128 | 9181 | 9235 | 9289 | 9342 | 9396 | 9449 | 9503 |
| 12 | 9556 | 9610 | 9663 | 9716 | 9770 | 9823 | 9877 | 9930 | 9984 | *0037 |
| 13 | 91 0091 | 0144 | 0197 | 0251 | 0304 | 0358 | 0411 | 0464 | 0518 | 0571 |
| 14 | 0624 | 0678 | 0731 | 0784 | 0838 | 0891 | 0944 | 0998 | 1051 | 1104 |
| 15 | 1158 | 1211 | 1264 | 1317 | 1371 | 1424 | 1477 | 1530 | 1584 | 1637 |
| 16 | 1690 | 1743 | 1797 | 1850 | 1903 | 1956 | 2009 | 2063 | 2116 | 2169 |
| 17 | 2222 | 2275 | 2328 | 2381 | 2435 | 2488 | 2541 | 2594 | 2647 | 2700 |
| 18 | 2753 | 2806 | 2839 | 2913 | 2966 | 3019 | 3072 | 3125 | 3178 | 3231 |
| 19 | 3284 | 3337 | 3390 | 3443 | 3496 | 3549 | 3602 | 3655 | 3708 | 3761 |
| 820 | 91 3814 | 3867 | 3920 | 3973 | 4026 | 4079 | 4132 | 4184 | 4237 | 4290 |
| 21 | 4343 | 4396 | 4449 | 4502 | 4555 | 4608 | 4660 | 4713 | 4766 | 4819 |
| 22 | 4872 | 4925 | 4977 | 5030 | 5083 | 5136 | 5189 | 5241 | 5294 | 5347 |
| 23 | 5400 | 5453 | 5505 | 5558 | 5611 | 5664 | 5716 | 5769 | 5822 | 5875 |
| 24 | 5927 | 5980 | 6033 | 6085 | 6138 | 6191 | 6243 | 6296 | 6349 | 6401 |
| 25 | 6454 | 6507 | 6559 | 6612 | 6664 | 6717 | 6770 | 6822 | 6875 | 6927 |
| 26 | 6980 | 7033 | 7085 | 7138 | 7190 | 7243 | 7295 | 7348 | 7400 | 7453 |
| 27 | 7506 | 7558 | 7611 | 7663 | 7716 | 7768 | 7820 | 7873 | 7925 | 7978 |
| 28 | 8030 | 8083 | 8135 | 8188 | 8240 | 8293 | 8345 | 8397 | 8450 | 8502 |
| 29 | 8555 | 8607 | 8659 | 8712 | 8764 | 8816 | 8869 | 8921 | 8973 | 9026 |
| 830 | 91 9078 | 9130 | 9183 | 9235 | 9287 | 9340 | 9392 | 9444 | 9496 | 9549 |
| 31 | 9601 | 9653 | 9706 | 9758 | 9810 | 9862 | 9914 | 9967 | *0019 | *0071 |
| 32 | 92 0123 | 0176 | 0228 | 0280 | 0332 | 0384 | 0436 | 0489 | 0541 | 0593 |
| 33 | 0645 | 0697 | 0749 | 0801 | 0853 | 0906 | 0958 | 1010 | 1062 | 1114 |
| 34 | 1166 | 1218 | 1270 | 1322 | 1374 | 1426 | 1478 | 1530 | 1582 | 1634 |
| 35 | 1686 | 1738 | 1790 | 1842 | 1894 | 1946 | 1998 | 2050 | 2102 | 2154 |
| 36 | 2206 | 2258 | 2310 | 2362 | 2414 | 2466 | 2518 | 2570 | 2622 | 2674 |
| 37 | 2725 | 2777 | 2829 | 2881 | 2933 | 2985 | 3037 | 3089 | 3140 | 3192 |
| 38 | 3244 | 3296 | 3348 | 3399 | 3451 | 3503 | 3555 | 3607 | 3658 | 3710 |
| 39 | 3762 | 3814 | 3865 | 3917 | 3969 | 4021 | 4072 | 4124 | 4176 | 4228 |
| 840 | 92 4279 | 4331 | 4383 | 4434 | 4486 | 4538 | 4589 | 4641 | 4693 | 4744 |
| 41 | 4796 | 4848 | 4899 | 4951 | 5003 | 5054 | 5106 | 5157 | 5209 | 5261 |
| 42 | 5312 | 5364 | 5415 | 5467 | 5518 | 5570 | 5621 | 5673 | 5725 | 5776 |
| 43 | 5828 | 5879 | 5931 | 5982 | 6034 | 6085 | 6137 | 6188 | 6240 | 6291 |
| 44 | 6342 | 6394 | 6445 | 6497 | 6548 | 6600 | 6651 | 6702 | 6754 | 6805 |
| 45 | 6857 | 6908 | 6959 | 7011 | 7062 | 7114 | 7165 | 7216 | 7268 | 7319 |
| 46 | 7370 | 7422 | 7473 | 7524 | 7576 | 7627 | 7678 | 7730 | 7781 | 7832 |
| 47 | 7883 | 7935 | 7986 | 8037 | 8088 | 8140 | 8191 | 8242 | 8293 | 8345 |
| 48 | 8396 | 8447 | 8498 | 8549 | 8601 | 8652 | 8703 | 8754 | 8805 | 8857 |
| 49 | 8909 | 8959 | 9010 | 9061 | 9112 | 9163 | 9215 | 9266 | 9317 | 9368 |
| 850 | 92 9419 | 9470 | 9521 | 9572 | 9623 | 9674 | 9725 | 9776 | 9827 | 9879 |

LOGARITHMS

PROPORTIONAL PARTS

54

| | |
|---|---|
| 1 | 5.4 |
| 2 | 10.8 |
| 3 | 16.2 |
| 4 | 21.6 |
| 5 | 27.0 |
| 6 | 32.4 |
| 7 | 37.8 |
| 8 | 43.2 |
| 9 | 48.6 |

53

| | |
|---|---|
| 1 | 5.3 |
| 2 | 10.6 |
| 3 | 15.9 |
| 4 | 21.2 |
| 5 | 26.5 |
| 6 | 31.8 |
| 7 | 37.1 |
| 8 | 42.4 |
| 9 | 47.7 |

52

| | |
|---|---|
| 1 | 5.2 |
| 2 | 10.4 |
| 3 | 15.6 |
| 4 | 20.8 |
| 5 | 26.0 |
| 6 | 31.2 |
| 7 | 36.4 |
| 8 | 41.6 |
| 9 | 46.8 |

51

| | |
|---|---|
| 1 | 5.1 |
| 2 | 10.2 |
| 3 | 15.3 |
| 4 | 20.4 |
| 5 | 25.5 |
| 6 | 30.6 |
| 7 | 35.7 |
| 8 | 40.8 |
| 9 | 45.9 |

850–900 **TABLE I**

| N | LOGARITHMS 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | PROPORTIONAL PARTS |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 51 |
| 850 | 92 9419 | 9470 | 9521 | 9572 | 9623 | 9674 | 9725 | 9776 | 9827 | 9879 | 1 5.1 |
| 51 | 9930 | 9981 | *0032 | *0083 | *0134 | *0185 | *0236 | *0287 | *0338 | *0389 | 2 10.2 |
| 52 | 93 0440 | 0491 | 0542 | 0592 | 0643 | 0694 | 0745 | 0796 | 0847 | 0898 | 3 15.3 |
| 53 | 0949 | 1000 | 1051 | 1102 | 1153 | 1204 | 1254 | 1305 | 1356 | 1407 | 4 20.4 |
| 54 | 1458 | 1509 | 1560 | 1610 | 1661 | 1712 | 1763 | 1814 | 1865 | 1915 | 5 25.5 |
| | | | | | | | | | | | 6 30.6 |
| 55 | 1966 | 2017 | 2068 | 2118 | 2169 | 2220 | 2271 | 2322 | 2372 | 2423 | 7 35.7 |
| 56 | 2474 | 2524 | 2575 | 2626 | 2677 | 2727 | 2778 | 2829 | 2879 | 2930 | 8 40.8 |
| 57 | 2981 | 3031 | 3082 | 3133 | 3183 | 3234 | 3285 | 3335 | 3386 | 3437 | 9 45.9 |
| 58 | 3487 | 3538 | 3589 | 3639 | 3690 | 3740 | 3791 | 3841 | 3892 | 3943 | |
| 59 | 3993 | 4044 | 4094 | 4145 | 4195 | 4246 | 4296 | 4347 | 4397 | 4448 | |
| | | | | | | | | | | | 50 |
| 860 | 93 4498 | 4549 | 4599 | 4650 | 4700 | 4751 | 4801 | 4852 | 4902 | 4953 | |
| 61 | 5003 | 5054 | 5104 | 5154 | 5205 | 5255 | 5306 | 5356 | 5406 | 5457 | |
| 62 | 5507 | 5558 | 5608 | 5658 | 5709 | 5759 | 5809 | 5860 | 5910 | 5960 | 1 5.0 |
| 63 | 6011 | 6061 | 6111 | 6162 | 6212 | 6262 | 6313 | 6363 | 6413 | 6463 | 2 10.0 |
| 64 | 6514 | 6564 | 6614 | 6665 | 6715 | 6765 | 6815 | 6865 | 6916 | 6966 | 3 15.0 |
| | | | | | | | | | | | 4 20.0 |
| 65 | 7016 | 7066 | 7117 | 7167 | 7217 | 7267 | 7317 | 7367 | 7418 | 7468 | 5 25.0 |
| 66 | 7518 | 7568 | 7618 | 7668 | 7718 | 7769 | 7819 | 7869 | 7919 | 7969 | 6 30.0 |
| 67 | 8019 | 8069 | 8119 | 8169 | 8219 | 8269 | 8320 | 8370 | 8420 | 8470 | 7 35.9 |
| 68 | 8520 | 8570 | 8620 | 8670 | 8720 | 8770 | 8820 | 8870 | 8920 | 8970 | 8 40.0 |
| 69 | 9020 | 9070 | 9120 | 9170 | 9220 | 9270 | 9320 | 9369 | 9419 | 9469 | 9 45.0 |
| | | | | | | | | | | | |
| 870 | 93 9519 | 9569 | 9619 | 9669 | 9719 | 9769 | 9819 | 9869 | 9918 | 9968 | |
| 71 | 94 0018 | 0068 | 0118 | 0168 | 0218 | 0267 | 0317 | 0367 | 0417 | 0467 | 49 |
| 72 | 0516 | 0566 | 0616 | 0666 | 0716 | 0765 | 0815 | 0865 | 0915 | 0964 | |
| 73 | 1014 | 1064 | 1114 | 1163 | 1213 | 1263 | 1313 | 1362 | 1412 | 1462 | |
| 74 | 1511 | 1561 | 1611 | 1660 | 1710 | 1760 | 1809 | 1859 | 1909 | 1958 | 1 4.9 |
| | | | | | | | | | | | 2 9.8 |
| 75 | 2008 | 2058 | 2107 | 2157 | 2207 | 2256 | 2306 | 2355 | 2405 | 2455 | 3 14.7 |
| 76 | 2504 | 2554 | 2603 | 2653 | 2702 | 2752 | 2801 | 2851 | 2901 | 2950 | 4 19.6 |
| 77 | 3000 | 3049 | 3099 | 3148 | 3198 | 3247 | 3297 | 3346 | 3396 | 3445 | 5 24.5 |
| 78 | 3495 | 3544 | 3593 | 3643 | 3692 | 3742 | 3791 | 3841 | 3890 | 3939 | 6 29.4 |
| 79 | 3989 | 4038 | 4088 | 4137 | 4186 | 4236 | 4285 | 4335 | 4384 | 4433 | 7 34.3 |
| | | | | | | | | | | | 8 39.2 |
| 880 | 94 4483 | 4532 | 4581 | 4631 | 4680 | 4729 | 4779 | 4828 | 4877 | 4927 | 9 44.1 |
| 81 | 4976 | 5025 | 5074 | 5124 | 5173 | 5222 | 5272 | 5321 | 5370 | 5419 | |
| 82 | 5469 | 5518 | 5567 | 5616 | 5665 | 5715 | 5764 | 5813 | 5862 | 5912 | |
| 83 | 5961 | 6010 | 6059 | 6108 | 6157 | 6207 | 6256 | 6305 | 6354 | 6403 | 48 |
| 84 | 6452 | 6501 | 6551 | 6600 | 6649 | 6698 | 6747 | 6796 | 6845 | 6894 | |
| | | | | | | | | | | | |
| 85 | 6943 | 6992 | 7041 | 7090 | 7140 | 7189 | 7238 | 7287 | 7336 | 7385 | |
| 86 | 7434 | 7483 | 7532 | 7581 | 7630 | 7679 | 7728 | 7777 | 7826 | 7875 | 1 4.8 |
| 87 | 7924 | 7973 | 8022 | 8070 | 8119 | 8168 | 8217 | 8266 | 8315 | 8364 | 2 9.6 |
| 88 | 8413 | 8462 | 8511 | 8560 | 8609 | 8657 | 8706 | 8755 | 8804 | 8853 | 3 14.4 |
| 89 | 8902 | 8951 | 8999 | 9048 | 9097 | 9146 | 9195 | 9244 | 9292 | 9341 | 4 19.2 |
| | | | | | | | | | | | 5 24.0 |
| 890 | 94 9390 | 9439 | 9488 | 9536 | 9585 | 9634 | 9683 | 9731 | 9780 | 9829 | 6 28.8 |
| 91 | 9878 | 9926 | 9975 | *0024 | *0073 | *0121 | *0170 | *0219 | *0267 | *0316 | 7 33.6 |
| 92 | 95 0365 | 0414 | 0462 | 0511 | 0560 | 0608 | 0657 | 0706 | 0754 | 0803 | 8 38.4 |
| 93 | 0851 | 0900 | 0949 | 0997 | 1046 | 1095 | 1143 | 1192 | 1240 | 1289 | 9 43.2 |
| 94 | 1338 | 1386 | 1435 | 1483 | 1532 | 1580 | 1629 | 1677 | 1726 | 1775 | |
| | | | | | | | | | | | |
| 95 | 1823 | 1872 | 1920 | 1969 | 2017 | 2066 | 2114 | 2163 | 2211 | 2260 | |
| 96 | 2308 | 2356 | 2405 | 2453 | 2502 | 2550 | 2599 | 2647 | 2696 | 2744 | |
| 97 | 2792 | 2841 | 2889 | 2938 | 2986 | 3034 | 3083 | 3131 | 3180 | 3228 | |
| 98 | 3276 | 3325 | 3373 | 3421 | 3470 | 3518 | 3566 | 3615 | 3663 | 3711 | |
| 99 | 3760 | 3808 | 3856 | 3905 | 3953 | 4001 | 4049 | 4098 | 4146 | 4194 | |
| | | | | | | | | | | | |
| 900 | 95 4243 | 4291 | 4339 | 4387 | 4435 | 4484 | 4532 | 4580 | 4628 | 4677 | |

**900–950** **TABLE I**

| N | LOGARITHMS 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | PROPORTIONAL PARTS | 43 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 900 | 95 4243 | 4291 | 4339 | 4387 | 4435 | 4484 | 4532 | 4580 | 4628 | 4677 | 1 | 4.8 |
| 01 | 4725 | 4773 | 4821 | 4869 | 4918 | 4966 | 5014 | 5062 | 5110 | 5158 | 2 | 9.6 |
| 02 | 5207 | 5255 | 5303 | 5351 | 5399 | 5447 | 5495 | 5543 | 5592 | 5640 | 3 | 14.4 |
| 03 | 5688 | 5736 | 5784 | 5832 | 5880 | 5928 | 5976 | 6024 | 6072 | 6120 | 4 | 19.2 |
| 54 | 6168 | 6216 | 6265 | 6313 | 6361 | 6409 | 6457 | 6505 | 6553 | 6601 | 5 | 24.0 |
| | | | | | | | | | | | 6 | 28.8 |
| 05 | 6649 | 6697 | 6745 | 6793 | 6840 | 6888 | 6936 | 6984 | 7032 | 7080 | 7 | 33.6 |
| 06 | 7128 | 7176 | 7224 | 7272 | 7320 | 7368 | 7416 | 7464 | 7512 | 7559 | 8 | 38.4 |
| 07 | 7607 | 7655 | 7703 | 7751 | 7799 | 7847 | 7894 | 7942 | 7990 | 8038 | 9 | 43.2 |
| 08 | 8086 | 8134 | 8181 | 8229 | 8277 | 8325 | 8373 | 8421 | 8468 | 8516 | | |
| 09 | 8564 | 8612 | 8659 | 8707 | 8755 | 8803 | 8850 | 8898 | 8946 | 8994 | | |
| | | | | | | | | | | | | 47 |
| 910 | 95 9041 | 9089 | 9137 | 9185 | 9232 | 9280 | 9328 | 9375 | 9423 | 9471 | | |
| 11 | 9518 | 9566 | 9614 | 9661 | 9709 | 9757 | 9804 | 9852 | 9900 | 9947 | | |
| 12 | 9995 | *0042 | *0090 | *0138 | *0185 | *0233 | *0280 | *0328 | *0376 | *0423 | 1 | 4.7 |
| 13 | 96 0471 | 0518 | 0566 | 0613 | 0661 | 0709 | 0756 | 0804 | 0851 | 0899 | 2 | 9.4 |
| 14 | 0946 | 0994 | 1041 | 1089 | 1136 | 1184 | 1231 | 1279 | 1326 | 1374 | 3 | 14.1 |
| | | | | | | | | | | | 4 | 18.8 |
| 15 | 1421 | 1469 | 1516 | 1563 | 1611 | 1658 | 1706 | 1753 | 1801 | 1848 | 5 | 23.5 |
| 16 | 1895 | 1943 | 1990 | 2038 | 2085 | 2132 | 2180 | 2227 | 2275 | 2322 | 6 | 28.2 |
| 17 | 2369 | 2417 | 2464 | 2511 | 2559 | 2606 | 2653 | 2701 | 2748 | 2795 | 7 | 32.9 |
| 18 | 2843 | 2890 | 2937 | 2985 | 3032 | 3079 | 3126 | 3174 | 3221 | 3268 | 8 | 37.6 |
| 19 | 3316 | 3363 | 3410 | 3457 | 3504 | 3552 | 3599 | 3646 | 3693 | 3741 | 9 | 42.3 |
| | | | | | | | | | | | | |
| 920 | 96 3788 | 3835 | 3882 | 3929 | 3977 | 4024 | 4071 | 4118 | 4165 | 4212 | | |
| 21 | 4260 | 4307 | 4354 | 4401 | 4448 | 4495 | 4542 | 4590 | 4637 | 4684 | | 46 |
| 22 | 4731 | 4778 | 4825 | 4872 | 4919 | 4966 | 5013 | 5061 | 5108 | 5155 | | |
| 23 | 5202 | 5249 | 5296 | 5343 | 5390 | 5437 | 5484 | 5531 | 5578 | 5625 | | |
| 24 | 5672 | 5719 | 5766 | 5813 | 5860 | 5907 | 5954 | 6001 | 6048 | 6095 | | |
| | | | | | | | | | | | 1 | 4.6 |
| 25 | 6142 | 6189 | 6236 | 6283 | 6329 | 6376 | 6423 | 6470 | 6517 | 6564 | 2 | 9.2 |
| 26 | 6611 | 6658 | 6705 | 6752 | 6799 | 6845 | 6892 | 6939 | 6986 | 7033 | 3 | 13.8 |
| 27 | 7080 | 7127 | 7173 | 7220 | 7267 | 7314 | 7361 | 7408 | 7454 | 7501 | 4 | 18.4 |
| 28 | 7548 | 7595 | 7642 | 7688 | 7735 | 7782 | 7829 | 7875 | 7922 | 7969 | 5 | 23.0 |
| 29 | 8016 | 8062 | 8109 | 8156 | 8203 | 8249 | 8296 | 8343 | 8390 | 8436 | 6 | 27.6 |
| | | | | | | | | | | | 7 | 32.2 |
| 930 | 96 8483 | 8530 | 8576 | 8623 | 8670 | 8716 | 8763 | 8810 | 8856 | 8903 | 8 | 36.8 |
| 31 | 8950 | 8996 | 9043 | 9090 | 9136 | 9183 | 9229 | 9276 | 9323 | 9369 | 9 | 41.4 |
| 32 | 9416 | 9463 | 9509 | 9556 | 9602 | 9649 | 9695 | 9742 | 9789 | 9835 | | |
| 33 | 9882 | 9928 | 9975 | *0021 | *0068 | *0114 | *0161 | *0207 | *0254 | *0300 | | |
| 34 | 97 0347 | 0393 | 0440 | 0486 | 0533 | 0579 | 0626 | 0672 | 0719 | 0765 | | |
| | | | | | | | | | | | | |
| 35 | 0812 | 0858 | 0904 | 0951 | 0997 | 1044 | 1090 | 1137 | 1183 | 1229 | | |
| 36 | 1276 | 1322 | 1369 | 1415 | 1461 | 1508 | 1554 | 1601 | 1647 | 1693 | | |
| 37 | 1740 | 1786 | 1832 | 1879 | 1925 | 1971 | 2018 | 2064 | 2110 | 2157 | | |
| 38 | 2203 | 2249 | 2295 | 2342 | 2388 | 2434 | 2481 | 2527 | 2573 | 2619 | | |
| 39 | 2666 | 2712 | 2758 | 2804 | 2851 | 2897 | 2943 | 2989 | 3035 | 3082 | | |
| | | | | | | | | | | | | |
| 940 | 97 3128 | 3174 | 3220 | 3266 | 3313 | 3359 | 3405 | 3451 | 3497 | 3543 | | |
| 41 | 3590 | 3636 | 3682 | 3728 | 3774 | 3820 | 3866 | 3913 | 3959 | 4005 | | |
| 42 | 4051 | 4097 | 4143 | 4189 | 4235 | 4281 | 4327 | 4374 | 4420 | 4466 | | |
| 43 | 4512 | 4558 | 4604 | 4650 | 4696 | 4742 | 4788 | 4834 | 4880 | 4926 | | |
| 44 | 4972 | 5018 | 5064 | 5110 | 5156 | 5202 | 5248 | 5294 | 5340 | 5386 | | |
| | | | | | | | | | | | | |
| 45 | 5432 | 5478 | 5524 | 5570 | 5616 | 5662 | 5707 | 5753 | 5799 | 5845 | | |
| 46 | 5891 | 5937 | 5983 | 6029 | 6075 | 6121 | 6167 | 6212 | 6258 | 6304 | | |
| 47 | 6350 | 6396 | 6442 | 6488 | 6533 | 6579 | 6625 | 6671 | 6717 | 6763 | | |
| 48 | 6808 | 6854 | 6900 | 6946 | 6992 | 7037 | 7083 | 7129 | 7175 | 7220 | | |
| 49 | 7266 | 7312 | 7358 | 7403 | 7449 | 7495 | 7541 | 7586 | 7632 | 7678 | | |
| | | | | | | | | | | | | |
| 950 | 97 7724 | 7769 | 7815 | 7861 | 7906 | 7952 | 7998 | 8043 | 8089 | 8135 | | |

950–1000

## TABLE I

| N | LOGARITHMS 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 950 | 97 7724 | 7769 | 7815 | 7861 | 7906 | 7952 | 7998 | 8043 | 8089 | 8135 |
| 51 | 8181 | 8226 | 8272 | 8317 | 8363 | 8409 | 8454 | 8500 | 8546 | 8591 |
| 52 | 8637 | 8683 | 8728 | 8774 | 8819 | 8865 | 8911 | 8956 | 9002 | 9047 |
| 53 | 9093 | 9138 | 9184 | 9230 | 9275 | 9321 | 9366 | 9412 | 9457 | 9503 |
| 54 | 9548 | 9594 | 9639 | 9685 | 9730 | 9776 | 9821 | 9867 | 9912 | 9958 |
| 55 | 98 0003 | 0049 | 0094 | 0140 | 0185 | 0231 | 0276 | 0322 | 0367 | 0412 |
| 56 | 0458 | 0503 | 0549 | 0594 | 0640 | 0685 | 0730 | 0776 | 0821 | 0867 |
| 57 | 0912 | 0957 | 1003 | 1048 | 1093 | 1139 | 1184 | 1229 | 1275 | 1320 |
| 58 | 1366 | 1411 | 1456 | 1501 | 1547 | 1592 | 1637 | 1683 | 1728 | 1773 |
| 59 | 1819 | 1864 | 1909 | 1954 | 2000 | 2045 | 2090 | 2135 | 2181 | 2226 |
| 960 | 98 2271 | 2316 | 2362 | 2407 | 2452 | 2497 | 2543 | 2588 | 2633 | 2678 |
| 61 | 2723 | 2769 | 2814 | 2859 | 2904 | 2949 | 2994 | 3040 | 3085 | 3130 |
| 62 | 3175 | 3220 | 3265 | 3310 | 3356 | 3401 | 3446 | 3491 | 3536 | 3581 |
| 63 | 3626 | 3671 | 3716 | 3762 | 3807 | 3852 | 3897 | 3942 | 3987 | 4032 |
| 64 | 4077 | 4122 | 4167 | 4212 | 4257 | 4302 | 4347 | 4392 | 4437 | 4482 |
| 65 | 4527 | 4572 | 4617 | 4662 | 4707 | 4752 | 4797 | 4842 | 4887 | 4932 |
| 66 | 4977 | 5022 | 5067 | 5112 | 5157 | 5202 | 5247 | 5292 | 5337 | 5382 |
| 67 | 5426 | 5471 | 5516 | 5561 | 5606 | 5651 | 5696 | 5741 | 5786 | 5830 |
| 68 | 5875 | 5920 | 5965 | 6010 | 6055 | 6100 | 6144 | 6189 | 6234 | 6279 |
| 69 | 6324 | 6369 | 6413 | 6458 | 6503 | 6548 | 6593 | 6637 | 6682 | 6727 |
| 970 | 98 6772 | 6817 | 6861 | 6906 | 6951 | 6996 | 7040 | 7085 | 7130 | 7175 |
| 71 | 7219 | 7264 | 7309 | 7353 | 7398 | 7443 | 7488 | 7532 | 7577 | 7622 |
| 72 | 7666 | 7711 | 7756 | 7800 | 7845 | 7890 | 7934 | 7979 | 8024 | 8068 |
| 73 | 8113 | 8157 | 8202 | 8247 | 8291 | 8336 | 8381 | 8425 | 8470 | 8514 |
| 74 | 8559 | 8604 | 8648 | 8693 | 8737 | 8782 | 8826 | 8871 | 8916 | 8960 |
| 75 | 9005 | 9049 | 9094 | 9138 | 9183 | 9227 | 9272 | 9316 | 9361 | 9405 |
| 76 | 9450 | 9494 | 9539 | 9583 | 9628 | 9672 | 9717 | 9761 | 9806 | 9850 |
| 77 | 9895 | 9939 | 9983 | *0028 | *0072 | *0117 | *0161 | *0206 | *0250 | *0294 |
| 78 | 99 0339 | 0383 | 0428 | 0472 | 0516 | 0561 | 0605 | 0650 | 0694 | 0738 |
| 79 | 0783 | 0827 | 0871 | 0916 | 0960 | 1004 | 1049 | 1093 | 1137 | 1182 |
| 980 | 99 1226 | 1270 | 1315 | 1359 | 1403 | 1448 | 1492 | 1336 | 1580 | 1625 |
| 81 | 1669 | 1713 | 1758 | 1802 | 1846 | 1890 | 1935 | 1979 | 2023 | 2067 |
| 82 | 2111 | 2156 | 2200 | 2244 | 2288 | 2333 | 2377 | 2421 | 2465 | 2509 |
| 83 | 2554 | 2598 | 2642 | 2686 | 2730 | 2774 | 2819 | 2863 | 2907 | 2951 |
| 84 | 2995 | 3039 | 3083 | 3127 | 3172 | 3216 | 3260 | 3304 | 3348 | 3392 |
| 85 | 3436 | 3480 | 3524 | 3568 | 3613 | 3657 | 3701 | 3745 | 3789 | 3833 |
| 86 | 3877 | 3921 | 3965 | 4009 | 4053 | 4097 | 4141 | 4185 | 4229 | 4273 |
| 87 | 4317 | 4361 | 4405 | 4449 | 4493 | 4537 | 4581 | 4625 | 4669 | 4713 |
| 88 | 4757 | 4801 | 4845 | 4889 | 4933 | 4977 | 5021 | 5065 | 5108 | 5152 |
| 89 | 5196 | 5240 | 5284 | 5328 | 5372 | 5416 | 5460 | 5504 | 5547 | 5591 |
| 990 | 99 5635 | 5679 | 5723 | 5767 | 5811 | 5854 | 5898 | 5942 | 5986 | 6030 |
| 91 | 6074 | 6117 | 6161 | 6205 | 6249 | 6293 | 6337 | 6380 | 6424 | 6468 |
| 92 | 6512 | 6555 | 6599 | 6643 | 6687 | 6731 | 6774 | 6818 | 6862 | 6906 |
| 93 | 6949 | 6993 | 7037 | 7080 | 7124 | 7168 | 7212 | 7255 | 7299 | 7343 |
| 94 | 7386 | 7430 | 7474 | 7517 | 7561 | 7605 | 7648 | 7692 | 7736 | 7779 |
| 95 | 7823 | 7867 | 7910 | 7954 | 7998 | 8041 | 8085 | 8129 | 8172 | 8216 |
| 96 | 8259 | 8303 | 8347 | 8390 | 8434 | 8477 | 8521 | 8564 | 8608 | 8652 |
| 97 | 8695 | 8739 | 8782 | 8826 | 8869 | 8913 | 8956 | 9000 | 9043 | 9087 |
| 98 | 9131 | 9174 | 9218 | 9261 | 9305 | 9348 | 9392 | 9435 | 9479 | 9522 |
| 99 | 9565 | 9609 | 9652 | 9696 | 9739 | 9783 | 9826 | 9870 | 9913 | 9957 |
| 1000 | | | | | | | | | | |

| PROPORTIONAL PARTS | 45 |
|---|---|
| 1 | 4.5 |
| 2 | 9.0 |
| 3 | 13.5 |
| 4 | 18.0 |
| 5 | 22.5 |
| 6 | 27.0 |
| 8 | 31.5 |
| 8 | 36.0 |
| 9 | 40.5 |

| | 44 |
|---|---|
| 1 | 4.4 |
| 2 | 8.8 |
| 3 | 13.2 |
| 4 | 17.6 |
| 5 | 22.0 |
| 6 | 26.4 |
| 7 | 30.8 |
| 8 | 35.2 |
| 9 | 39.9 |

| | 43 |
|---|---|
| 1 | 4.3 |
| 2 | 8.6 |
| 3 | 12.9 |
| 4 | 17.2 |
| 5 | 21.5 |
| 6 | 25.8 |
| 7 | 30.1 |
| 8 | 34.4 |
| 9 | 38.7 |

## TABLE I (Supplement)

**1000–1050** LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1000 | 000 0000 | 0434 | 0869 | 1303 | 1737 | 2171 | 2605 | 3039 | 3473 | 3907 |
| 1001 | 4341 | 4775 | 5208 | 5642 | 6076 | 6510 | 6943 | 7377 | 7810 | 8244 |
| 1002 | 8677 | 9111 | 9544 | 9977 | *0411 | *0844 | *1277 | *1710 | *2143 | *2576 |
| 1003 | 001 3009 | 3442 | 3875 | 4308 | 4741 | 5174 | 5607 | 6039 | 6472 | 6905 |
| 1004 | 7337 | 7770 | 8202 | 8635 | 9067 | 9499 | 9932 | *0364 | *0796 | *1228 |
| 1005 | 002 1661 | 2093 | 2525 | 2957 | 3389 | 3821 | 4253 | 4685 | 5116 | 5548 |
| 1006 | 5980 | 6411 | 6843 | 7275 | 7706 | 8138 | 8569 | 9001 | 9432 | 9863 |
| 1007 | 003 0295 | 0726 | 1157 | 1588 | 2019 | 2451 | 2882 | 3313 | 3744 | 4174 |
| 1008 | 4605 | 5036 | 5467 | 5898 | 5328 | 6759 | 7190 | 7620 | 8051 | 8481 |
| 1009 | 8912 | 9342 | 9772 | *0203 | *0633 | *1063 | *1493 | *1924 | *2354 | *2784 |
| 1010 | 004 3214 | 3644 | 4074 | 4504 | 4933 | 5363 | 5793 | 6223 | 6652 | 7082 |
| 1011 | 7512 | 7941 | 8371 | 8800 | 9229 | 9659 | *0088 | *0517 | *0947 | *1376 |
| 1012 | 005 1805 | 2234 | 2663 | 3092 | 3521 | 3950 | 4379 | 4808 | 5237 | 5666 |
| 1013 | 6094 | 6523 | 6952 | 7380 | 7809 | 8238 | 8666 | 9094 | 9523 | 9951 |
| 1014 | 006 0380 | 0808 | 1236 | 1664 | 2092 | 2521 | 2949 | 3377 | 3805 | 4233 |
| 1515 | 4660 | 5088 | 5516 | 5944 | 6372 | 6799 | 7227 | 7655 | 8082 | 8510 |
| 1016 | 8937 | 9365 | 9792 | *0219 | *0647 | *1074 | *1501 | *1928 | *2355 | *2782 |
| 1017 | 007 3210 | 3637 | 4064 | 4490 | 4917 | 5344 | 5771 | 6198 | 6624 | 7051 |
| 1018 | 7478 | 7904 | 8331 | 8757 | 9184 | 9610 | *0037 | *0463 | *0889 | *1316 |
| 1019 | 008 1742 | 2168 | 2594 | 3020 | 3446 | 3872 | 4298 | 4724 | 5150 | 5576 |
| 1020 | 6002 | 6427 | 6853 | 7279 | 7704 | 8130 | 8556 | 8981 | 9407 | 9832 |
| 1021 | 009 0257 | 0683 | 1108 | 1533 | 1959 | 2384 | 2809 | 3234 | 3659 | 4084 |
| 1022 | 4509 | 4934 | 5359 | 5784 | 6208 | 6633 | 7058 | 7483 | 7907 | 8332 |
| 1023 | 8756 | 9181 | 9605 | *0030 | *0454 | *0878 | *1303 | *1727 | * 2151 | *2575 |
| 1024 | 010 3000 | 3424 | 3848 | 4272 | 4696 | 5120 | 5544 | 5967 | 6391 | 6815 |
| 1025 | 7239 | 7662 | 8086 | 8510 | 8933 | 9357 | 9780 | *0204 | *0627 | *1050 |
| 1026 | 011 1474 | 1897 | 2320 | 2743 | 3166 | 3590 | 4013 | 4436 | 4859 | 5282 |
| 1027 | 5704 | 6127 | 6550 | 6973 | 7396 | 7818 | 8241 | 8664 | 9086 | 9509 |
| 1028 | 9931 | *0354 | *0776 | *1198 | *1621 | *2043 | *2465 | *2887 | *3310 | *3732 |
| 1029 | 012 4154 | 4576 | 4998 | 5420 | 5842 | 6264 | 6685 | 7107 | 7529 | 7951 |
| 1030 | 8372 | 8794 | 9215 | 9637 | *0059 | *0480 | *0901 | *1323 | *1744 | *2165 |
| 1031 | 013 2587 | 3008 | 3429 | 3850 | 4271 | 4692 | 5113 | 5534 | 5955 | 6376 |
| 1032 | 6797 | 7218 | 7639 | 8059 | 8480 | 8901 | 9321 | 9742 | *0162 | *0583 |
| 1033 | 014 1003 | 1424 | 1844 | 2264 | 2685 | 3105 | 3525 | 3945 | 4365 | 4785 |
| 1034 | 5205 | 5625 | 6045 | 6465 | 6885 | 7305 | 7725 | 8144 | 8564 | 8984 |
| 1035 | 9403 | 9823 | *0243 | *0662 | *1082 | *1501 | *1920 | *2340 | *2759 | *3178 |
| 1036 | 015 3598 | 4017 | 4436 | 4855 | 5274 | 5693 | 6112 | 6531 | 6950 | 7369 |
| 1037 | 7788 | 8206 | 8625 | 9044 | 9462 | 9881 | *0300 | *0718 | *1137 | *1555 |
| 1038 | 016 1974 | 2392 | 2810 | 3229 | 3647 | 4065 | 4483 | 4901 | 5319 | 5737 |
| 1039 | 6155 | 6573 | 6991 | 7409 | 7827 | 8245 | 8663 | 9080 | 9498 | 9916 |
| 1040 | 017 0333 | 0751 | 1168 | 1586 | 2003 | 2421 | 2838 | 3256 | 3673 | 4090 |
| 1041 | 4507 | 4924 | 5342 | 5759 | 6176 | 6593 | 7010 | 7427 | 7844 | 8260 |
| 1042 | 8677 | 9094 | 9511 | 9927 | *0344 | *0761 | *1177 | *1594 | *2010 | *2427 |
| 1043 | 018 2843 | 3259 | 3676 | 4092 | 4508 | 4925 | 5341 | 5757 | 6173 | 6589 |
| 1044 | 7005 | 7421 | 7837 | 8253 | 8669 | 9084 | 9500 | 9916 | *0332 | *0747 |
| 1045 | 019 1163 | 1578 | 1994 | 2410 | 2825 | 3240 | 3656 | 4071 | 4486 | 4902 |
| 1046 | 5317 | 5732 | 6147 | 6562 | 6977 | 7392 | 7807 | 8222 | 8637 | 9052 |
| 1047 | 9467 | 9882 | *0296 | *0711 | *1126 | *1540 | *1955 | *2369 | *2784 | *3198 |
| 1048 | 020 3613 | 4027 | 4442 | 4856 | 5270 | 5684 | 6099 | 6513 | 6927 | 7341 |
| 1049 | 7755 | 8169 | 8583 | 8997 | 9411 | 9824 | *0238 | *0652 | *1066 | *1479 |
| 1050 | 021 1893 | 2307 | 2720 | 3134 | 3547 | 3961 | 4374 | 4787 | 5201 | 5614 |

## TABLE I (Supplement)

**1050–1100** LOGARITHMS

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **1050** | 021 1893 | 2307 | 2720 | 3134 | 3547 | 3961 | 4374 | 4787 | 5201 | 5614 |
| **1051** | 6027 | 6440 | 6854 | 7267 | 7680 | 8093 | 8506 | 8919 | 9332 | 9745 |
| **1052** | 022 0157 | 0570 | 0983 | 1396 | 1808 | 2221 | 2634 | 3046 | 3459 | 3871 |
| **1053** | 4284 | 4696 | 5109 | 5521 | 5933 | 6345 | 6758 | 7170 | 7582 | 7994 |
| **1054** | 8406 | 8818 | 9230 | 9642 | *0054 | *0466 | *0878 | *1289 | *1701 | *2113 |
| **1055** | 023 2525 | 2936 | 3348 | 3759 | 4171 | 4582 | 4994 | 5405 | 5817 | 6228 |
| **1056** | 6639 | 7050 | 7462 | 7873 | 8284 | 8695 | 9106 | 9517 | 9928 | *0339 |
| **1057** | 024 0750 | 1161 | 1572 | 1982 | 2393 | 2804 | 3214 | 3625 | 4036 | 4446 |
| **1058** | 4857 | 5267 | 5678 | 6088 | 6498 | 6909 | 7319 | 7729 | 8139 | 8549 |
| **1059** | 8960 | 9370 | 9780 | *0190 | *0600 | *1010 | *1419 | *1829 | *2239 | *2649 |
| **1060** | 025 3059 | 3468 | 3878 | 4288 | 4697 | 5107 | 5516 | 5926 | 6335 | 6744 |
| **1061** | 7154 | 7563 | 7972 | 8382 | 8791 | 9200 | 9609 | *0018 | *0427 | *0836 |
| **1062** | 026 1245 | 1654 | 2063 | 2472 | 2881 | 3289 | 3698 | 4107 | 4515 | 4924 |
| **1063** | 5333 | 5741 | 6150 | 6558 | 6967 | 7375 | 7783 | 8192 | 8600 | 9008 |
| **1064** | 9416 | 9824 | *0233 | *0641 | *1049 | *1457 | *1865 | *2273 | *2680 | *3088 |
| **1065** | 027 3496 | 3904 | 4312 | 4719 | 5127 | 5535 | 5942 | 6350 | 6757 | 7165 |
| **1066** | 7572 | 7979 | 8387 | 8794 | 9201 | 9609 | *0016 | *0423 | *0830 | *1237 |
| **1067** | 028 1644 | 2051 | 2458 | 2865 | 3272 | 3679 | 4086 | 4492 | 4899 | 5306 |
| **1068** | 5713 | 6119 | 6526 | 6932 | 7339 | 7745 | 8152 | 8558 | 8964 | 9371 |
| **1069** | 9777 | *0183 | *0590 | *0996 | *1402 | *1808 | *2214 | *2620 | *3026 | *3432 |
| **1070** | 029 3838 | 4244 | 4649 | 5055 | 5461 | 5867 | 6272 | 6678 | 7084 | 7489 |
| **1071** | 7895 | 8300 | 8706 | 9111 | 9516 | 9922 | *0327 | *0732 | *1138 | *1543 |
| **1072** | 030 1948 | 2353 | 2758 | 3163 | 3568 | 3973 | 4378 | 4783 | 5188 | 5592 |
| **1073** | 5997 | 6402 | 6807 | 7211 | 7616 | 8020 | 8425 | 8830 | 9234 | 9638 |
| **1074** | 031 0043 | 0447 | 0851 | 1256 | 1660 | 2064 | 2468 | 2872 | 3277 | 3681 |
| **1075** | 4085 | 4489 | 4893 | 5296 | 5700 | 6104 | 6508 | 6912 | 7315 | 7719 |
| **1076** | 8123 | 8526 | 8930 | 9333 | 9737 | *0140 | *0544 | *0947 | *1350 | *1754 |
| **1077** | 032 2157 | 2560 | 2963 | 3367 | 3770 | 4173 | 4576 | 4979 | 5382 | 5785 |
| **1078** | 6188 | 6590 | 6993 | 7396 | 7799 | 8201 | 8604 | 9007 | 9409 | 9812 |
| **1079** | 033 0214 | 0617 | 1019 | 1422 | 1824 | 2226 | 2629 | 3031 | 3433 | 3835 |
| **1080** | 4238 | 4640 | 5042 | 5444 | 5846 | 6248 | 6650 | 7052 | 7453 | 7855 |
| **1081** | 8257 | 8659 | 9060 | 9462 | 9864 | *0265 | *0667 | *1068 | *1470 | *1871 |
| **1082** | 034 2273 | 2674 | 3075 | 3477 | 3878 | 4279 | 4680 | 5081 | 5482 | 5884 |
| **1083** | 6285 | 6686 | 7087 | 7487 | 7888 | 8289 | 8690 | 9091 | 9491 | 9892 |
| **1084** | 035 0293 | 0693 | 1094 | 1495 | 1895 | 2296 | 2696 | 3096 | 3497 | 3897 |
| **1085** | 4297 | 4698 | 5098 | 5498 | 5898 | 6298 | 6698 | 7098 | 7498 | 7898 |
| **1086** | 8298 | 8698 | 9098 | 9498 | 9898 | *0297 | *0697 | *1097 | *1496 | *1896 |
| **1087** | 036 2295 | 2695 | 3094 | 3494 | 3893 | 4293 | 4692 | 5091 | 5491 | 5890 |
| **1088** | 6289 | 6688 | 7087 | 7486 | 7885 | 8284 | 8683 | 9082 | 9481 | 9880 |
| **1089** | 037 0279 | 0678 | 1076 | 1475 | 1874 | 2272 | 2671 | 3070 | 3468 | 3867 |
| **1090** | 4265 | 4663 | 5062 | 5460 | 5858 | 6257 | 6655 | 7053 | 7451 | 7849 |
| **1091** | 8248 | 8646 | 9044 | 9442 | 9839 | *0237 | *0635 | *1033 | *1431 | *1829 |
| **1092** | 038 2226 | 2624 | 3022 | 3419 | 3817 | 4214 | 4612 | 5009 | 5407 | 5804 |
| **1093** | 6202 | 6599 | 6996 | 7393 | 7791 | 8188 | 8585 | 8982 | 9379 | 9776 |
| **1094** | 039 0173 | 0570 | 0967 | 1364 | 1761 | 2158 | 2554 | 2951 | 3348 | 3745 |
| **1095** | 4141 | 4538 | 4934 | 5331 | 5727 | 6124 | 6520 | 6917 | 7313 | 7709 |
| **1096** | 8106 | 8502 | 8898 | 9294 | 9690 | *0086 | *0482 | *0878 | *1274 | *1670 |
| **1097** | 040 2066 | 2462 | 2858 | 3254 | 3650 | 4045 | 4441 | 4837 | 5232 | 5628 |
| **1098** | 6023 | 6419 | 6814 | 7210 | 7605 | 8001 | 8396 | 8791 | 9187 | 9582 |
| **1099** | 9977 | *0372 | *0767 | *1162 | *1557 | *1952 | *2347 | *2342 | *3137 | *3532 |
| **1100** | 041 3927 | 4322 | 4716 | 5111 | 5506 | 5900 | 6295 | 6690 | 7084 | 7479 |

**TABLE II. The Number of Each Day of the Year Counting from January 1**

| Day of Month | Jan. | Feb. | Mar. | April | May | June | July | Aug. | Sept. | Oct. | Nov. | Dec. | Day of Month |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 32 | 60 | 91 | 121 | 152 | 182 | 213 | 244 | 274 | 305 | 335 | 1 |
| 2 | 2 | 33 | 61 | 92 | 122 | 153 | 183 | 214 | 245 | 275 | 306 | 336 | 2 |
| 3 | 3 | 34 | 62 | 93 | 123 | 154 | 184 | 215 | 246 | 276 | 307 | 337 | 3 |
| 4 | 4 | 35 | 63 | 94 | 124 | 155 | 185 | 216 | 247 | 277 | 308 | 338 | 4 |
| 5 | 5 | 36 | 64 | 95 | 125 | 156 | 186 | 217 | 248 | 278 | 309 | 339 | 5 |
| 6 | 6 | 37 | 65 | 96 | 126 | 157 | 187 | 218 | 249 | 279 | 310 | 340 | 6 |
| 7 | 7 | 38 | 66 | 97 | 127 | 158 | 188 | 219 | 250 | 280 | 311 | 341 | 7 |
| 8 | 8 | 39 | 67 | 98 | 128 | 159 | 189 | 220 | 251 | 281 | 312 | 342 | 8 |
| 9 | 9 | 40 | 68 | 99 | 129 | 160 | 190 | 221 | 252 | 282 | 313 | 343 | 9 |
| 10 | 10 | 41 | 69 | 100 | 130 | 161 | 191 | 222 | 253 | 283 | 314 | 344 | 10 |
| 11 | 11 | 42 | 70 | 101 | 131 | 162 | 192 | 223 | 254 | 284 | 315 | 345 | 11 |
| 12 | 12 | 43 | 71 | 102 | 132 | 163 | 193 | 224 | 255 | 285 | 316 | 346 | 12 |
| 13 | 13 | 44 | 72 | 103 | 133 | 164 | 194 | 225 | 256 | 286 | 317 | 347 | 13 |
| 14 | 14 | 45 | 73 | 104 | 134 | 165 | 195 | 226 | 257 | 287 | 318 | 348 | 14 |
| 15 | 15 | 46 | 74 | 105 | 135 | 166 | 196 | 227 | 258 | 288 | 319 | 349 | 15 |
| 16 | 16 | 47 | 75 | 106 | 136 | 167 | 197 | 228 | 259 | 289 | 320 | 350 | 16 |
| 17 | 17 | 48 | 76 | 107 | 137 | 168 | 198 | 229 | 260 | 290 | 321 | 351 | 17 |
| 18 | 18 | 49 | 77 | 108 | 138 | 169 | 199 | 230 | 261 | 291 | 322 | 352 | 18 |
| 19 | 19 | 50 | 78 | 109 | 139 | 170 | 200 | 231 | 262 | 292 | 323 | 353 | 19 |
| 20 | 20 | 51 | 79 | 110 | 140 | 171 | 201 | 232 | 263 | 293 | 324 | 354 | 20 |
| 21 | 21 | 52 | 80 | 111 | 141 | 172 | 202 | 233 | 264 | 294 | 325 | 355 | 21 |
| 22 | 22 | 53 | 81 | 112 | 142 | 173 | 203 | 234 | 265 | 295 | 326 | 356 | 22 |
| 23 | 23 | 54 | 82 | 113 | 143 | 174 | 204 | 235 | 266 | 296 | 327 | 357 | 23 |
| 24 | 24 | 55 | 83 | 114 | 144 | 175 | 205 | 236 | 267 | 297 | 328 | 358 | 24 |
| 25 | 25 | 56 | 84 | 115 | 145 | 176 | 206 | 237 | 268 | 298 | 329 | 359 | 25 |
| 26 | 26 | 57 | 85 | 116 | 146 | 177 | 207 | 238 | 269 | 299 | 330 | 360 | 26 |
| 27 | 27 | 58 | 86 | 117 | 147 | 178 | 208 | 239 | 270 | 300 | 331 | 361 | 27 |
| 28 | 28 | 59 | 87 | 118 | 148 | 179 | 209 | 240 | 271 | 301 | 332 | 362 | 28 |
| 29 | 29 | | 88 | 119 | 149 | 180 | 210 | 241 | 272 | 302 | 333 | 363 | 29 |
| 30 | 30 | | 89 | 120 | 150 | 181 | 211 | 242 | 273 | 303 | 334 | 364 | 30 |
| 31 | 31 | | 90 | | 151 | | 212 | 243 | | 304 | | 365 | 31 |

Note. — In leap years, after February 28, add 1 to the tabular number.

## TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 1 | 1.0033 3333 | 1.0041 6667 | 1.0050 0000 | 1.0087 5000 | 1.0100 0000 |
| 2 | 1.0066 7778 | 1.0083 5069 | 1.0100 2500 | 1.0175 7656 | 1.0201 0000 |
| 3 | 1.0100 3337 | 1.0125 5216 | 1.0150 7513 | 1.0264 8036 | 1.0303 0100 |
| 4 | 1.0134 0015 | 1.0167 7112 | 1.0201 5050 | 1.0354 6206 | 1.0406 0401 |
| 5 | 1.0167 7815 | 1.0210 0767 | 1.0252 5125 | 1.0445 2235 | 1.0510 1005 |
| 6 | 1.0201 6741 | 1.0252 6187 | 1.0303 7751 | 1.0536 6192 | 1.0615 2015 |
| 7 | 1.0235 6797 | 1.0295 3379 | 1.0355 2940 | 1.0628 8147 | 1.0721 3535 |
| 8 | 1.0269 7986 | 1.0338 2352 | 1.0407 0704 | 1.0721 8168 | 1.0828 5671 |
| 9 | 1.0304 0313 | 1.0381 3111 | 1.0459 1058 | 1.0815 6327 | 1.0936 8527 |
| 10 | 1.0338 3780 | 1.0424 5666 | 1.0511 4013 | 1.0910 2695 | 1.1046 2213 |
| 11 | 1.0372 8393 | 1.0468 0023 | 1.0563 9583 | 1.1005 7343 | 1.1156 6835 |
| 12 | 1.0407 4154 | 1.0511 6190 | 1.0616 7781 | 1.1102 0345 | 1.1268 2503 |
| 13 | 1.0442 1068 | 1.0555 4174 | 1.0669 8620 | 1.1199 1773 | 1.1380 9328 |
| 14 | 1.0476 9138 | 1.0599 3983 | 1.0723 2113 | 1.1297 1701 | 1.1494 7421 |
| 15 | 1.0511 8369 | 1.0643 5625 | 1.0776 8274 | 1.1396 0203 | 1.1609 6896 |
| 16 | 1.0546 8763 | 1.0687 9106 | 1.0830 7115 | 1.1495 7355 | 1.1725 7864 |
| 17 | 1.0582 0326 | 1.0732 4436 | 1.0884 8651 | 1.1596 3232 | 1.1843 0443 |
| 18 | 1.0617 3060 | 1.0777 1621 | 1.0939 2894 | 1.1697 7910 | 1.1961 4748 |
| 19 | 1.0652 6971 | 1.0822 0670 | 1.0993 9858 | 1.1800 1467 | 1.2081 0895 |
| 20 | 1.0688 2060 | 1.0867 1589 | 1.1048 9558 | 1.1903 3980 | 1.2201 9004 |
| 21 | 1.0723 8334 | 1.0912 4387 | 1.1104 2006 | 1.2007 5527 | 1.2323 9194 |
| 22 | 1.0759 5795 | 1.0957 9072 | 1.1159 7216 | 1.2112 6188 | 1.2447 1586 |
| 23 | 1.0795 4448 | 1.1003 5652 | 1.1215 5202 | 1.2218 6042 | 1.2571 6302 |
| 24 | 1.0831 4296 | 1.1049 4134 | 1.1271 5978 | 1.2325 5170 | 1.2697 3465 |
| 25 | 1.0867 5344 | 1.1095 4526 | 1.1327 9558 | 1.2433 3653 | 1.2824 3200 |
| 26 | 1.0903 7595 | 1.1141 6836 | 1.1384 5955 | 1.2542 1572 | 1.2952 5631 |
| 27 | 1.0940 1053 | 1.1188 1073 | 1.1441 5185 | 1.2651 9011 | 1.3082 0888 |
| 28 | 1.0976 5724 | 1.1234 7244 | 1.1498 7261 | 1.2762 6052 | 1.3212 9097 |
| 29 | 1.1013 1609 | 1.1281 5358 | 1.1556 2197 | 1.2874 2780 | 1.3345 0388 |
| 30 | 1.1049 8715 | 1.1328 5422 | 1.1614 0008 | 1.2986 9280 | 1.3478 4892 |
| 31 | 1.1086 7044 | 1.1375 7444 | 1.1672 0708 | 1.3100 5636 | 1.3613 2740 |
| 32 | 1.1123 6601 | 1.1423 1434 | 1.1730 4312 | 1.3215 1935 | 1.3749 4068 |
| 33 | 1.1160 7389 | 1.1470 7398 | 1.1789 0833 | 1.3330 8265 | 1.3886 9009 |
| 34 | 1.1197 9414 | 1.1518 5346 | 1.1848 0288 | 1.3447 4712 | 1.4025 7699 |
| 35 | 1.1235 2679 | 1.1566 5284 | 1.1907 2689 | 1.3565 1366 | 1.4166 0276 |
| 36 | 1.1272 7187 | 1.1614 7223 | 1.1966 8052 | 1.3683 8315 | 1.4307 6878 |
| 37 | 1.1310 2945 | 1.1663 1170 | 1.2026 6393 | 1.3803 5650 | 1.4450 7647 |
| 38 | 1.1347 9955 | 1.1711 7133 | 1.2086 7725 | 1.3924 3462 | 1.4595 2724 |
| 39 | 1.1385 8221 | 1.1760 5121 | 1.2147 2063 | 1.4046 1843 | 1.4741 2251 |
| 40 | 1.1423 7748 | 1.1809 5142 | 1.2207 9424 | 1.4146 0884 | 1.4888 6373 |
| 41 | 1.1461 8541 | 1.1858 7206 | 1.2268 9821 | 1.4293 0679 | 1.5037 5237 |
| 42 | 1.1500 0603 | 1.1908 1319 | 1.2330 3270 | 1.4418 1322 | 1.5187 8989 |
| 43 | 1.1538 3938 | 1.1957 7491 | 1.2391 9786 | 1.4544 2909 | 1.5339 7779 |
| 44 | 1.1576 8551 | 1.2007 5731 | 1.2453 9385 | 1.4671 5534 | 1.5493 1757 |
| 45 | 1.1615 4446 | 1.2057 6046 | 1.2516 2082 | 1.4799 9295 | 1.5648 1075 |
| 46 | 1.1654 1628 | 1.2107 8446 | 1.2578 7892 | 1.4929 4289 | 1.5804 5885 |
| 47 | 1.1693 0100 | 1.2158 2940 | 1.2641 6832 | 1.5060 0614 | 1.5962 6344 |
| 48 | 1.1731 9867 | 1.2208 9536 | 1.2704 8916 | 1.5191 8370 | 1.6122 2608 |
| 49 | 1.1771 0933 | 1.2259 8242 | 1.2768 4161 | 1.5324 7655 | 1.6283 4834 |
| 50 | 1.1810 3303 | 1.2310 9068 | 1.2832 2581 | 1.5458 8572 | 1.6446 3182 |

## TABLE III. The Amount of 1 at Compound Interest

$(1 + i)^n$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 51 | 1.1849 6981 | 1.2362 2002 | 1.2896 4194 | 1.5594 1222 | 1.6610 7814 |
| 52 | 1.1889 1971 | 1.2413 7114 | 1.2960 9015 | 1.5730 5708 | 1.6776 8892 |
| 53 | 1.1928 8277 | 1.2465 4352 | 1.3025 7060 | 1.5868 2133 | 1.6944 6581 |
| 54 | 1.1968 5905 | 1.2517 3745 | 1.3090 8346 | 1.6007 0602 | 1.7114 1047 |
| 55 | 1.2008 4858 | 1.2569 5302 | 1.3156 2887 | 1.6147 1219 | 1.7285 2457 |
| 56 | 1.2048 5141 | 1.2621 9033 | 1.3222 0702 | 1.6288 4093 | 1.7458 0982 |
| 57 | 1.2088 6758 | 1.2674 4946 | 1.3288 1805 | 1.6430 9328 | 1.7632 6792 |
| 58 | 1.2128 9714 | 1.2727 3050 | 1.3354 6214 | 1.6574 7035 | 1.7809 0060 |
| 59 | 1.2169 4013 | 1.2780 3354 | 1.3421 3946 | 1.6719 7322 | 1.7987 0960 |
| 60 | 1.2209 9659 | 1.2833 5868 | 1.3488 5015 | 1.6866 0298 | 1.8166 9670 |
| 61 | 1.2250 6658 | 1.2887 0601 | 1.3555 9440 | 1.7013 6076 | 1.8348 6367 |
| 62 | 1.2291 5014 | 1.2940 7561 | 1.3623 7238 | 1.7162 4766 | 1.8532 1230 |
| 63 | 1.2332 4730 | 1.2994 6760 | 1.3691 8424 | 1.7312 6483 | 1.8717 4443 |
| 64 | 1.2373 5813 | 1.3048 8204 | 1.3760 3016 | 1.7464 1340 | 1.8904 6187 |
| 65 | 1.2414 8266 | 1.3103 1905 | 1.3829 1031 | 1.7616 9452 | 1.9093 6649 |
| 66 | 1.2456 2093 | 1.3157 7872 | 1.3898 2486 | 1.7771 0934 | 1.9284 6015 |
| 67 | 1.2497 7300 | 1.3212 6113 | 1.3967 7399 | 1.7926 5905 | 1.9477 4475 |
| 68 | 1.2539 3891 | 1.3267 6638 | 1.4037 5785 | 1.8083 4482 | 1.9672 2220 |
| 69 | 1.2581 1871 | 1.3322 9458 | 1.4107 7664 | 1.8241 6783 | 1.9868 9442 |
| 70 | 1.2623 1244 | 1.3378 4580 | 1.4178 3053 | 1.8401 2930 | 2.0067 6337 |
| 71 | 1.2665 2015 | 1.3434 2016 | 1.4249 1968 | 1.8562 3043 | 2.0268 3100 |
| 72 | 1.2707 4188 | 1.3490 1774 | 1.4320 4428 | 1.8724 7245 | 2.0470 9931 |
| 73 | 1.2749 7769 | 1.3546 3865 | 1.4392 0450 | 1.8888 5658 | 2.0675 7031 |
| 74 | 1.2792 2761 | 1.3602 8298 | 1.4464 0052 | 1.9053 8408 | 2.0882 4601 |
| 75 | 1.2834 9170 | 1.3659 5082 | 1.4536 3252 | 1.9220 5619 | 2.1091 2847 |
| 76 | 1.2877 7001 | 1.3716 4229 | 1.4609 0069 | 1.9388 7418 | 2.1302 1975 |
| 77 | 1.2920 6258 | 1.3773 5746 | 1.4682 0519 | 1.9558 3933 | 2.1515 2195 |
| 78 | 1.2963 6945 | 1.3830 9645 | 1.4755 4622 | 1.9729 5292 | 2.1730 3717 |
| 79 | 1.3006 9068 | 1.3888 5935 | 1.4829 2395 | 1.9902 1626 | 2.1947 6754 |
| 80 | 1.3050 2632 | 1.3946 4627 | 1.4903 3857 | 2.0076 3066 | 2.2167 1522 |
| 81 | 1.3093 7641 | 1.4004 5729 | 1.4977 9026 | 2.0251 9742 | 2.2388 8237 |
| 82 | 1.3137 4099 | 1.4062 9253 | 1.5052 7921 | 2.0429 1790 | 2.2612 7119 |
| 83 | 1.3181 2013 | 1.4121 5209 | 1.5128 0561 | 2.0607 9343 | 2.2838 8390 |
| 84 | 1.3225 1386 | 1.4180 3605 | 1.5203 6964 | 2.0788 2537 | 2.3067 2274 |
| 85 | 1.3269 2224 | 1.4239 4454 | 1.5279 7148 | 2.0970 1510 | 2.3297 8997 |
| 86 | 1.3313 4532 | 1.4298 7764 | 1.5356 1134 | 2.1153 6398 | 2.3530 8787 |
| 87 | 1.3357 8314 | 1.4358 3546 | 1.5432 8940 | 2.1338 7341 | 2.3766 1875 |
| 88 | 1.3402 3575 | 1.4418 1811 | 1.5510 0585 | 2.1525 4481 | 2.4003 8494 |
| 89 | 1.3447 0320 | 1.4478 2568 | 1.5587 6087 | 2.1713 7957 | 2.4243 8879 |
| 90 | 1.3491 8554 | 1.4538 5829 | 1.5665 5468 | 2.1903 7914 | 2.4486 3267 |
| 91 | 1.3536 8283 | 1.4599 1603 | 1.5743 8745 | 2.2095 4496 | 2.4731 1900 |
| 92 | 1.3581 9510 | 1.4659 9902 | 1.5822 5939 | 2.2288 7848 | 2.4978 5019 |
| 93 | 1.3627 2242 | 1.4721 0735 | 1.5901 7069 | 2.2483 8117 | 2.5228 2869 |
| 94 | 1.3672 6483 | 1.4782 4113 | 1.5981 2154 | 2.2680 5450 | 2.5480 5698 |
| 95 | 1.3718 2238 | 1.4844 0047 | 1.6061 1215 | 2.2878 9998 | 2.5735 3755 |
| 96 | 1.3763 9512 | 1.4905 8547 | 1.6141 4271 | 2.3079 1910 | 2.5992 7293 |
| 97 | 1.3809 8310 | 1.4967 9624 | 1.6222 1342 | 2.3281 1340 | 2.6252 6565 |
| 98 | 1.3855 8638 | 1.5030 3289 | 1.6303 2449 | 2.3484 8439 | 2.6515 1831 |
| 99 | 1.3902 0500 | 1.5092 9553 | 1.6384 7611 | 2.3690 3363 | 2.6780 3349 |
| 100 | 1.3948 3902 | 1.5155 8426 | 1.6466 6849 | 2.3862 6267 | 2.7048 1383 |

## TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 1 | 1.0112 5000 | 1.0125 0000 | 1.0137 5000 | 1.0150 0000 | 1.0175 0000 |
| 2 | 1.0226 2656 | 1.0251 5625 | 1.0276 8906 | 1.0302 2500 | 1.0353 0625 |
| 3 | 1.0341 3111 | 1.0379 7070 | 1.0418 1979 | 1.0456 7838 | 1.0534 2411 |
| 4 | 1.0457 6509 | 1.0509 4534 | 1.0561 4481 | 1.0613 6355 | 1.0718 5903 |
| 5 | 1.0575 2994 | 1.0640 8215 | 1.0706 6680 | 1.0772 8400 | 1.0906 1656 |
| 6 | 1.0694 2716 | 1.0773 8318 | 1.0853 8847 | 1.0934 4326 | 1.1097 0235 |
| 7 | 1.0814 5821 | 1.0908 5047 | 1.1003 1256 | 1.1098 4491 | 1.1291 2215 |
| 8 | 1.0936 2462 | 1.1044 8610 | 1.1154 4186 | 1.1264 9259 | 1.1488 8178 |
| 9 | 1.1059 2789 | 1.1182 9218 | 1.1307 7918 | 1.1433 8998 | 1.1689 8721 |
| 10 | 1.1183 6958 | 1.1322 7083 | 1.1463 2740 | 1.1605 4083 | 1.1894 4449 |
| 11 | 1.1309 5124 | 1.1464 2422 | 1.1620 8940 | 1.1779 4894 | 1.2102 5977 |
| 12 | 1.1436 7444 | 1.1607 5452 | 1.1780 6813 | 1.1956 1817 | 1.2314 3931 |
| 13 | 1.1565 4078 | 1.1752 6395 | 1.1942 6656 | 1.2135 5244 | 1.2529 8950 |
| 14 | 1.1695 5186 | 1.1899 5475 | 1.2106 8773 | 1.2317 5573 | 1.2749 1682 |
| 15 | 1.1827 0932 | 1.2048 2918 | 1.2273 3469 | 1.2502 3207 | 1.2972 2786 |
| 16 | 1.1960 1480 | 1.2198 8955 | 1.2442 1054 | 1.2689 8555 | 1.3199 2935 |
| 17 | 1.2094 6997 | 1.2351 3817 | 1.2613 1843 | 1.2880 2033 | 1.3430 2811 |
| 18 | 1.2230 7650 | 1.2505 7739 | 1.2786 6156 | 1.3073 4064 | 1.3665 3111 |
| 19 | 1.2368 3611 | 1.2662 0961 | 1.2962 4316 | 1.3269 5075 | 1.3904 4540 |
| 20 | 1.2507 5052 | 1.2820 3723 | 1.3140 6650 | 1.3468 5501 | 1.4147 7820 |
| 21 | 1.2648 2143 | 1.2980 6270 | 1.3321 3492 | 1.3670 5783 | 1.4395 3681 |
| 22 | 1.2790 5071 | 1.3142 8848 | 1.3504 5177 | 1.3875 6370 | 1.4647 2871 |
| 23 | 1.2934 4003 | 1.3307 1709 | 1.3690 2048 | 1.4083 7715 | 1.4903 6146 |
| 24 | 1.3079 9123 | 1.3473 5105 | 1.3878 4451 | 1.4295 0281 | 1.5164 4279 |
| 25 | 1.3227 0613 | 1.3641 9294 | 1.4069 2738 | 1.4509 4535 | 1.5429 8054 |
| 26 | 1.3375 8657 | 1.3812 4535 | 1.4262 7263 | 1.4727 0953 | 1.5699 8269 |
| 27 | 1.3526 3442 | 1.3985 1092 | 1.4458 8388 | 1.4948 0018 | 1.5974 5739 |
| 28 | 1.3678 5156 | 1.4159 9230 | 1.4657 6478 | 1.5172 2218 | 1.6254 1290 |
| 29 | 1.3832 3989 | 1.4336 9221 | 1.4859 1905 | 1.5399 8051 | 1.6538 5762 |
| 30 | 1.3988 0134 | 1.4516 1336 | 1.5063 5043 | 1.5630 8022 | 1.6828 0013 |
| 31 | 1.4145 3785 | 1.4697 5853 | 1.5270 6275 | 1.5865 2642 | 1.7122 4913 |
| 32 | 1.4304 5140 | 1.4881 3051 | 1.5480 5986 | 1.6103 2432 | 1.7422 1349 |
| 33 | 1.4465 4398 | 1.5067 3214 | 1.5693 5469 | 1.6344 7918 | 1.7727 0223 |
| 34 | 1.4628 1760 | 1.5255 6629 | 1.5909 2419 | 1.6589 9637 | 1.8037 2452 |
| 35 | 1.4792 7430 | 1.5446 3587 | 1.6127 9940 | 1.6838 8132 | 1.8352 8970 |
| 36 | 1.4959 1613 | 1.5639 4382 | 1.6349 7539 | 1.7091 3954 | 1.8674 0727 |
| 37 | 1.5127 4519 | 1.5834 9312 | 1.6574 5630 | 1.7347 7663 | 1.9000 8689 |
| 38 | 1.5297 6357 | 1.6032 8678 | 1.6802 4633 | 1.7607 9828 | 1.9333 3841 |
| 39 | 1.5469 7341 | 1.6233 2787 | 1.7033 4971 | 1.7872 1025 | 1.9671 7184 |
| 40 | 1.5643 7687 | 1.6436 1946 | 1.7267 7077 | 1.8140 1841 | 2.0015 9734 |
| 41 | 1.5819 7611 | 1.6641 6471 | 1.7505 1387 | 1.8412 2868 | 2.0366 2530 |
| 42 | 1.5997 7334 | 1.6849 6677 | 1.7745 8343 | 1.8688 4712 | 2.0722 6624 |
| 43 | 1.6177 7079 | 1.7060 2885 | 1.7989 8396 | 1.8968 7982 | 2.1085 3090 |
| 44 | 1.6359 7071 | 1.7273 5421 | 1.8237 1999 | 1.9253 3302 | 2.1454 3019 |
| 45 | 1.6543 7538 | 1.7489 4614 | 1.8487 9614 | 1.9542 1301 | 2.1829 7522 |
| 46 | 1.6729 8710 | 1.7708 0797 | 1.8742 1708 | 1.9835 2621 | 2.2211 7728 |
| 47 | 1.6918 0821 | 1.7929 4306 | 1.8999 8757 | 2.0132 7910 | 2.2600 4789 |
| 48 | 1.7108 4105 | 1.8153 5485 | 1.9261 1240 | 2.0434 7829 | 2.2995 9872 |
| 49 | 1.7300 8801 | 1.8380 4679 | 1.9525 9644 | 2.0741 3046 | 2.3398 4170 |
| 50 | 1.7495 5150 | 1.8610 2237 | 1.9794 4464 | 2.1052 4242 | 2.3807 8893 |

## TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 51 | 1.7692 3395 | 1.8842 8515 | 2.0066 6201 | 2.1368 2106 | 2.4224 5274 |
| 52 | 1.7891 3784 | 1.9078 3872 | 2.0342 5361 | 2.1688 7337 | 2.4648 4566 |
| 53 | 1.8092 6564 | 1.9316 8670 | 2.0622 2460 | 2.2014 0647 | 2.5079 8046 |
| 54 | 1.8296 1988 | 1.9558 3279 | 2.0905 8019 | 2.2344 2757 | 2.5518 7012 |
| 55 | 1.8502 0310 | 1.9802 8070 | 2.1193 2566 | 2.2679 4398 | 2.5965 2785 |
| 56 | 1.8710 1788 | 2.0050 3420 | 2.1484 6639 | 2.3019 6314 | 2.6419 6708 |
| 57 | 1.8920 6684 | 2.0300 9713 | 2.1780 0780 | 2.3364 9259 | 2.6882 0151 |
| 58 | 1.9133 5259 | 2.0554 7335 | 2.2079 5541 | 2.3715 3998 | 2.7352 4503 |
| 59 | 1.9348 7780 | 2.0811 6676 | 2.2383 1480 | 2.4071 1308 | 2.7831 1182 |
| 60 | 1.9566 4518 | 2.1071 8135 | 2.2690 9163 | 2.4432 1978 | 2.8318 1628 |
| 61 | 1.9786 5744 | 2.1335 2111 | 2.3002 9164 | 2.4798 6807 | 2.8813 7306 |
| 62 | 2.0009 1733 | 2.1601 9013 | 2.3319 2065 | 2.5170 6609 | 2.9317 9709 |
| 63 | 2.0234 2765 | 2.1871 9250 | 2.3639 8456 | 2.5548 2208 | 2.9831 0354 |
| 64 | 2.0461 9121 | 2.2145 3241 | 2.3964 8934 | 2.5931 4442 | 3.0343 0785 |
| 65 | 2.6092 1087 | 2.2422 1407 | 2.4294 4107 | 2.6320 4158 | 3.0884 2574 |
| 66 | 2.0924 8949 | 2.2702 4174 | 2.4628 4589 | 2.6715 2221 | 3.1424 7319 |
| 67 | 2.1160 2999 | 2.2986 1976 | 2.4967 1002 | 2.7115 9504 | 3.1974 6647 |
| 68 | 2.1398 3533 | 2.3273 5251 | 2.5310 3978 | 2.7522 6896 | 3.2534 2213 |
| 69 | 2.1639 0848 | 2.3564 4442 | 2.5658 4158 | 2.7935 5300 | 3.3103 5702 |
| 70 | 2.1882 5245 | 2.3858 9997 | 2.6011 2190 | 2.8354 5629 | 3.3682 8827 |
| 71 | 2.2128 7029 | 2.4157 2372 | 2.6368 8732 | 2.8779 8814 | 3.4272 3331 |
| 72 | 2.2377 6508 | 2.4459 2027 | 2.6731 4453 | 2.9211 5796 | 3.4872 0990 |
| 73 | 2.2629 3994 | 2.4764 9427 | 2.7099 0026 | 2.9649 7533 | 3.5482 3607 |
| 74 | 2.2833 9801 | 2.5074 5045 | 2.7471 6139 | 3.0094 4996 | 3.6103 3020 |
| 75 | 2.3141 4249 | 2.5387 9358 | 2.7849 3486 | 3.0545 9171 | 3.6735 1098 |
| 76 | 2.3401 7659 | 2.5705 2850 | 2.8232 2771 | 3.1004 1059 | 3.7377 9742 |
| 77 | 2.3665 0358 | 2.6026 6011 | 2.8620 4710 | 3.1469 1674 | 3.8032 0888 |
| 78 | 2.3931 2675 | 2.6351 9336 | 2.9014 0024 | 3.1941 2050 | 3.8697 6503 |
| 79 | 2.4200 4942 | 2.6681 3327 | 2.9412 9450 | 3.2420 3230 | 3.9374 8592 |
| 80 | 2.4472 7498 | 2.7014 8494 | 2.9817 3730 | 3.2906 6279 | 4.0063 9192 |
| 81 | 2.4748 0682 | 2.7352 5350 | 3.0227 3618 | 3.3400 2273 | 4.0765 0378 |
| 82 | 2.5026 4840 | 2.7694 4417 | 3.0642 9881 | 3.3901 2307 | 4.1478 4260 |
| 83 | 2.5308 0319 | 2.8040 6222 | 3.1064 3291 | 3.4409 7492 | 4.2204 2984 |
| 84 | 2.5592 7473 | 2.8391 1300 | 3.1491 4637 | 3.4925 8954 | 4.2942 8737 |
| 85 | 2.5880 6657 | 2.8746 0191 | 3.1924 4713 | 3.5449 7838 | 4.3694 3740 |
| 86 | 2.6171 8232 | 2.9105 3444 | 3.2363 4328 | 3.5981 5306 | 4.4459 0255 |
| 87 | 2.6466 2562 | 2.9469 1612 | 3.2808 4300 | 3.6521 2535 | 4.5237 0584 |
| 88 | 2.6764 0016 | 2.9837 5257 | 3.3259 5459 | 3.7069 0723 | 4.6028 7070 |
| 89 | 2.7065 0966 | 3.0210 4948 | 3.3716 8646 | 3.7625 1084 | 4.6834 2093 |
| 90 | 2.7369 5789 | 3.0588 1260 | 3.4180 4715 | 3.8189 4851 | 4.7653 8080 |
| 91 | 2.7677 4867 | 3.0970 4775 | 3.4650 4530 | 3.8762 3273 | 4.8487 7496 |
| 92 | 2.7988 8584 | 3.1357 6085 | 3.5126 8967 | 3.9343 7622 | 4.9336 2853 |
| 93 | 2.8303 7331 | 3.1749 5786 | 3.5609 8916 | 3.9933 9187 | 5.0199 6703 |
| 94 | 2.8622 1501 | 3.2146 4483 | 3.6099 5276 | 4.0532 9275 | 5.1078 1645 |
| 95 | 2.8944 1492 | 3.2548 2789 | 3.6595 8961 | 4.1140 9214 | 5.1972 0324 |
| 96 | 2.9269 7709 | 3.2955 1324 | 3.7099 0897 | 4.1758 0352 | 5.2881 5429 |
| 97 | 2.9599 0559 | 3.3367 0716 | 3.7609 2021 | 4.2384 4057 | 5.3806 9699 |
| 98 | 2.9932 0452 | 3.3784 1600 | 3.8126 3287 | 4.3020 1718 | 5.4748 5919 |
| 99 | 3.0268 7807 | 3.4206 4620 | 3.8650 5657 | 4.3665 4744 | 5.5706 6923 |
| 100 | 3.0609 3045 | 3.4634 0427 | 3.9182 0110 | 4.4320 4565 | 5.6681 5594 |

## TABLE III. The Amount of 1 at Compound Interest

$(1 + i)^n$

| $n$ | 2% | $2\frac{1}{4}\%$ | $2\frac{1}{2}\%$ | $2\frac{3}{4}\%$ | 3% |
|---|---|---|---|---|---|
| 1 | 1.0200 0000 | 1.0225 0000 | 1.0250 0000 | 1.0275 0000 | 1.0300 0000 |
| 2 | 1.0404 0000 | 1.0455 0625 | 1.0506 2500 | 1.0557 5625 | 1.0609 0000 |
| 3 | 1.0612 0800 | 1.0690 3014 | 1.0768 9063 | 1.0847 8955 | 1.0927 2700 |
| 4 | 1.0824 3216 | 1.0930 8332 | 1.1038 1289 | 1.1146 2126 | 1.1255 0881 |
| 5 | 1.1040 8080 | 1.1176 7769 | 1.1314 0821 | 1.1452 7334 | 1.1592 7407 |
| 6 | 1.1261 6242 | 1.1428 2544 | 1.1596 9342 | 1.1767 6836 | 1.1940 5230 |
| 7 | 1.1486 8567 | 1.1685 3901 | 1.1886 8575 | 1.2091 2949 | 1.2298 7387 |
| 8 | 1.1716 5938 | 1.1948 3114 | 1.2184 0290 | 1.2423 8055 | 1.2667 8007 |
| 9 | 1.1950 9257 | 1.2217 1484 | 1.2488 6297 | 1.2765 4602 | 1.3047 7318 |
| 10 | 1.2189 9442 | 1.2492 0343 | 1.2800 8454 | 1.3116 5103 | 1.3439 1638 |
| 11 | 1.2433 7431 | 1.2773 1050 | 1.3120 8666 | 1.3477 2144 | 1.3842 3387 |
| 12 | 1.2682 4179 | 1.3060 4999 | 1.3448 8882 | 1.3847 8378 | 1.4257 6089 |
| 13 | 1.2936 0663 | 1.3354 3611 | 1.3785 1104 | 1.4228 6533 | 1.4685 3371 |
| 14 | 1.3194 7876 | 1.3654 8343 | 1.4129 7382 | 1.4619 9413 | 1.5125 8972 |
| 15 | 1.3458 6834 | 1.3962 0680 | 1.4482 9817 | 1.5021 9896 | 1.5579 6742 |
| 16 | 1.3727 8571 | 1.4276 2146 | 1.4845 0562 | 1.5435 0944 | 1.6047 0644 |
| 17 | 1.4002 4142 | 1.4597 4294 | 1.5216 1826 | 1.5859 5595 | 1.6528 4763 |
| 18 | 1.4282 4625 | 1.4925 8716 | 1.5596 5872 | 1.6295 6973 | 1.7024 3306 |
| 19 | 1.4568 1117 | 1.5261 7037 | 1.5986 5019 | 1.6743 8290 | 1.7535 0605 |
| 20 | 1.4859 4740 | 1.5605 0920 | 1.6386 1644 | 1.7204 2843 | 1.8061 1123 |
| 21 | 1.5156 6634 | 1.5956 2066 | 1.6795 8185 | 1.7677 4021 | 1.8602 9457 |
| 22 | 1.5459 7967 | 1.6315 2212 | 1.7215 7140 | 1.8163 5307 | 1.9161 0341 |
| 23 | 1.5768 9926 | 1.6682 3137 | 1.7646 1068 | 1.8663 0278 | 1.9735 8651 |
| 24 | 1.6084 3725 | 1.7057 6658 | 1.8087 2595 | 1.9176 2610 | 2.0327 9411 |
| 25 | 1.6406 0599 | 1.7441 4632 | 1.8539 4410 | 1.9703 6082 | 2.0937 7793 |
| 26 | 1.6734 1811 | 1.7833 8962 | 1.9002 9270 | 2.0245 4575 | 2.1565 9127 |
| 27 | 1.7068 8648 | 1.8235 1588 | 1.9478 0002 | 2.0802 2075 | 2.2212 8901 |
| 28 | 1.7410 2421 | 1.8645 4499 | 1.9964 9502 | 2.1374 2682 | 2.2879 2768 |
| 29 | 1.7758 4469 | 1.9064 9725 | 2.0464 0739 | 2.1962 0606 | 2.3565 6551 |
| 30 | 1.8113 6158 | 1.9493 9344 | 2.0975 6758 | 2.2566 0173 | 2.4272 6247 |
| 31 | 1.8475 8882 | 1.9932 5479 | 2.1500 0677 | 2.3186 5828 | 2.5000 8035 |
| 32 | 1.8845 4059 | 2.0381 0303 | 2.2037 5694 | 2.3824 2138 | 2.5750 8276 |
| 33 | 1.9222 3140 | 2.0839 6034 | 2.2588 5086 | 2.4479 3797 | 2.6523 3524 |
| 34 | 1.9606 7603 | 2.1308 4945 | 2.3153 2213 | 2.5152 5626 | 2.7319 0530 |
| 35 | 1.9998 8955 | 2.1787 9356 | 2.3732 0519 | 2.5844 2581 | 2.8138 6245 |
| 36 | 2.0398 8734 | 2.2278 1642 | 2.4325 3532 | 2.6554 9752 | 2.8982 7833 |
| 37 | 2.0806 8509 | 2.2779 4229 | 2.4933 4870 | 2.7285 2370 | 2.9852 2668 |
| 38 | 2.1222 9879 | 2.3291 9599 | 2.5556 8242 | 2.8035 5810 | 3.0747 8348 |
| 39 | 2.1647 4477 | 2.3816 0290 | 2.6195 7448 | 2.8806 5595 | 3.1670 2698 |
| 40 | 2.2080 3966 | 2.4351 8897 | 2.6850 6384 | 2.9598 7399 | 3.2620 3779 |
| 41 | 2.2522 0046 | 2.4899 8072 | 2.7521 9043 | 3.0412 7052 | 3.3598 9893 |
| 42 | 2.2972 4447 | 2.5460 0528 | 2.8209 9520 | 3.1249 0546 | 3.4606 9589 |
| 43 | 2.3431 8936 | 2.6032 9040 | 2.8915 2008 | 3.2108 4036 | 3.5645 1677 |
| 44 | 2.3900 5314 | 2.6618 6444 | 2.9638 0808 | 3.2991 3847 | 3.6714 5227 |
| 45 | 2.4378 5421 | 2.7217 5639 | 3.0379 0328 | 3.3898 6478 | 3.7815 9584 |
| 46 | 2.4866 1129 | 2.7829 9590 | 3.1138 5086 | 3.4830 8606 | 3.8950 4372 |
| 47 | 2.5363 4351 | 2.8456 1331 | 3.1916 9713 | 3.5788 7093 | 4.0118 9503 |
| 48 | 2.5870 7039 | 2.9096 3961 | 3.2714 8956 | 3.6772 8988 | 4.1322 5188 |
| 49 | 2.6388 1179 | 2.9751 0650 | 3.3532 7680 | 3.7784 1535 | 4.2562 1944 |
| 50 | 2.6915 8803 | 3.0420 4640 | 3.4371 0872 | 3.8823 2177 | 4.3839 0602 |

## TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

| $n$ | 2% | $2\frac{1}{4}$% | $2\frac{1}{2}$% | $2\frac{3}{4}$% | 3% |
|---|---|---|---|---|---|
| 51 | 2.7454 1979 | 3.1104 9244 | 3.5230 3644 | 3.9890 8562 | 4.5154 2320 |
| 52 | 2.8003 2819 | 3.1804 7852 | 3.6111 1235 | 4.0987 8547 | 4.6508 8590 |
| 53 | 2.8563 3475 | 3.2520 3929 | 3.7013 9016 | 4.2115 0208 | 4.7904 1247 |
| 54 | 2.9134 6144 | 3.3252 1017 | 3.7939 2491 | 4.3273 1838 | 4.9341 2485 |
| 55 | 2.9717 3067 | 3.4000 2740 | 3.8887 7303 | 4.4463 1964 | 5.0821 4859 |
| 56 | 3.0311 6529 | 3.4765 2802 | 3.9859 9236 | 4.5685 9343 | 5.2346 1305 |
| 57 | 3.0917 8859 | 3.5547 4990 | 4.0856 4217 | 4.6942 2975 | 5.3916 5144 |
| 58 | 3.1536 2436 | 3.6347 3177 | 4.1877 8322 | 4.8233 2107 | 5.5534 0098 |
| 59 | 3.2166 9685 | 3.7165 1324 | 4.2924 7780 | 4.9559 6239 | 5.7200 0301 |
| 60 | 3.2810 3079 | 3.8001 3479 | 4.3997 8975 | 5.0922 5136 | 5.8916 0310 |
| 61 | 3.3466 5140 | 3.8856 3782 | 4.5097 8449 | 5.2322 8827 | 6.0683 5120 |
| 62 | 3.4135 8443 | 3.9730 6467 | 4.6225 2910 | 5.3761 7620 | 6.2504 0173 |
| 63 | 3.4818 5612 | 4.0624 5862 | 4.7380 9233 | 5.5240 2105 | 6.4379 1379 |
| 64 | 3.5514 9324 | 4.1538 6394 | 4.8565 4464 | 5.6759 3162 | 6.6310 5120 |
| 65 | 3.6225 2311 | 4.2473 2588 | 4.9779 5826 | 5.8320 1974 | 6.8299 8273 |
| 66 | 3.6949 7357 | 4.3428 9071 | 5.1024 0721 | 5.9924 0029 | 7.0348 8222 |
| 67 | 3.7688 7304 | 4.4406 0576 | 5.2299 6739 | 6.1571 9130 | 7.2459 2868 |
| 68 | 3.8442 5050 | 4.5405 1939 | 5.3607 1658 | 6.3265 1406 | 7.4633 0654 |
| 69 | 3.9211 3551 | 4.6426 8107 | 5.4947 3449 | 6.5004 9319 | 7.6872 0574 |
| 70 | 3.9995 5822 | 4.7471 4140 | 5.6321 0286 | 6.6792 5676 | 7.9178 2191 |
| 71 | 4.0795 4939 | 4.8539 5208 | 5.7729 0543 | 6.8629 3632 | 8.1553 5657 |
| 72 | 4.1611 4038 | 4.9631 6600 | 5.9172 2806 | 7.0516 6706 | 8.4000 1727 |
| 73 | 4.2443 6318 | 5.0748 3723 | 6.0651 5876 | 7.2455 8791 | 8.6520 1778 |
| 74 | 4.3292 5045 | 5.1890 2107 | 6.2167 8773 | 7.4448 4158 | 8.9115 7832 |
| 75 | 4.4158 3546 | 5.3057 7405 | 6.3722 0743 | 7.6495 7472 | 9.1789 2567 |
| 76 | 4.5041 5216 | 5.4251 5396 | 6.5315 1261 | 7.8599 3802 | 9.4542 9344 |
| 77 | 4.5942 3521 | 5.5472 1993 | 6.6948 0043 | 8.0760 8632 | 9.7379 2224 |
| 78 | 4.6861 1991 | 5.6720 3237 | 6.8621 7044 | 8.2981 7869 | 10.0300 5991 |
| 79 | 4.7798 4231 | 5.7996 5310 | 7.0337 2470 | 8.5263 7861 | 10.3309 6171 |
| 80 | 4.8754 3916 | 5.9301 4530 | 7.2095 6782 | 8.7608 5402 | 10.6408 9056 |
| 81 | 4.9729 4794 | 6.0635 7357 | 7.3898 0701 | 9.0017 7751 | 10.9601 1727 |
| 82 | 5.0724 0690 | 6.2000 0397 | 7.5745 5219 | 9.2493 2639 | 11.2889 2079 |
| 83 | 5.1738 5504 | 6.3395 0406 | 7.7639 1599 | 9.5036 8286 | 11.6275 8842 |
| 84 | 5.2773 3214 | 6.4821 4290 | 7.9580 1389 | 9.7650 3414 | 11.9764 1607 |
| 85 | 5.3828 7878 | 6.6279 9112 | 8.1569 6424 | 10.0335 7258 | 12.3357 0855 |
| 86 | 5.4905 3636 | 6.7771 2092 | 8.3608 8834 | 10.3094 9583 | 12.7057 7981 |
| 87 | 5.6003 4708 | 6.9296 0614 | 8.5699 1055 | 10.5930 0696 | 13.0869 5320 |
| 88 | 5.7123 5402 | 7.0855 2228 | 8.7841 5832 | 10.8843 1465 | 13.4795 6180 |
| 89 | 5.8266 0110 | 7.2449 4653 | 9.0037 6228 | 11.1836 3331 | 13.8839 4865 |
| 90 | 5.9431 3313 | 7.4079 5782 | 9.2288 5633 | 11.4911 8322 | 14.3004 6711 |
| 91 | 6.0619 9579 | 7.5746 3688 | 9.4595 7774 | 11.8071 9076 | 14.7294 8112 |
| 92 | 6.1832 3570 | 7.7450 6621 | 9.6960 6718 | 12.1318 8851 | 15.1713 6556 |
| 93 | 6.3069 0042 | 7.9193 3020 | 9.9384 6886 | 12.4655 1544 | 15.6265 0652 |
| 94 | 6.4330 3843 | 8.0975 1512 | 10.1869 3058 | 12.8083 1711 | 16.0953 0172 |
| 95 | 6.5616 9920 | 8.2797 0921 | 10.4416 0385 | 13.1605 4584 | 16.5781 6077 |
| 96 | 6.6929 3318 | 8.4660 0267 | 10.7026 4395 | 13.5224 6085 | 17.0755 0559 |
| 97 | 6.8267 9184 | 8.6564 8773 | 10.9702 1004 | 13.8943 2852 | 17.5877 7076 |
| 98 | 6.9633 2768 | 8.8512 5871 | 11.2444 6530 | 14.2764 2255 | 18.1154 0388 |
| 99 | 7.1025 9423 | 9.0504 1203 | 11.5255 7693 | 14.6690 2417 | 18.6588 6600 |
| 100 | 7.2446 4612 | 9.2540 4630 | 11.8137 1635 | 15.0724 2234 | 19.2186 3198 |

## TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 1 | 1.0350 0000 | 1.0400 0000 | 1.0450 0000 | 1.0500 0000 | 1.0550 0000 |
| 2 | 1.0712 2500 | 1.0816 0000 | 1.0920 2500 | 1.1025 0000 | 1.1130 2500 |
| 3 | 1.1087 1788 | 1.1248 6400 | 1.1411 6613 | 1.1576 2500 | 1.1742 4138 |
| 4 | 1.1475 2300 | 1.1698 5856 | 1.1925 1860 | 1.2155 0625 | 1.2388 2465 |
| 5 | 1.1876 8631 | 1.2166 5290 | 1.2461 8194 | 1.2762 8156 | 1.3069 6001 |
| 6 | 1.2292 5533 | 1.2653 1902 | 1.3022 6012 | 1.3400 9564 | 1.3788 4281 |
| 7 | 1.2722 7926 | 1.3159 3178 | 1.3608 6183 | 1.4071 0042 | 1.4546 7916 |
| 8 | 1.3168 0904 | 1.3685 6905 | 1.4221 0061 | 1.4774 5544 | 1.5346 8651 |
| 9 | 1.3628 9735 | 1.4233 1181 | 1.4860 9514 | 1.5513 2822 | 1.6190 9427 |
| 10 | 1.4105 9876 | 1.4802 4428 | 1.5529 6942 | 1.6288 9463 | 1.7081 4446 |
| 11 | 1.4599 6972 | 1.5394 5406 | 1.6228 5305 | 1.7103 3936 | 1.8020 9240 |
| 12 | 1.5110 6866 | 1.6010 3222 | 1.6958 8143 | 1.7958 5633 | 1.9012 0749 |
| 13 | 1.5639 5606 | 1.6650 7351 | 1.7721 9610 | 1.8856 4914 | 2.0057 7390 |
| 14 | 1.6186 9452 | 1.7316 7645 | 1.8519 4492 | 1.9799 3160 | 2.1160 9146 |
| 15 | 1.6753 4883 | 1.8009 4351 | 1.9352 8244 | 2.0789 2818 | 2.2324 7649 |
| 16 | 1.7339 8604 | 1.8729 8125 | 2.0223 7015 | 2.1828 7459 | 2.3552 6270 |
| 17 | 1.7946 7555 | 1.9479 0050 | 2.1133 7681 | 2.2920 1832 | 2.4848 0215 |
| 18 | 1.8574 8920 | 2.0258 1652 | 2.2084 7877 | 2.4066 1923 | 2.6214 6627 |
| 19 | 1.9225 0132 | 2.1068 4918 | 2.3078 6031 | 2.5269 5020 | 2.7656 4691 |
| 20 | 1.9897 8886 | 2.1911 2314 | 2.4117 1402 | 2.6532 9771 | 2.9177 5749 |
| 21 | 2.0594 3147 | 2.2787 6807 | 2.5202 4116 | 2.7859 6259 | 3.0782 3415 |
| 22 | 2.1315 1158 | 2.3699 1879 | 2.6336 5201 | 2.9252 6072 | 3.2475 3703 |
| 23 | 2.2061 1448 | 2.4647 1554 | 2.7521 6635 | 3.0715 2376 | 3.4261 5157 |
| 24 | 2.2833 2849 | 2.5633 0416 | 2.8760 1383 | 3.2250 9994 | 3.6145 8990 |
| 25 | 2.3632 4498 | 2.6658 3633 | 3.0054 3446 | 3.3863 5494 | 3.8133 9235 |
| 26 | 2.4459 5856 | 2.7724 6978 | 3.1406 7901 | 3.5556 7269 | 4.0231 2893 |
| 27 | 2.5315 6711 | 2.8833 6858 | 3.2820 0956 | 3.7334 5632 | 4.2444 0102 |
| 28 | 2.6201 7196 | 2.9987 0332 | 3.4296 9999 | 3.9201 2914 | 4.4778 4307 |
| 29 | 2.7118 7798 | 3.1186 5145 | 3.5840 3649 | 4.1161 3560 | 4.7241 2444 |
| 30 | 2.8067 9370 | 3.2433 9751 | 3.7453 1813 | 4.3219 4238 | 4.9839 5129 |
| 31 | 2.9050 3148 | 3.3731 3341 | 3.9138 5745 | 4.5380 3949 | 5.2580 6861 |
| 32 | 3.0067 0759 | 3.5080 5875 | 4.0899 8104 | 4.7649 4147 | 5.5472 6238 |
| 33 | 3.1119 4235 | 3.6483 8110 | 4.2740 3018 | 5.0031 8854 | 5.8523 6181 |
| 34 | 3.2208 6033 | 3.7943 1634 | 4.4663 6154 | 5.2533 4797 | 6.1742 4171 |
| 35 | 3.3335 9045 | 3.9460 8899 | 4.6673 4781 | 5.5160 1537 | 6.5138 2501 |
| 36 | 3.4502 6611 | 4.1039 3255 | 4.8773 7846 | 5.7918 1614 | 6.8720 8538 |
| 37 | 3.5710 2543 | 4.2680 8986 | 5.0968 6049 | 6.0814 0694 | 7.2500 5008 |
| 38 | 3.6960 1132 | 4.4388 1345 | 5.3262 1921 | 6.3854 7729 | 7.6488 0283 |
| 39 | 3.8253 7171 | 4.6163 6599 | 5.5658 9908 | 6.7047 5115 | 8.0694 8699 |
| 40 | 3.9592 5972 | 4.8010 2063 | 5.8163 6454 | 7.0399 8871 | 8.5133 0877 |
| 41 | 4.0978 3381 | 4.9930 6145 | 6.0781 0094 | 7.3919 8815 | 8.9815 4076 |
| 42 | 4.2412 5799 | 5.1927 8391 | 6.3516 1548 | 7.7615 8756 | 9.4755 2550 |
| 43 | 4.3897 0202 | 5.4004 9527 | 6.6374 3818 | 8.1496 6693 | 9.9966 7940 |
| 44 | 4.5433 4160 | 5.6165 1508 | 6.9361 2290 | 8.5571 5028 | 10.5464 9677 |
| 45 | 4.7023 5855 | 5.8411 7568 | 7.2482 4843 | 8.9850 0779 | 11.1265 5409 |
| 46 | 4.8669 4110 | 6.0748 2271 | 7.5744 1961 | 9.4342 5818 | 11.7385 1456 |
| 47 | 5.0372 8404 | 6.3178 1562 | 7.9152 6849 | 9.9059 7109 | 12.3841 3287 |
| 48 | 5.2135 8898 | 6.5705 2824 | 8.2714 5557 | 10.4012 6965 | 13.0652 6017 |
| 49 | 5.3960 5459 | 6.8333 4937 | 8.6436 7107 | 10.9213 3313 | 13.7838 4948 |
| 50 | 5.5849 2686 | 7.1066 8335 | 9.0326 3627 | 11.4673 9979 | 14.5419 6120 |

## TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 51 | 5.7803 9930 | 7.3909 5068 | 9.4391 0490 | 12.0407 6978 | 15.3417 6907 |
| 52 | 5.9827 1327 | 7.6865 8871 | 9.8638 6463 | 12.6428 0826 | 16.1855 6637 |
| 53 | 6.1921 0824 | 7.9940 5226 | 10.3077 3853 | 13.2749 4868 | 17.0757 7252 |
| 54 | 6.4088 3202 | 8.3138 1435 | 10.7715 8677 | 13.9386 9611 | 18.0149 4001 |
| 55 | 6.6331 4114 | 8.6463 6692 | 11.2563 0817 | 14.6356 3092 | 19.0057 6171 |
| 56 | 6.8653 0108 | 8.9922 2160 | 11.7628 4204 | 15.3674 1246 | 20.0510 7860 |
| 57 | 7.1055 8662 | 9.3519 1046 | 12.2921 6993 | 16.1357 8309 | 21.1538 8793 |
| 58 | 7.3542 8215 | 9.7259 8688 | 12.8453 1758 | 16.9425 7224 | 22.3173 5176 |
| 59 | 7.6116 8203 | 10.1150 2635 | 13.4233 5687 | 17.7897 0085 | 23.5448 0611 |
| 60 | 7.8780 9090 | 10.5196 2741 | 14.0274 0793 | 18.6791 8589 | 24.8397 7045 |
| 61 | 8.1538 2408 | 10.9404 1250 | 14.6586 4129 | 19.6131 4519 | 26.2059 5782 |
| 62 | 8.4392 0793 | 11.3780 2900 | 15.3182 8014 | 20.5938 0245 | 27.6472 8550 |
| 63 | 8.7345 8020 | 11.8331 5016 | 16.0076 0275 | 21.6234 9257 | 29.1678 8620 |
| 64 | 9.0402 9051 | 12.3064 7617 | 16.7279 4487 | 22.7046 6720 | 30.7721 1994 |
| 65 | 9.3567 0068 | 12.7987 3522 | 17.4807 0239 | 23.8399 0056 | 32.4645 8654 |
| 66 | 9.6841 8520 | 13.3106 8463 | 18.2673 3400 | 25.0318 9559 | 34.2501 3880 |
| 67 | 10.0231 3168 | 13.8431 1201 | 19.0893 6403 | 26.2834 9037 | 36.1338 9643 |
| 68 | 10.3739 4129 | 14.3968 3649 | 19.9483 8541 | 27.5976 6488 | 38.1212 6074 |
| 69 | 10.7370 2024 | 14.9727 0995 | 20.8460 6276 | 28.9775 4813 | 40.2179 3008 |
| 70 | 11.1128 2526 | 15.5716 1835 | 21.7841 3558 | 30.4264 2554 | 42.4299 1623 |
| 71 | 11.5017 7414 | 16.1944 8308 | 22.7644 2168 | 31.9477 4681 | 44.7635 6163 |
| 72 | 11.9043 3624 | 16.8422 6241 | 23.7888 2066 | 33.5451 3415 | 47.2255 5751 |
| 73 | 12.3209 8801 | 17.5159 5290 | 24.8593 1759 | 35.2223 9086 | 49.8229 6318 |
| 74 | 12.7522 2259 | 18.2165 9102 | 25.9779 8688 | 36.9835 1040 | 52.5632 2615 |
| 75 | 13.1985 5038 | 18.9452 5466 | 27.1469 9629 | 38.8326 8592 | 55.4542 0359 |
| 76 | 13.6604 9964 | 19.7030 6485 | 28.3686 1112 | 40.7743 2022 | 58.5041 8479 |
| 77 | 14.1386 1713 | 20.4911 8744 | 29.6451 9862 | 42.8130 3623 | 61.7219 1495 |
| 78 | 14.6334 6873 | 21.3108 3494 | 30.9792 3256 | 44.9536 8804 | 65.1166 2027 |
| 79 | 15.1456 4013 | 22.1632 6834 | 32.3732 9802 | 47.2013 7244 | 68.6980 3439 |
| 80 | 15.6757 3754 | 23.0497 9907 | 33.8300 9643 | 49.5614 4107 | 72.4764 2628 |
| 81 | 16.2243 8835 | 23.9717 9103 | 35.3524 5077 | 52.0395 1312 | 76.4626 2973 |
| 82 | 16.7922 4195 | 24.9306 6267 | 36.9433 1106 | 54.6414 8878 | 80.6680 7436 |
| 83 | 17.3799 7041 | 25.9278 8918 | 38.6057 6006 | 57.3735 6322 | 85.1048 1845 |
| 84 | 17.9882 6938 | 26.9650 0475 | 40.3430 1926 | 60.2422 4138 | 89.7855 8347 |
| 85 | 18.6178 5881 | 28.0436 0494 | 42.1584 5513 | 63.2543 5344 | 94.7237 9056 |
| 86 | 19.2694 8387 | 29.1653 4914 | 44.0555 8561 | 66.4170 7112 | 99.9335 9904 |
| 87 | 19.9439 1580 | 30.3319 6310 | 46.0380 8696 | 69.7379 2467 | 105.4299 4698 |
| 88 | 20.6419 5285 | 31.5452 4163 | 48.1098 0087 | 73.2248 2091 | 111.2285 9407 |
| 89 | 21.3644 2120 | 32.8070 5129 | 50.2747 4191 | 76.8860 6195 | 117.3461 6674 |
| 90 | 22.1121 7595 | 34.1193 3334 | 52.5371 0530 | 80.7303 6505 | 123.8002 0591 |
| 91 | 22.8861 0210 | 35.4841 0668 | 54.9012 7503 | 84.7668 8330 | 130.6092 1724 |
| 92 | 23.6871 1568 | 36.9034 7094 | 57.3718 3241 | 89.0052 2747 | 137.7927 2419 |
| 93 | 24.5161 6473 | 38.3796 0978 | 59.9535 6487 | 93.4554 8884 | 145.3713 2402 |
| 94 | 25.3742 3049 | 39.9147 9417 | 62.6514 7529 | 98.1282 6328 | 153.3667 4684 |
| 95 | 26.2623 2856 | 41.5113 8594 | 65.4707 9168 | 103.0346 7645 | 161.8019 1791 |
| 96 | 27.1815 1006 | 43.1718 4138 | 68.4169 7730 | 108.1864 1027 | 170.7010 2340 |
| 97 | 28.1328 6291 | 44.8987 1503 | 71.4957 4128 | 113.5957 3078 | 180.0895 7969 |
| 98 | 29.1175 1311 | 46.6946 6363 | 74.7130 4964 | 119.2755 1732 | 189.9945 0657 |
| 99 | 30.1366 2607 | 48.5624 5018 | 78.0751 3687 | 125.2392 9319 | 200.4442 0443 |
| 100 | 31.1914 0798 | 50.5049 4818 | 81.5885 1803 | 131.5012 5785 | 211.4686 3567 |

## TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

| $n$ | 6% | $6\frac{1}{2}$% | 7% | $7\frac{1}{2}$% | 8% |
|---|---|---|---|---|---|
| 1 | 1.0600 0000 | 1.0650 0000 | 1.0700 0000 | 1.0750 0000 | 1.0800 0000 |
| 2 | 1.1236 0000 | 1.1342 2500 | 1.1449 0000 | 1.1556 2500 | 1.1664 0000 |
| 3 | 1.1910 1600 | 1.2079 4963 | 1.2250 4300 | 1.2422 9688 | 1.2597 1200 |
| 4 | 1.2624 7696 | 1.2864 6635 | 1.3107 9601 | 1.3354 6914 | 1.3604 8896 |
| 5 | 1.3382 2558 | 1.3700 8666 | 1.4025 5173 | 1.4356 2933 | 1.4693 2808 |
| 6 | 1.4185 1911 | 1.4591 4230 | 1.5007 3035 | 1.5433 0153 | 1.5868 7432 |
| 7 | 1.5036 3026 | 1.5539 8655 | 1.6057 8148 | 1.6590 4914 | 1.7138 2427 |
| 8 | 1.5938 4807 | 1.6549 9567 | 1.7181 8618 | 1.7834 7783 | 1.8509 3021 |
| 9 | 1.6894 7896 | 1.7625 7039 | 1.8384 5921 | 1.9172 3866 | 1.9990 0463 |
| 10 | 1.7908 4770 | 1.8771 3747 | 1.9671 5136 | 2.0610 3156 | 2.1589 2500 |
| 11 | 1.8982 9856 | 1.9991 5140 | 2.1048 5195 | 2.2156 0893 | 2.3316 3900 |
| 12 | 2.0121 9647 | 2.1290 9624 | 2.2521 9159 | 2.3817 7960 | 2.5181 7012 |
| 13 | 2.1329 2826 | 2.2674 8750 | 2.4098 4500 | 2.5604 1307 | 2.7196 2373 |
| 14 | 2.2609 0396 | 2.4148 7418 | 2.5785 3415 | 2.7524 4405 | 2.9371 9362 |
| 15 | 2.3965 5819 | 2.5718 4102 | 2.7590 3154 | 2.9588 7735 | 3.1721 6911 |
| 16 | 2.5403 5168 | 2.7390 1067 | 2.9521 6375 | 3.1807 9315 | 3.4259 4264 |
| 17 | 2.6927 7279 | 2.9170 4637 | 3.1588 1521 | 3.4193 5264 | 3.7000 1805 |
| 18 | 2.8543 3915 | 3.1066 5438 | 3.3799 3228 | 3.6758 0409 | 3.9960 1950 |
| 19 | 3.0255 9950 | 3.3085 8691 | 3.6165 2754 | 3.9514 8940 | 4.3157 0106 |
| 20 | 3.2071 3547 | 3.5236 4506 | 3.8696 8446 | 4.2478 5110 | 4.6609 5714 |
| 21 | 3.3995 6360 | 3.7526 8199 | 4.1405 6237 | 4.5664 3993 | 5.0338 3372 |
| 22 | 3.6035 3742 | 3.9966 0632 | 4.4304 0174 | 4.9089 2293 | 5.4365 4041 |
| 23 | 3.8197 4066 | 4.2563 8573 | 4.7405 2986 | 5.2770 9215 | 5.8714 6365 |
| 24 | 4.0489 3464 | 4.5330 5081 | 5.0723 6695 | 5.6728 7406 | 6.3411 8074 |
| 25 | 4.2918 7072 | 4.8276 9911 | 5.4274 3264 | 6.0983 3961 | 6.8484 7520 |
| 26 | 4.5493 8296 | 5.1414 9955 | 5.8073 5292 | 6.5557 1508 | 7.3963 5321 |
| 27 | 4.8223 4594 | 5.4756 9702 | 6.2138 6763 | 7.0473 9371 | 7.9880 6147 |
| 28 | 5.1116 8670 | 5.8316 1733 | 6.6488 3836 | 7.5759 4824 | 8.6271 0639 |
| 29 | 5.4183 8790 | 6.2106 7245 | 7.1142 5705 | 8.1441 4436 | 9.3172 7490 |
| 30 | 5.7434 9117 | 6.6143 6616 | 7.6122 5504 | 8.7549 5519 | 10.0626 5689 |
| 31 | 6.0881 0064 | 7.0442 9996 | 8.1451 1290 | 9.4115 7683 | 10.8676 6944 |
| 32 | 6.4533 8668 | 7.5021 7946 | 8.7152 7080 | 10.1174 4509 | 11.7370 8300 |
| 33 | 6.8405 8988 | 7.9898 2113 | 9.3253 3975 | 10.8762 5347 | 12.6760 4964 |
| 34 | 7.2510 2528 | 8.5091 5950 | 9.9781 1354 | 11.6919 7248 | 13.6901 3361 |
| 35 | 7.6860 8679 | 9.0622 5487 | 10.6765 8148 | 12.5688 7042 | 14.7853 4429 |
| 36 | 8.1472 5200 | 9.6513 0143 | 11.4239 4219 | 13.5115 3570 | 15.9681 7184 |
| 37 | 8.6360 8712 | 10.2786 3603 | 12.2236 1814 | 14.5249 0088 | 17.2456 2558 |
| 38 | 9.1542 5235 | 10.9467 4737 | 13.0792 7141 | 15.6142 6844 | 18.6252 7563 |
| 39 | 9.7035 0749 | 11.6582 8595 | 13.9948 2041 | 16.7853 3858 | 20.1152 9768 |
| 40 | 10.2857 1794 | 12.4160 7453 | 14.9744 5784 | 18.0442 3897 | 21.7245 2150 |
| 41 | 10.9028 6101 | 13.2231 1938 | 16.0226 6989 | 19.3975 5689 | 23.4624 8322 |
| 42 | 11.5570 3267 | 14.0826 2214 | 17.1442 5678 | 20.8523 7366 | 25.3394 8187 |
| 43 | 12.2504 5463 | 14.9979 9258 | 18.3443 5475 | 22.4163 0168 | 27.3666 4042 |
| 44 | 12.9854 8191 | 15.9728 6209 | 19.6284 5959 | 24.0975 2431 | 29.5559 7166 |
| 45 | 13.7646 1083 | 17.0110 9813 | 21.0024 5176 | 25.9048 3863 | 31.9204 4939 |
| 46 | 14.5904 8748 | 18.1168 1951 | 22.4726 2338 | 27.8477 0153 | 34.4740 8534 |
| 47 | 15.4659 1673 | 19.2944 1278 | 24.0457 0702 | 29.9362 7915 | 37.2320 1217 |
| 48 | 16.3938 7173 | 20.5485 4961 | 25.7289 0651 | 32.1815 0008 | 40.2105 7314 |
| 49 | 17.3775 0403 | 21.8842 0533 | 27.5299 2997 | 34.5951 1259 | 43.4274 1899 |
| 50 | 18.4201 5427 | 23.3066 7868 | 29.4570 2506 | 37.1897 4603 | 46.9016 1251 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 1 | 0.9966 7774 | 0.9958 5062 | 0.9950 2488 | 0.9913 2590 | 0.9900 9901 |
| 2 | 0.9933 6652 | 0.9917 1846 | 0.9900 7450 | 0.9827 2704 | 0.9802 9605 |
| 3 | 0.9900 6630 | 0.9876 0345 | 0.9851 4876 | 0.9742 0276 | 0.9705 9015 |
| 4 | 0.9867 7704 | 0.9835 0551 | 0.9802 4752 | 0.9657 5243 | 0.9609 8034 |
| 5 | 0.9834 9871 | 0.9794 2457 | 0.9753 7067 | 0.9573 7539 | 0.9514 6569 |
| 6 | 0.9802 3127 | 0.9753 6057 | 0.9705 1808 | 0.9490 7102 | 0.9420 4524 |
| 7 | 0.9769 7469 | 0.9713 1343 | 0.9656 8963 | 0.9408 3868 | 0.9327 1805 |
| 8 | 0.9737 2893 | 0.9672 8308 | 0.9608 8520 | 0.9326 7775 | 0.9234 8322 |
| 9 | 0.9794 9395 | 0.9632 6946 | 0.9561 0468 | 0.9245 8761 | 0.9143 3982 |
| 10 | 0.9672 6972 | 0.9592 7249 | 0.9513 4794 | 0.9165 6765 | 0.9052 8695 |
| 11 | 0.9640 5620 | 0.9552 9211 | 0.9466 1489 | 0.9086 1724 | 0.8963 2372 |
| 12 | 0.9608 5335 | 0.9513 2824 | 0.9419 0534 | 0.9007 3581 | 0.8874 4923 |
| 13 | 0.9576 6115 | 0.9473 8082 | 0.9372 1924 | 0.9029 2273 | 0.8786 6260 |
| 14 | 0.9544 7955 | 0.9434 4978 | 0.9325 5646 | 0.8851 7743 | 0.8699 6297 |
| 15 | 0.9513 0852 | 0.9395 3505 | 0.9279 1688 | 0.8774 9931 | 0.8613 4947 |
| 16 | 0.9481 4803 | 0.9356 3656 | 0.9233 0037 | 0.8698 8779 | 0.8528 2126 |
| 17 | 0.9449 9803 | 0.9317 5425 | 0.9187 0684 | 0.8623 4230 | 0.8443 7749 |
| 18 | 0.9418 5851 | 0.9278 8805 | 0.9141 3616 | 0.8548 6225 | 0.8360 1731 |
| 19 | 0.9387 2941 | 0.9240 3789 | 0.9095 8822 | 0.8474 4709 | 0.8277 3992 |
| 20 | 0.9356 1071 | 0.9202 0371 | 0.9050 6290 | 0.8400 9624 | 0.8195 4447 |
| 21 | 0.9325 0236 | 0.9163 8544 | 0.9005 6010 | 0.8328 0917 | 0.8114 3017 |
| 22 | 0.9294 0435 | 0.9125 8301 | 0.8960 7971 | 0.8255 8530 | 0.8033 9621 |
| 23 | 0.9263 1663 | 0.9087 9636 | 0.8916 2160 | 0.8184 2409 | 0.7954 4179 |
| 24 | 0.9232 3916 | 0.9050 2542 | 0.8871 8567 | 0.8113 2499 | 0.7875 6613 |
| 25 | 0.9201 7192 | 0.9012 7012 | 0.8827 7181 | 0.8042 8748 | 0.7797 6844 |
| 26 | 0.9171 1487 | 0.8975 3041 | 0.8783 7991 | 0.7973 1101 | 0.7720 4796 |
| 27 | 0.9140 6798 | 0.8938 0622 | 0.8740 0986 | 0.7903 9505 | 0.7644 0392 |
| 28 | 0.9110 3121 | 0.8900 9748 | 0.8696 6155 | 0.7835 3908 | 0.7568 3557 |
| 29 | 0.9080 0453 | 0.8864 0413 | 0.8653 3488 | 0.7767 4258 | 0.7493 4215 |
| 30 | 0.9049 8790 | 0.8827 2610 | 0.8610 2973 | 0.7700 0504 | 0.7419 2292 |
| 31 | 0.9019 8130 | 0.8790 6334 | 0.8567 4600 | 0.7633 2594 | 0.7345 7715 |
| 32 | 0.8989 8468 | 0.8754 1577 | 0.8524 8358 | 0.7567 0477 | 0.7273 0411 |
| 33 | 0.8959 9802 | 0.8717 8334 | 0.8482 4237 | 0.7501 4104 | 0.7201 0307 |
| 34 | 0.8930 2128 | 0.8681 6599 | 0.8440 2226 | 0.7436 3424 | 0.7129 7334 |
| 35 | 0.8900 5444 | 0.8645 6364 | 0.8398 2314 | 0.7371 8388 | 0.7059 1420 |
| 36 | 0.8870 9745 | 0.8609 7624 | 0.8356 4492 | 0.7307 9847 | 0.6989 2495 |
| 37 | 0.8841 5028 | 0.8574 0372 | 0.8314 8748 | 0.7244 5053 | 0.6920 0490 |
| 38 | 0.8812 1290 | 0.8538 4603 | 0.8273 5073 | 0.7181 6657 | 0.6851 5337 |
| 39 | 0.8782 8528 | 0.8503 0310 | 0.8232 3455 | 0.7119 3712 | 0.6783 6967 |
| 40 | 0.8753 6739 | 0.8467 7487 | 0.8191 3886 | 0.7057 6171 | 0.6716 5314 |
| 41 | 0.8724 5920 | 0.8432 6128 | 0.8150 6354 | 0.6996 3986 | 0.6650 0311 |
| 42 | 0.8695 6066 | 0.8397 6227 | 0.8110 0850 | 0.6935 7111 | 0.6584 1892 |
| 43 | 0.8666 7175 | 0.8362 7778 | 0.8069 7363 | 0.6875 5500 | 0.6518 9992 |
| 44 | 0.8637 9245 | 0.8328 0775 | 0.8029 5884 | 0.6815 9108 | 0.6454 4546 |
| 45 | 0.8609 2270 | 0.8293 5211 | 0.7989 6402 | 0.6756 7889 | 0.6390 5492 |
| 46 | 0.8580 6249 | 0.8259 1082 | 0.7949 8907 | 0.6698 1798 | 0.6327 2764 |
| 47 | 0.8552 1179 | 0.8224 8380 | 0.7910 3390 | 0.6640 0792 | 0.6264 6301 |
| 48 | 0.8523 7055 | 0.8190 7100 | 0.7870 9841 | 0.6582 4824 | 0.6202 6041 |
| 49 | 0.8495 3876 | 0.8156 7237 | 0.7831 8250 | 0.6525 3853 | 0.6141 1921 |
| 50 | 0.8467 1637 | 0.8122 8784 | 0.7792 8607 | 0.6468 7835 | 0.6080 3882 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 51 | 0.8439 0336 | 0.8089 1735 | 0.7754 0902 | 0.6412 6726 | 0.6020 1864 |
| 52 | 0.8410 9969 | 0.8055 6084 | 0.7715 5127 | 0.6357 0484 | 0.5960 5806 |
| 53 | 0.8383 0534 | 0.8022 1827 | 0.7677 1270 | 0.6301 9067 | 0.5901 5649 |
| 54 | 0.8355 2027 | 0.7988 8956 | 0.7638 9324 | 0.6247 2433 | 0.5843 1336 |
| 55 | 0.8327 4446 | 0.7955 7467 | 0.7600 9277 | 0.6193 0541 | 0.5785 2808 |
| 56 | 0.8299 7787 | 0.7922 7353 | 0.7563 1122 | 0.6139 3349 | 0.5728 0008 |
| 57 | 0.8272 2047 | 0.7889 8608 | 0.7525 4847 | 0.6086 0817 | 0.5671 2879 |
| 58 | 0.8244 7222 | 0.7857 1228 | 0.7488 0445 | 0.6033 2904 | 0.5615 1365 |
| 59 | 0.8217 3311 | 0.7824 5207 | 0.7450 7906 | 0.5980 9571 | 0.5559 5411 |
| 60 | 0.8190 0310 | 0.7792 0538 | 0.7413 7220 | 0.5929 0776 | 0.5504 4962 |
| 61 | 0.8162 8216 | 0.7759 7216 | 0.7376 8378 | 0.5877 6482 | 0.5449 9962 |
| 62 | 0.8135 7026 | 0.7727 5236 | 0.7340 1371 | 0.5826 6649 | 0.5396 0358 |
| 63 | 0.8108 6737 | 0.7695 4591 | 0.7303 6190 | 0.5776 1238 | 0.5342 6097 |
| 64 | 0.8081 7346 | 0.7663 5278 | 0.7267 2826 | 0.5726 0211 | 0.5289 7126 |
| 65 | 0.8054 8850 | 0.7631 7289 | 0.7231 1269 | 0.5676 3530 | 0.5237 3392 |
| 66 | 0.8028 1246 | 0.7600 0620 | 0.7195 1512 | 0.5627 1158 | 0.5185 4844 |
| 67 | 0.8001 4531 | 0.7568 5265 | 0.7159 3544 | 0.5578 3056 | 0.5134 1429 |
| 68 | 0.7974 8702 | 0.7537 1218 | 0.7123 7357 | 0.5529 9188 | 0.5083 3099 |
| 69 | 0.7948 3756 | 0.7505 8474 | 0.7088 2943 | 0.5481 9517 | 0.5032 9801 |
| 70 | 0.7921 9690 | 0.7474 7028 | 0.7053 0291 | 0.5434 4007 | 0.4983 1486 |
| 71 | 0.7895 6502 | 0.7443 6874 | 0.7017 9394 | 0.5387 2622 | 0.4933 8105 |
| 72 | 0.7869 4188 | 0.7412 8008 | 0.6983 0243 | 0.5340 5325 | 0.4884 9609 |
| 73 | 0.7843 2745 | 0.7382 0423 | 0.6948 2829 | 0.5294 2082 | 0.4836 5949 |
| 74 | 0.7817 2171 | 0.7351 4114 | 0.6913 7143 | 0.5248 2857 | 0.4788 7078 |
| 75 | 0.7791 2463 | 0.7320 9076 | 0.6879 3177 | 0.5202 7615 | 0.4741 2949 |
| 76 | 0.7765 3618 | 0.7290 5304 | 0.6845 0923 | 0.5157 6322 | 0.4694 3514 |
| 77 | 0.7739 5632 | 0.7260 2792 | 0.6811 0371 | 0.5112 8944 | 0.4647 8726 |
| 78 | 0.7713 8504 | 0.7230 1536 | 0.6777 1513 | 0.5068 5447 | 0.4601 8541 |
| 79 | 0.7688 2230 | 0.7200 1529 | 0.6743 4342 | 0.5024 5796 | 0.4556 2912 |
| 80 | 0.7662 6807 | 0.7170 2768 | 0.6709 8847 | 0.4980 9959 | 0.4511 1794 |
| 81 | 0.7637 2233 | 0.7140 5246 | 0.6676 5022 | 0.4937 7902 | 0.4466 5142 |
| 82 | 0.7611 8505 | 0.7110 8959 | 0.6643 2858 | 0.4894 9593 | 0.4422 2913 |
| 83 | 0.7586 5619 | 0.7081 3901 | 0.6610 2346 | 0.4852 4999 | 0.4378 5063 |
| 84 | 0.7561 3574 | 0.7052 0067 | 0.6577 3479 | 0.4810 4089 | 0.4335 1547 |
| 85 | 0.7536 2366 | 0.7022 7453 | 0.6544 6248 | 0.4768 6829 | 0.4292 2324 |
| 86 | 0.7511 1993 | 0.6993 6052 | 0.6512 0644 | 0.4727 3188 | 0.4249 7350 |
| 87 | 0.7486 2451 | 0.6964 5861 | 0.6479 6661 | 0.4686 3136 | 0.4207 6585 |
| 88 | 0.7461 3739 | 0.6935 6874 | 0.6447 4290 | 0.4645 6640 | 0.4165 9985 |
| 89 | 0.7436 5853 | 0.6906 9086 | 0.6415 3522 | 0.4605 3671 | 0.4124 7510 |
| 90 | 0.7411 8790 | 0.6878 2493 | 0.6383 4350 | 0.4565 4197 | 0.4083 9119 |
| 91 | 0.7387 2548 | 0.6849 7088 | 0.6351 6766 | 0.4525 8187 | 0.4043 4771 |
| 92 | 0.7362 7125 | 0.6821 2868 | 0.6320 0763 | 0.4486 5613 | 0.4003 4427 |
| 93 | 0.7338 2516 | 0.6792 9827 | 0.6288 6331 | 0.4447 6444 | 0.3963 8046 |
| 94 | 0.7313 8720 | 0.6764 7960 | 0.6257 3464 | 0.4409 0651 | 0.3924 5590 |
| 95 | 0.7389 5735 | 0.6736 7263 | 0.6226 2153 | 0.4370 8204 | 0.3885 7020 |
| 96 | 0.7265 3556 | 0.6708 7731 | 0.6195 2391 | 0.4332 9075 | 0.3847 2297 |
| 97 | 0.7241 2182 | 0.6680 9359 | 0.6164 4170 | 0.4295 3234 | 0.3809 1383 |
| 98 | 0.7217 1610 | 0.6653 2141 | 0.6133 7483 | 0.4258 0654 | 0.3771 4241 |
| 99 | 0.7193 1837 | 0.6625 6074 | 0.6103 2321 | 0.4221 1305 | 0.3734 0832 |
| 100 | 0.7169 2861 | 0.6598 1153 | 0.6072 8678 | 0.4184 5159 | 0.3697 1121 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v = (1 + i)^{-n}$$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 1 | 0.9888 7515 | 0.9876 5432 | 0.9864 3650 | 0.9852 2167 | 0.9828 0098 |
| 2 | 0.9778 7407 | 0.9754 6106 | 0.9730 5696 | 0.9706 6175 | 0.9658 9777 |
| 3 | 0.9669 9537 | 0.9634 1833 | 0.9598 5890 | 0.9563 1699 | 0.9492 8528 |
| 4 | 0.9562 3770 | 0.9515 2428 | 0.9468 3986 | 0.9421 8423 | 0.9329 5851 |
| 5 | 0.9455 9970 | 0.9397 7706 | 0.9339 9739 | 0.9282 6033 | 0.9169 1254 |
| 6 | 0.9350 8005 | 0.9281 7488 | 0.9213 2912 | 0.9145 4219 | 0.9011 4254 |
| 7 | 0.9246 7743 | 0.9167 1593 | 0.9088 3267 | 0.9010 2679 | 0.8856 4378 |
| 8 | 0.9143 9054 | 0.9053 9845 | 0.8965 0571 | 0.8877 1112 | 0.8704 1157 |
| 9 | 0.9042 1808 | 0.8942 2069 | 0.8843 4596 | 0.8745 9224 | 0.8554 4135 |
| 10 | 0.8941 5881 | 0.8831 8093 | 0.8723 5113 | 0.8616 6723 | 0.8407 2860 |
| 11 | 0.8842 1142 | 0.8722 7746 | 0.8605 1899 | 0.8489 3323 | 0.8262 6889 |
| 12 | 0.8743 7470 | 0.8615 0860 | 0.8488 4734 | 0.8363 8742 | 0.8120 5788 |
| 13 | 0.8646 4742 | 0.8508 7269 | 0.8373 3400 | 0.8240 2702 | 0.7980 9128 |
| 14 | 0.8550 2835 | 0.8403 6800 | 0.8259 7682 | 0.8118 4928 | 0.7843 6490 |
| 15 | 0.8455 1629 | 0.8299 9318 | 0.8147 7368 | 0.7998 5150 | 0.7708 7459 |
| 16 | 0.8361 1005 | 0.8197 4635 | 0.8037 2250 | 0.7880 3104 | 0.7576 1631 |
| 17 | 0.8268 0846 | 0.8096 2602 | 0.7928 2120 | 0.7763 8526 | 0.7445 8605 |
| 18 | 0.8176 1034 | 0.7996 3064 | 0.7820 6777 | 0.7649 1159 | 0.7317 7990 |
| 19 | 0.8085 1455 | 0.7897 5866 | 0.7714 6020 | 0.7536 0747 | 0.7191 9401 |
| 20 | 0.7995 1995 | 0.7800 0855 | 0.7609 9649 | 0.7424 7042 | 0.7068 2458 |
| 21 | 0.7906 2542 | 0.7703 7881 | 0.7506 7472 | 0.7314 9795 | 0.6946 6789 |
| 22 | 0.7818 2983 | 0.7608 6796 | 0.7404 9294 | 0.7206 8763 | 0.6827 2028 |
| 23 | 0.7731 3210 | 0.7514 7453 | 0.7304 4926 | 0.7100 3708 | 0.6709 7817 |
| 24 | 0.7645 3112 | 0.7421 9707 | 0.7205 4181 | 0.6995 4392 | 0.6594 3800 |
| 25 | 0.7560 2583 | 0.7330 3414 | 0.7107 6874 | 0.6892 0583 | 0.6480 9632 |
| 26 | 0.7476 1516 | 0.7239 8434 | 0.7011 2823 | 0.6790 2052 | 0.6369 4970 |
| 27 | 0.7392 9806 | 0.7150 4626 | 0.6916 1847 | 0.6689 8574 | 0.6259 9479 |
| 28 | 0.7310 7348 | 0.7062 1853 | 0.6822 3771 | 0.6590 9925 | 0.6152 2829 |
| 29 | 0.7229 4040 | 0.6974 9978 | 0.6729 8417 | 0.6493 5887 | 0.6046 4697 |
| 30 | 0.7148 9780 | 0.6888 8867 | 0.6638 5615 | 0.6397 6243 | 0.5942 4764 |
| 31 | 0.7069 4467 | 0.6803 8387 | 0.6548 5194 | 0.6303 0781 | 0.5840 2716 |
| 32 | 0.6990 8002 | 0.6719 8407 | 0.6459 6985 | 0.6209 9292 | 0.5739 8247 |
| 33 | 0.6913 0287 | 0.6636 8797 | 0.6372 0824 | 0.6118 1568 | 0.5641 1053 |
| 34 | 0.6836 1223 | 0.6554 9429 | 0.6285 6546 | 0.6027 7407 | 0.5544 0839 |
| 35 | 0.6760 0715 | 0.6474 0177 | 0.6200 3991 | 0.5938 6608 | 0.5448 7311 |
| 36 | 0.6684 8667 | 0.6394 0916 | 0.6116 3000 | 0.5850 8974 | 0.5355 0183 |
| 37 | 0.6610 4986 | 0.6315 1522 | 0.6033 3416 | 0.5764 4309 | 0.5262 9172 |
| 38 | 0.6536 9578 | 0.6237 1873 | 0.5951 5083 | 0.5679 2423 | 0.5172 4002 |
| 39 | 0.6464 2352 | 0.6160 1850 | 0.5870 7850 | 0.5595 3126 | 0.5083 4400 |
| 40 | 0.6392 3216 | 0.6084 1334 | 0.5791 1566 | 0.5512 6232 | 0.4996 0098 |
| 41 | 0.6321 2080 | 0.6009 0206 | 0.5712 6083 | 0.5431 1559 | 0.4910 0834 |
| 42 | 0.6250 8855 | 0.5934 8352 | 0.5635 1253 | 0.5350 8925 | 0.4825 6348 |
| 43 | 0.6181 3454 | 0.5861 5656 | 0.5558 6933 | 0.5271 8153 | 0.4742 6386 |
| 44 | 0.6112 5789 | 0.5789 2006 | 0.5483 2979 | 0.5193 9067 | 0.4661 0699 |
| 45 | 0.6044 5774 | 0.5717 7290 | 0.5408 9252 | 0.5117 1494 | 0.4580 9040 |
| 46 | 0.5977 3324 | 0.5647 1397 | 0.5335 5612 | 0.5041 5265 | 0.4502 1170 |
| 47 | 0.5910 8355 | 0.5577 4219 | 0.5263 1923 | 0.4967 0212 | 0.4424 6850 |
| 48 | 0.5845 0784 | 0.5508 5649 | 0.5191 8050 | 0.4893 6170 | 0.4348 5848 |
| 49 | 0.5780 0528 | 0.5440 5579 | 0.5121 3860 | 0.4821 2975 | 0.4273 7934 |
| 50 | 0.5715 7506 | 0.5373 3905 | 0.5051 9220 | 0.4750 0468 | 0.4200 2883 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v = (1 + i)^{-n}$$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 51 | 0.5652 1637 | 0.5307 0524 | 0.4983 4003 | 0.4679 8491 | 0.4128 0475 |
| 52 | 0.5589 2843 | 0.5241 5332 | 0.4915 8079 | 0.4610 6887 | 0.4057 0492 |
| 53 | 0.5527 1044 | 0.5176 8229 | 0.4849 1323 | 0.4542 5505 | 0.3987 2719 |
| 54 | 0.5465 6162 | 0.5112 9115 | 0.4783 3611 | 0.4475 4192 | 0.3918 6947 |
| 55 | 0.5404 8120 | 0.5049 7892 | 0.4718 4820 | 0.4409 2800 | 0.3851 2970 |
| 56 | 0.5344 6843 | 0.4987 4461 | 0.4654 4829 | 0.4344 1182 | 0.3785 0585 |
| 57 | 0.5285 2256 | 0.4925 8727 | 0.4591 3518 | 0.4279 9194 | 0.3719 9592 |
| 58 | 0.5226 4282 | 0.4865 0594 | 0.4529 0770 | 0.4216 6694 | 0.3655 9796 |
| 59 | 0.5168 2850 | 0.4804 9970 | 0.4467 6468 | 0.4154 3541 | 0.3593 1003 |
| 60 | 0.5110 7887 | 0.4745 6760 | 0.4407 0499 | 0.4092 9597 | 0.3531 3025 |
| 61 | 0.5053 9319 | 0.4687 0874 | 0.4347 2749 | 0.4032 4726 | 0.3470 5676 |
| 62 | 0.4997 7077 | 0.4629 2222 | 0.4288 3106 | 0.3972 8794 | 0.3410 8772 |
| 63 | 0.4942 1090 | 0.4572 0713 | 0.4230 1461 | 0.3914 1669 | 0.3352 2135 |
| 64 | 0.4887 1288 | 0.4515 6259 | 0.4172 7705 | 0.3856 3221 | 0.3294 5587 |
| 65 | 0.4832 7602 | 0.4459 8775 | 0.4116 1731 | 0.3799 3321 | 0.3237 8956 |
| 66 | 0.4778 9965 | 0.4404 8173 | 0.4060 3434 | 0.3743 1843 | 0.3182 2069 |
| 67 | 0.4725 8309 | 0.4350 4368 | 0.4005 2709 | 0.3687 8663 | 0.3127 4761 |
| 68 | 0.4673 2568 | 0.4296 7277 | 0.3950 9454 | 0.3633 3658 | 0.3073 6866 |
| 69 | 0.4621 2675 | 0.4243 6817 | 0.3897 3568 | 0.3579 6708 | 0.3020 8222 |
| 70 | 0.4569 8566 | 0.4191 2905 | 0.3844 4949 | 0.3526 7692 | 0.2968 8670 |
| 71 | 0.4519 0177 | 0.4139 5462 | 0.3792 3501 | 0.3474 6495 | 0.2917 8054 |
| 72 | 0.4468 7443 | 0.4088 4407 | 0.3740 9126 | 0.3423 3000 | 0.2867 6221 |
| 73 | 0.4419 0302 | 0.4037 9661 | 0.3690 1727 | 0.3372 7093 | 0.2818 3018 |
| 74 | 0.4369 8692 | 0.3988 1147 | 0.3640 1210 | 0.3322 8663 | 0.2769 8298 |
| 75 | 0.4321 2551 | 0.3938 8787 | 0.3590 7483 | 0.3273 7599 | 0.2722 1914 |
| 76 | 0.4273 1818 | 0.3890 2506 | 0.3542 0451 | 0.3225 3793 | 0.2675 3724 |
| 77 | 0.4225 6433 | 0.3842 2228 | 0.3494 0026 | 0.3177 7136 | 0.2629 3586 |
| 78 | 0.4178 6337 | 0.3794 7879 | 0.3446 6117 | 0.3130 7523 | 0.2584 1362 |
| 79 | 0.4132 1470 | 0.3747 9387 | 0.3399 8636 | 0.3084 4850 | 0.2539 6916 |
| 80 | 0.4086 1775 | 0.3701 6679 | 0.3353 7495 | 0.3038 9015 | 0.2496 0114 |
| 81 | 0.4040 7194 | 0.3655 9683 | 0.3308 2609 | 0.2993 9916 | 0.2453 0825 |
| 82 | 0.3995 7670 | 0.3610 8329 | 0.3263 3893 | 0.2949 7454 | 0.2410 8919 |
| 83 | 0.3951 3148 | 0.3566 2547 | 0.3219 1263 | 0.2906 1531 | 0.2369 4269 |
| 84 | 0.3907 3570 | 0.3522 2268 | 0.3175 4637 | 0.2863 2050 | 0.2328 6751 |
| 85 | 0.3863 8882 | 0.3478 7426 | 0.3132 3933 | 0.2820 8917 | 0.2288 6242 |
| 86 | 0.3820 9031 | 0.3435 7951 | 0.3089 9071 | 0.2779 2036 | 0.2249 2621 |
| 87 | 0.3778 3961 | 0.3393 3779 | 0.3047 9971 | 0.2738 1316 | 0.2210 5770 |
| 88 | 0.3736 3621 | 0.3351 4843 | 0.3006 6556 | 0.2697 6666 | 0.2172 5572 |
| 89 | 0.3694 7956 | 0.3310 1080 | 0.2965 8748 | 0.2657 7997 | 0.2135 1914 |
| 90 | 0.3653 6916 | 0.3269 2425 | 0.2925 6472 | 0.2618 5218 | 0.2098 4682 |
| 91 | 0.3613 0448 | 0.3228 8814 | 0.2885 9652 | 0.2579 8245 | 0.2062 3766 |
| 92 | 0.3572 8503 | 0.3189 0187 | 0.2846 8214 | 0.2541 6990 | 0.2026 9057 |
| 93 | 0.3533 1029 | 0.3149 6481 | 0.2808 2085 | 0.2504 1369 | 0.1992 0450 |
| 94 | 0.3493 7976 | 0.3110 7636 | 0.2770 1194 | 0.2467 1300 | 0.1957 7837 |
| 95 | 0.3454 9297 | 0.3072 3591 | 0.2732 5468 | 0.2430 6699 | 0.1924 1118 |
| 96 | 0.3416 4941 | 0.3034 4287 | 0.2695 4839 | 0.2394 7487 | 0.1891 0190 |
| 97 | 0.3378 4861 | 0.2996 9666 | 0.2658 9237 | 0.2359 3583 | 0.1858 4953 |
| 98 | 0.3340 9010 | 0.2959 9670 | 0.2622 8594 | 0.2324 4909 | 0.1826 5310 |
| 99 | 0.3303 7340 | 0.2923 4242 | 0.2587 2843 | 0.2290 1389 | 0.1795 1165 |
| 100 | 0.3266 9805 | 0.2887 3326 | 0.2552 1916 | 0.2256 2944 | 0.1764 2422 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

| $n$ | 2% | $2\frac{1}{4}$% | $2\frac{1}{2}$% | $2\frac{3}{4}$% | 3% |
|---|---|---|---|---|---|
| 1 | 0.9803 9216 | 0.9779 9511 | 0.9756 0976 | 0.9732 3601 | 0.9708 7379 |
| 2 | 0.9611 6878 | 0.9564 7444 | 0.9518 1440 | 0.9471 8833 | 0.9425 9591 |
| 3 | 0.9423 2233 | 0.9354 2732 | 0.9285 9941 | 0.9218 3779 | 0.9151 4166 |
| 4 | 0.9238 4543 | 0.9148 4335 | 0.9059 5064 | 0.8971 6573 | 0.8884 8705 |
| 5 | 0.9057 3081 | 0.8947 1232 | 0.8838 5429 | 0.8731 5400 | 0.8626 0878 |
| 6 | 0.8879 7138 | 0.8750 2427 | 0.8622 9687 | 0.8497 8491 | 0.8374 8426 |
| 7 | 0.8705 6018 | 0.8557 6946 | 0.8412 6524 | 0.8270 4128 | 0.8130 9151 |
| 8 | 0.8534 9037 | 0.8369 3835 | 0.8207 4657 | 0.8049 0635 | 0.7894 0923 |
| 9 | 0.8367 5527 | 0.8185 2161 | 0.8007 2836 | 0.7833 6385 | 0.7664 1673 |
| 10 | 0.8203 4830 | 0.8005 1013 | 0.7811 9840 | 0.7623 9791 | 0.7440 9391 |
| 11 | 0.8042 6304 | 0.7828 9499 | 0.7621 4478 | 0.7419 9310 | 0.7224 2128 |
| 12 | 0.7884 9318 | 0.7656 6748 | 0.7435 5589 | 0.7221 3440 | 0.7013 7988 |
| 13 | 0.7730 3253 | 0.7488 1905 | 0.7254 2038 | 0.7028 0720 | 0.6809 5134 |
| 14 | 0.7578 7502 | 0.7323 4137 | 0.7077 2720 | 0.6839 9728 | 0.6611 1781 |
| 15 | 0.7430 1473 | 0.7162 2628 | 0.6904 6556 | 0.6656 9078 | 0.6418 6195 |
| 16 | 0.7284 4581 | 0.7004 6580 | 0.6736 2493 | 0.6478 7424 | 0.6231 6694 |
| 17 | 0.7141 6256 | 0.6850 5212 | 0.6571 9506 | 0.6305 3454 | 0.6050 1645 |
| 18 | 0.7001 5937 | 0.6699 7763 | 0.6411 6591 | 0.6136 5892 | 0.5873 9461 |
| 19 | 0.6864 3076 | 0.6552 3484 | 0.6255 2772 | 0.5972 3496 | 0.5702 8603 |
| 20 | 0.6729 7133 | 0.6408 1647 | 0.6102 7094 | 0.5812 5057 | 0.5536 7575 |
| 21 | 0.6597 7582 | 0.6267 1538 | 0.5953 8629 | 0.5656 9398 | 0.5375 4928 |
| 22 | 0.6468 3904 | 0.6129 2457 | 0.5808 6467 | 0.5505 5375 | 0.5218 9250 |
| 23 | 0.6341 5592 | 0.5994 3724 | 0.5666 9724 | 0.5358 1874 | 0.5066 9175 |
| 24 | 0.6217 2149 | 0.5862 4668 | 0.5528 7535 | 0.5214 7809 | 0.4919 3374 |
| 25 | 0.6095 3087 | 0.5733 4639 | 0.5393 9059 | 0.5075 2126 | 0.4776 0557 |
| 26 | 0.5975 7928 | 0.5607 2997 | 0.5262 3472 | 0.4939 3796 | 0.4636 9473 |
| 27 | 0.5858 6204 | 0.5483 9117 | 0.5133 9973 | 0.4807 1821 | 0.4501 8906 |
| 28 | 0.5743 7455 | 0.5363 2388 | 0.5008 7778 | 0.4678 5227 | 0.4370 7675 |
| 29 | 0.5631 1231 | 0.5245 2213 | 0.4886 6125 | 0.4553 3068 | 0.4243 4636 |
| 30 | 0.5520 7089 | 0.5129 8008 | 0.4767 4269 | 0.4431 4421 | 0.4119 8676 |
| 31 | 0.5412 4597 | 0.5016 9201 | 0.4651 1481 | 0.4312 8301 | 0.3999 8715 |
| 32 | 0.5306 3330 | 0.4906 5233 | 0.4537 7055 | 0.4197 4103 | 0.3883 3703 |
| 33 | 0.5202 2873 | 0.4798 5558 | 0.4427 0298 | 0.4085 0708 | 0.3770 2625 |
| 34 | 0.5100 2817 | 0.4692 9641 | 0.4319 0534 | 0.3975 7380 | 0.3660 4490 |
| 35 | 0.5000 2761 | 0.4589 6960 | 0.4213 7107 | 0.3869 3314 | 0.3553 8340 |
| 36 | 0.4902 2315 | 0.4488 7002 | 0.4110 9372 | 0.3765 7727 | 0.3450 3243 |
| 37 | 0.4806 1093 | 0.4389 9268 | 0.4010 6705 | 0.3664 9856 | 0.3349 8294 |
| 38 | 0.4711 8719 | 0.4293 3270 | 0.3912 8492 | 0.3566 8959 | 0.3252 2615 |
| 39 | 0.4619 4822 | 0.4198 8528 | 0.3817 4139 | 0.3471 4316 | 0.3157 5355 |
| 40 | 0.4528 9042 | 0.4106 4575 | 0.3724 3062 | 0.3378 5222 | 0.3065 5684 |
| 41 | 0.4440 1021 | 0.4016 0954 | 0.3633 4695 | 0.3288 0995 | 0.2976 2800 |
| 42 | 0.4353 0413 | 0.3927 7216 | 0.3544 8483 | 0.3200 0968 | 0.2889 5922 |
| 43 | 0.4267 6875 | 0.3841 2925 | 0.3458 3886 | 0.3114 4495 | 0.2805 4294 |
| 44 | 0.4184 0074 | 0.3756 7653 | 0.3374 0376 | 0.3031 0944 | 0.2723 7178 |
| 45 | 0.4101 9680 | 0.3674 0981 | 0.3291 7440 | 0.2949 9702 | 0.2644 3862 |
| 46 | 0.4021 5373 | 0.3593 2500 | 0.3211 4576 | 0.2871 0172 | 0.2567 3653 |
| 47 | 0.3942 6836 | 0.3514 1809 | 0.3133 1294 | 0.2794 1773 | 0.2492 5876 |
| 48 | 0.3865 3761 | 0.3436 8518 | 0.3056 7116 | 0.2719 3940 | 0.2419 9880 |
| 49 | 0.3789 5844 | 0.3361 2242 | 0.2982 1576 | 0.2646 6122 | 0.2349 5029 |
| 50 | 0.3715 2788 | 0.3287 2608 | 0.2909 4221 | 0.2575 7783 | 0.2281 0708 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

| $n$ | 2% | $2\frac{1}{4}\%$ | $2\frac{1}{2}\%$ | $2\frac{3}{4}\%$ | 3% |
|---|---|---|---|---|---|
| 51 | 0.3642 4302 | 0.3214 9250 | 0.2838 4606 | 0.2506 8402 | 0.2214 6318 |
| 52 | 0.3571 0100 | 0.3144 1810 | 0.2769 2298 | 0.2439 7471 | 0.2150 1280 |
| 53 | 0.3500 9902 | 0.3074 9936 | 0.2701 6876 | 0.2374 4497 | 0.2087 5029 |
| 54 | 0.3432 3433 | 0.3007 3287 | 0.2635 7928 | 0.2310 9000 | 0.2026 7019 |
| 55 | 0.3365 0425 | 0.2941 1528 | 0.2571 5052 | 0.2249 0511 | 0.1967 6717 |
| 56 | 0.3299 0613 | 0.2876 4330 | 0.2508 7855 | 0.2188 8575 | 0.1910 3609 |
| 57 | 0.3234 3738 | 0.2813 1374 | 0.2447 5956 | 0.2130 2749 | 0.1854 7193 |
| 58 | 0.3170 9547 | 0.2751 2347 | 0.2387 8982 | 0.2073 2603 | 0.1800 6984 |
| 59 | 0.3108 7791 | 0.2690 6940 | 0.2329 6568 | 0.2017 7716 | 0.1748 2508 |
| 60 | 0.3047 8227 | 0.2631 4856 | 0.2272 8359 | 0.1963 7679 | 0.1697 3309 |
| 61 | 0.2988 0614 | 0.2573 5801 | 0.2217 4009 | 0.1911 2097 | 0.1647 8941 |
| 62 | 0.2929 4720 | 0.2516 9487 | 0.2163 3179 | 0.1860 0581 | 0.1599 8972 |
| 63 | 0.2872 0314 | 0.2461 5635 | 0.2110 5541 | 0.1810 2755 | 0.1553 2982 |
| 64 | 0.2815 7170 | 0.2407 3971 | 0.2059 0771 | 0.1761 8253 | 0.1508 0565 |
| 65 | 0.2760 5069 | 0.2354 4226 | 0.2008 8557 | 0.1714 6718 | 0.1464 1325 |
| 66 | 0.2706 3793 | 0.2302 6138 | 0.1959 8593 | 0.1668 7804 | 0.1421 4879 |
| 67 | 0.2653 3130 | 0.2251 9450 | 0.1912 0578 | 0.1624 1172 | 0.1380 0853 |
| 68 | 0.2601 2873 | 0.2202 3912 | 0.1865 4223 | 0.1580 6493 | 0.1339 8887 |
| 69 | 0.2550 2817 | 0.2153 9278 | 0.1819 9241 | 0.1538 3448 | 0.1300 8628 |
| 70 | 0.2500 2761 | 0.2106 5309 | 0.1775 5358 | 0.1497 1726 | 0.1262 9736 |
| 71 | 0.2451 2511 | 0.2060 1769 | 0.1732 2300 | 0.1457 1023 | 0.1226 1880 |
| 72 | 0.2403 1874 | 0.2014 8429 | 0.1689 9805 | 0.1418 1044 | 0.1190 4737 |
| 73 | 0.2356 0661 | 0.1970 5065 | 0.1648 7615 | 0.1380 1503 | 0.1155 7998 |
| 74 | 0.2309 8687 | 0.1927 1458 | 0.1608 5478 | 0.1343 2119 | 0.1122 1357 |
| 75 | 0.2264 5771 | 0.1884 7391 | 0.1569 3149 | 0.1307 2622 | 0.1089 4521 |
| 76 | 0.2220 1737 | 0.1843 2657 | 0.1531 0389 | 0.1272 2747 | 0.1057 7205 |
| 77 | 0.2176 6408 | 0.1802 7048 | 0.1493 6965 | 0.1238 2235 | 0.1026 9131 |
| 78 | 0.2133 9616 | 0.1763 0365 | 0.1457 2649 | 0.1205 0837 | 0.0997 0030 |
| 79 | 0.2092 1192 | 0.1724 2411 | 0.1421 7218 | 0.1172 8309 | 0.0967 9641 |
| 80 | 0.2051 0973 | 0.1686 2993 | 0.1387 0457 | 0.1141 4412 | 0.0939 7710 |
| 81 | 0.2010 8797 | 0.1649 1925 | 0.1353 2153 | 0.1110 8917 | 0.0912 3990 |
| 82 | 0.1971 4507 | 0.1612 9022 | 0.1320 2101 | 0.1081 1598 | 0.0885 8243 |
| 83 | 0.1932 7948 | 0.1577 4105 | 0.1288 0098 | 0.1052 2237 | 0.0860 0236 |
| 84 | 0.1894 8968 | 0.1542 6997 | 0.1256 5949 | 0.1024 0620 | 0.0834 9743 |
| 85 | 0.1857 7420 | 0.1508 7528 | 0.1225 9463 | 0.0996 6540 | 0.0810 6547 |
| 86 | 0.1821 3157 | 0.1475 5528 | 0.1196 0452 | 0.0969 9795 | 0.0787 0434 |
| 87 | 0.1785 6036 | 0.1443 0835 | 0.1166 8733 | 0.0944 0190 | 0.0764 1198 |
| 88 | 0.1750 5918 | 0.1411 3286 | 0.1138 4130 | 0.0918 7533 | 0.0741 8639 |
| 89 | 0.1716 2665 | 0.1380 2724 | 0.1110 6468 | 0.0894 1638 | 0.0720 2562 |
| 90 | 0.1682 6142 | 0.1349 8997 | 0.1083 5579 | 0.0870 2324 | 0.0699 2779 |
| 91 | 0.1649 6217 | 0.1320 1953 | 0.1057 1296 | 0.0846 9415 | 0.0678 9105 |
| 92 | 0.1617 2762 | 0.1291 1445 | 0.1031 3460 | 0.0824 2740 | 0.0659 1364 |
| 93 | 0.1585 5649 | 0.1262 7331 | 0.1006 1912 | 0.0802 2131 | 0.0639 9383 |
| 94 | 0.1554 4754 | 0.1234 9468 | 0.0981 6500 | 0.0780 7427 | 0.0621 2993 |
| 95 | 0.1523 9955 | 0.1207 7719 | 0.0957 7073 | 0.0759 8469 | 0.0603 2032 |
| 96 | 0.1494 1132 | 0.1181 1950 | 0.0934 3486 | 0.0739 5104 | 0.0585 6342 |
| 97 | 0.1464 8169 | 0.1155 2029 | 0.0911 5596 | 0.0719 7181 | 0.0568 5769 |
| 98 | 0.1436 0950 | 0.1129 7828 | 0.0889 3264 | 0.0700 4556 | 0.0552 0164 |
| 99 | 0.1407 9363 | 0.1104 9221 | 0.0867 6355 | 0.0681 7086 | 0.0535 9383 |
| 100 | 0.1380 3297 | 0.1080 6084 | 0.0846 4537 | 0.0663 4634 | 0.0520 3284 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 1 | 0.9661 8357 | 0.9615 3846 | 0.9569 3780 | 0.9523 8095 | 0.9478 6730 |
| 2 | 0.9335 1070 | 0.9245 5621 | 0.9157 2995 | 0.9070 2948 | 0.8984 5242 |
| 3 | 0.9019 4271 | 0.8889 9636 | 0.8762 9660 | 0.8638 3760 | 0.8516 1366 |
| 4 | 0.8714 4223 | 0.8548 0419 | 0.8385 6134 | 0.8227 0247 | 0.8072 1674 |
| 5 | 0.8419 7317 | 0.8219 2711 | 0.8024 5105 | 0.7835 2617 | 0.7651 3435 |
| 6 | 0.8135 0064 | 0.7903 1453 | 0.7678 9574 | 0.7462 1540 | 0.7252 4583 |
| 7 | 0.7859 9096 | 0.7599 1781 | 0.7348 2846 | 0.7106 8133 | 0.6874 3681 |
| 8 | 0.7594 1156 | 0.7306 9021 | 0.7031 8513 | 0.6768 3936 | 0.6515 9887 |
| 9 | 0.7337 3097 | 0.7025 8674 | 0.6729 0443 | 0.6446 0892 | 0.6176 2926 |
| 10 | 0.7089 1881 | 0.6755 6417 | 0.6439 2768 | 0.6139 1325 | 0.5854 3058 |
| 11 | 0.6849 4571 | 0.6495 8093 | 0.6161 9874 | 0.5846 7929 | 0.5549 1050 |
| 12 | 0.6617 8330 | 0.6245 9705 | 0.5896 6386 | 0.5568 3742 | 0.5259 8152 |
| 13 | 0.6394 0415 | 0.6005 7409 | 0.5642 7164 | 0.5303 2135 | 0.4985 6068 |
| 14 | 0.6177 8179 | 0.5774 7508 | 0.5399 7286 | 0.5050 6795 | 0.4725 6937 |
| 15 | 0.5968 9062 | 0.5552 6450 | 0.5167 2044 | 0.4810 1710 | 0.4479 3305 |
| 16 | 0.5767 0591 | 0.5339 0818 | 0.4944 6932 | 0.4581 1152 | 0.4245 8109 |
| 17 | 0.5572 0378 | 0.5133 7325 | 0.4731 7639 | 0.4362 9669 | 0.4024 4653 |
| 18 | 0.5383 6114 | 0.4936 2812 | 0.4528 0037 | 0.4155 2065 | 0.3814 6590 |
| 19 | 0.5201 5569 | 0.4746 4242 | 0.4333 0179 | 0.3957 3396 | 0.3615 7906 |
| 20 | 0.5025 6588 | 0.4563 8695 | 0.4146 4286 | 0.3768 8948 | 0.3427 2896 |
| 21 | 0.4855 7090 | 0.4388 3360 | 0.3967 8743 | 0.3589 4236 | 0.3248 6158 |
| 22 | 0.4691 5063 | 0.4219 5539 | 0.3797 0089 | 0.3418 4987 | 0.3079 2567 |
| 23 | 0.4532 8563 | 0.4057 2633 | 0.3633 5013 | 0.3255 7131 | 0.2918 7267 |
| 24 | 0.4379 5713 | 0.3901 2147 | 0.3477 0347 | 0.3100 6791 | 0.2766 5656 |
| 25 | 0.4231 4699 | 0.3751 1680 | 0.3327 3060 | 0.2953 0277 | 0.2622 3370 |
| 26 | 0.4088 3767 | 0.3606 8923 | 0.3184 0248 | 0.2812 4073 | 0.2485 6275 |
| 27 | 0.3950 1224 | 0.3468 1657 | 0.3046 9137 | 0.2678 4832 | 0.2356 0450 |
| 28 | 0.3816 5434 | 0.3334 7747 | 0.2915 7069 | 0.2550 9364 | 0.2233 2181 |
| 29 | 0.3687 4815 | 0.3206 5141 | 0.2790 1502 | 0.2429 4632 | 0.2116 7944 |
| 30 | 0.3562 7841 | 0.3083 1867 | 0.2670 0002 | 0.2313 7745 | 0.2006 4402 |
| 31 | 0.3442 3035 | 0.2964 6026 | 0.2555 0241 | 0.2203 5947 | 0.1901 8390 |
| 32 | 0.3325 8971 | 0.2850 5794 | 0.2444 9991 | 0.2098 6617 | 0.1802 6910 |
| 33 | 0.3213 4271 | 0.2740 9417 | 0.2339 7121 | 0.1998 7254 | 0.1708 7119 |
| 34 | 0.3104 7605 | 0.2635 5209 | 0.2238 9589 | 0.1903 5480 | 0.1619 6321 |
| 35 | 0.2999 7686 | 0.2534 1547 | 0.2142 5444 | 0.1812 9029 | 0.1535 1963 |
| 36 | 0.2898 3272 | 0.2436 6872 | 0.2050 2817 | 0.1726 5741 | 0.1455 1624 |
| 37 | 0.2800 3161 | 0.2342 9685 | 0.1961 9921 | 0.1644 3563 | 0.1379 3008 |
| 38 | 0.2705 6194 | 0.2252 8543 | 0.1877 5044 | 0.1566 0536 | 0.1307 3941 |
| 39 | 0.2614 1250 | 0.2166 2061 | 0.1796 6549 | 0.1491 4797 | 0.1239 2362 |
| 40 | 0.2525 7247 | 0.2082 8904 | 0.1719 2870 | 0.1420 4568 | 0.1174 6314 |
| 41 | 0.2440 3137 | 0.2002 7793 | 0.1645 2507 | 0.1352 8160 | 0.1113 3947 |
| 42 | 0.2357 7910 | 0.1925 7493 | 0.1574 4026 | 0.1288 3962 | 0.1055 3504 |
| 43 | 0.2278 0590 | 0.1851 6820 | 0.1506 6054 | 0.1227 0440 | 0.1000 3322 |
| 44 | 0.2201 0231 | 0.1780 4635 | 0.1441 7276 | 0.1168 6133 | 0.0948 1822 |
| 45 | 0.2126 5924 | 0.1711 9841 | 0.1379 6437 | 0.1112 9651 | 0.0898 7509 |
| 46 | 0.2054 6787 | 0.1646 1386 | 0.1320 2332 | 0.1059 9668 | 0.0851 8965 |
| 47 | 0.1985 1968 | 0.1582 8256 | 0.1263 3810 | 0.1009 4921 | 0.0807 4849 |
| 48 | 0.1918 0645 | 0.1521 9476 | 0.1208 9771 | 0.0961 4211 | 0.0765 3885 |
| 49 | 0.1853 2024 | 0.1463 4112 | 0.1156 9158 | 0.0915 6391 | 0.0725 4867 |
| 50 | 0.1790 5337 | 0.1407 1262 | 0.1107 0965 | 0.0872 0373 | 0.0687 6652 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 51 | 0.1729 9843 | 0.1353 0059 | 0.1059 4225 | 0.0830 5117 | 0.0651 8153 |
| 52 | 0.1671 4824 | 0.1300 9672 | 0.1013 8014 | 0.0790 9635 | 0.0617 8344 |
| 53 | 0.1614 9589 | 0.1250 9300 | 0.0970 1449 | 0.0753 2986 | 0.0585 6250 |
| 54 | 0.1560 3467 | 0.1202 8173 | 0.0928 3683 | 0.0717 4272 | 0.0555 0948 |
| 55 | 0.1507 5814 | 0.1156 5551 | 0.0888 3907 | 0.0683 2640 | 0.0526 1562 |
| 56 | 0.1456 6004 | 0.1112 0722 | 0.0850 1347 | 0.0650 7276 | 0.0498 7263 |
| 57 | 0.1407 3433 | 0.1069 3002 | 0.0813 5260 | 0.0619 7406 | 0.0472 7263 |
| 58 | 0.1359 7520 | 0.1028 1733 | 0.0778 4938 | 0.0590 2291 | 0.0448 0818 |
| 59 | 0.1313 7701 | 0.0988 6282 | 0.0744 9701 | 0.0562 1230 | 0.0424 7221 |
| 60 | 0.1269 3431 | 0.0950 6040 | 0.0712 8901 | 0.0535 3552 | 0.0402 5802 |
| 61 | 0.1226 4184 | 0.0914 0423 | 0.0682 1915 | 0.0509 8621 | 0.0381 5926 |
| 62 | 0.1184 9453 | 0.0878 8868 | 0.0652 8148 | 0.0485 5830 | 0.0361 6992 |
| 63 | 0.1144 8747 | 0.0845 0835 | 0.0624 7032 | 0.0462 4600 | 0.0342 8428 |
| 64 | 0.1106 1591 | 0.0812 5803 | 0.0597 8021 | 0.0440 4381 | 0.0324 9695 |
| 65 | 0.1068 7528 | 0.0781 3272 | 0.0572 0594 | 0.0419 4648 | 0.0308 0279 |
| 66 | 0.1032 6114 | 0.0751 2762 | 0.0547 4253 | 0.0399 4903 | 0.0291 9696 |
| 67 | 0.0997 6922 | 0.0722 3809 | 0.0523 8519 | 0.0380 4670 | 0.0276 7485 |
| 68 | 0.0963 9538 | 0.0694 5970 | 0.0501 2937 | 0.0362 3495 | 0.0262 3208 |
| 69 | 0.0931 3563 | 0.0667 8818 | 0.0479 7069 | 0.0345 0948 | 0.0248 6453 |
| 70 | 0.0899 8612 | 0.0642 1940 | 0.0459 0497 | 0.0328 6617 | 0.0235 6828 |
| 71 | 0.0869 4311 | 0.0617 4942 | 0.0439 2820 | 0.0313 0111 | 0.0223 3960 |
| 72 | 0.0840 0300 | 0.0593 7445 | 0.0420 3655 | 0.0298 1058 | 0.0211 7498 |
| 73 | 0.0811 6232 | 0.0570 9081 | 0.0402 2637 | 0.0283 9103 | 0.0200 7107 |
| 74 | 0.0784 1770 | 0.0548 9501 | 0.0384 9413 | 0.0270 3908 | 0.0190 2471 |
| 75 | 0.0757 6590 | 0.0527 8367 | 0.0368 3649 | 0.0257 5150 | 0.0180 3290 |
| 76 | 0.0732 0376 | 0.0507 5353 | 0.0352 5023 | 0.0245 2524 | 0.0170 9279 |
| 77 | 0.0707 2827 | 0.0488 0147 | 0.0337 3228 | 0.0233 5737 | 0.0162 0170 |
| 78 | 0.0683 3650 | 0.0469 2449 | 0.0322 7969 | 0.0222 4512 | 0.0153 5706 |
| 79 | 0.0660 2560 | 0.0451 1970 | 0.0308 8965 | 0.0211 8582 | 0.0145 5646 |
| 80 | 0.0637 9285 | 0.0433 8433 | 0.0295 5948 | 0.0201 7698 | 0.0137 9759 |
| 81 | 0.0616 3561 | 0.0417 1570 | 0.0282 8658 | 0.0192 1617 | 0.0130 7828 |
| 82 | 0.0595 5131 | 0.0401 1125 | 0.0270 6850 | 0.0183 0111 | 0.0123 9648 |
| 83 | 0.0575 3750 | 0.0385 6851 | 0.0259 0287 | 0.0174 2963 | 0.0117 5022 |
| 84 | 0.0555 9178 | 0.0370 8510 | 0.0247 8744 | 0.0165 9965 | 0.0111 3765 |
| 85 | 0.0537 1187 | 0.0356 5875 | 0.0237 2003 | 0.0158 0919 | 0.0105 5701 |
| 86 | 0.0518 9553 | 0.0342 8726 | 0.0226 9860 | 0.0150 5637 | 0.0100 0664 |
| 87 | 0.0501 4060 | 0.0329 6852 | 0.0217 2115 | 0.0143 3940 | 0.0094 8497 |
| 88 | 0.0484 4503 | 0.0317 0050 | 0.0207 8579 | 0.0136 5657 | 0.0089 9049 |
| 89 | 0.0468 0679 | 0.0304 8125 | 0.0198 9070 | 0.0130 0626 | 0.0085 2180 |
| 90 | 0.0452 2395 | 0.0293 0890 | 0.0190 3417 | 0.0123 8691 | 0.0080 7753 |
| 91 | 0.0436 9464 | 0.0281 8163 | 0.0182 1451 | 0.0117 9706 | 0.0076 5643 |
| 92 | 0.0422 1704 | 0.0270 9772 | 0.0174 3016 | 0.0112 3530 | 0.0072 5728 |
| 93 | 0.0407 8941 | 0.0260 5550 | 0.0166 7958 | 0.0107 0028 | 0.0068 7894 |
| 94 | 0.0394 1006 | 0.0250 5337 | 0.0159 6132 | 0.0101 9074 | 0.0065 2032 |
| 95 | 0.0380 7735 | 0.0240 8978 | 0.0152 7399 | 0.0097 0547 | 0.0061 8040 |
| 96 | 0.0367 8971 | 0.0231 6325 | 0.0146 1626 | 0.0092 4331 | 0.0058 5820 |
| 97 | 0.0355 4562 | 0.0222 7235 | 0.0139 8685 | 0.0088 0315 | 0.0055 5279 |
| 98 | 0.0343 4359 | 0.0214 1572 | 0.0133 8454 | 0.0083 8395 | 0.0052 6331 |
| 99 | 0.0331 8221 | 0.0205 9204 | 0.0128 0817 | 0.0079 8471 | 0.0049 8892 |
| 100 | 0.0320 6011 | 0.0198 0004 | 0.0122 5663 | 0.0076 0449 | 0.0047 2883 |

## TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

| $n$ | 6% | $6\frac{1}{2}$% | 7% | $7\frac{1}{2}$% | 8% |
|---|---|---|---|---|---|
| 1 | 0.9433 9623 | 0.9389 6714 | 0.9345 7944 | 0.9302 3256 | 0.9259 2593 |
| 2 | 0.8899 9644 | 0.8816 5928 | 0.8734 3872 | 0.8653 3261 | 0.8573 3882 |
| 3 | 0.8396 1928 | 0.8278 4909 | 0.8162 9788 | 0.8049 6057 | 0.7938 3224 |
| 4 | 0.7920 9366 | 0.7773 2309 | 0.7628 9521 | 0.7488 0053 | 0.7350 2985 |
| 5 | 0.7472 5817 | 0.7298 8084 | 0.7129 8618 | 0.6965 5863 | 0.6805 8320 |
| 6 | 0.7049 6054 | 0.6853 3412 | 0.6663 4222 | 0.6479 6152 | 0.6301 6963 |
| 7 | 0.6650 5711 | 0.6435 0621 | 0.6227 4974 | 0.6027 5490 | 0.5834 9040 |
| 8 | 0.6274 1237 | 0.6042 3119 | 0.5820 0910 | 0.5607 0223 | 0.5402 6888 |
| 9 | 0.5918 9846 | 0.5673 5323 | 0.5439 3374 | 0.5215 8347 | 0.5002 4897 |
| 10 | 0.5583 9478 | 0.5327 2604 | 0.5083 4929 | 0.4851 9393 | 0.4631 9349 |
| 11 | 0.5267 8753 | 0.5002 1224 | 0.4750 9280 | 0.4513 4319 | 0.4288 8286 |
| 12 | 0.4969 6936 | 0.4696 8285 | 0.4440 1196 | 0.4198 5413 | 0.3971 1376 |
| 13 | 0.4688 3902 | 0.4410 1676 | 0.4149 6445 | 0.3905 6198 | 0.3676 9792 |
| 14 | 0.4423 0096 | 0.4141 0025 | 0.3878 1724 | 0.3633 1347 | 0.3404 6104 |
| 15 | 0.4172 6506 | 0.3888 2652 | 0.3624 4602 | 0.3379 6602 | 0.3152 4170 |
| 16 | 0.3936 4628 | 0.3650 9533 | 0.3387 3460 | 0.3143 8699 | 0.2918 9047 |
| 17 | 0.3713 6442 | 0.3428 1251 | 0.3165 7439 | 0.2924 5302 | 0.2702 6895 |
| 18 | 0.3503 4379 | 0.3218 8969 | 0.2958 6392 | 0.2720 4932 | 0.2502 4903 |
| 19 | 0.3305 1301 | 0.3022 4384 | 0.2765 0832 | 0.2530 6913 | 0.2317 1206 |
| 20 | 0.3118 0473 | 0.2837 9703 | 0.2584 1900 | 0.2354 1315 | 0.2145 4821 |
| 21 | 0.2941 5540 | 0.2664 7608 | 0.2415 1309 | 0.2189 8897 | 0.1986 5575 |
| 22 | 0.2775 0510 | 0.2502 1228 | 0.2257 1317 | 0.2037 1067 | 0.1839 4051 |
| 23 | 0.2617 9726 | 0.2349 4111 | 0.2109 4688 | 0.1894 9830 | 0.1703 1528 |
| 24 | 0.2469 7855 | 0.2206 0198 | 0.1971 4662 | 0.1762 7749 | 0.1576 9934 |
| 25 | 0.2329 9863 | 0.2071 3801 | 0.1842 4918 | 0.1639 7906 | 0.1460 1790 |
| 26 | 0.2198 1003 | 0.1944 9579 | 0.1721 9549 | 0.1525 3866 | 0.1352 0176 |
| 27 | 0.2073 6795 | 0.1826 2515 | 0.1609 3037 | 0.1418 9643 | 0.1251 8682 |
| 28 | 0.1956 3014 | 0.1714 7902 | 0.1504 0221 | 0.1319 9668 | 0.1159 1372 |
| 29 | 0.1845 5674 | 0.1610 1316 | 0.1405 6282 | 0.1227 8761 | 0.1073 2752 |
| 30 | 0.1741 1013 | 0.1511 8607 | 0.1313 6712 | 0.1142 2103 | 0.0993 7733 |
| 31 | 0.1642 5484 | 0.1419 5875 | 0.1227 7301 | 0.1062 5212 | 0.0920 1605 |
| 32 | 0.1549 5740 | 0.1332 9460 | 0.1147 4113 | 0.0988 3918 | 0.0852 0005 |
| 33 | 0.1461 8622 | 0.1251 5925 | 0.1072 3470 | 0.0919 4343 | 0.0788 8893 |
| 34 | 0.1379 1153 | 0.1175 2042 | 0.1002 1934 | 0.0855 2877 | 0.0730 4531 |
| 35 | 0.1301 0522 | 0.1103 4781 | 0.0936 6294 | 0.0795.6164 | 0.0676 3454 |
| 36 | 0.1227 4077 | 0.1036 1297 | 0.0875 3546 | 0.0740 1083 | 0.0626 2458 |
| 37 | 0.1157 9318 | 0.0972 8917 | 0.0818 0884 | 0.0688 4729 | 0.0579 8572 |
| 38 | 0.1092 3885 | 0.0913 5134 | 0.0764 5686 | 0.0640 4399 | 0.0536 9048 |
| 39 | 0.1030 5552 | 0.0857 7590 | 0.0714 5501 | 0.0595 7580 | 0.0497 1341 |
| 40 | 0.0972 2219 | 0.0805 4075 | 0.0667 8038 | 0.0554 1935 | 0.0460 3093 |
| 41 | 0.0917 1905 | 0.0756 2512 | 0.0624 1157 | 0.0515 5288 | 0.0426 2123 |
| 42 | 0.0865 2740 | 0.0710 0950 | 0.0583 2857 | 0.0479 5617 | 0.0394 6411 |
| 43 | 0.0816 2962 | 0.0666 7559 | 0.0545 1268 | 0.0446 1039 | 0.0365 4084 |
| 44 | 0.0770 0908 | 0.0626 0619 | 0.0509 4643 | 0.0414 9804 | 0.0338 3411 |
| 45 | 0.0726 5007 | 0.0587 8515 | 0.0476 1349 | 0.0386 0283 | 0.0313 2788 |
| 46 | 0.0685 3781 | 0.0551 9733 | 0.0444 9859 | 0.0359 0961 | 0.0290 0730 |
| 47 | 0.0646 5831 | 0.0518 2848 | 0.0415 8747 | 0.0334 0428 | 0.0268 5861 |
| 48 | 0.0609 9840 | 0.0486 6524 | 0.0388 6679 | 0.0310 7375 | 0.0248 6908 |
| 49 | 0.0575 4566 | 0.0456 9506 | 0.0363 2410 | 0.0289 0582 | 0.0230 2693 |
| 50 | 0.0542 8836 | 0.0429 0616 | 0.0339 4776 | 0.0268 8913 | 0.0213 2123 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 1 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 |
| 2 | 2.0033 3333 | 2.0041 6667 | 2.0050 0000 | 2.0087 5000 | 2.0100 0000 |
| 3 | 3.0100 1111 | 3.0125 1736 | 3.0150 2500 | 3.0263 2656 | 3.0301 0000 |
| 4 | 4.0200 4448 | 4.0250 6952 | 4.0301 0013 | 4.0528 0692 | 4.0604 0100 |
| 5 | 5.0334 4463 | 5.0418 4064 | 5.0502 5063 | 5.0882 6898 | 5.1010 0501 |
| 6 | 6.0502 2278 | 6.0628 4831 | 6.0755 0188 | 6.1327 9133 | 6.1520 1506 |
| 7 | 7.0703 9019 | 7.0881 1018 | 7.1058 7939 | 7.1864 5326 | 7.2135 3521 |
| 8 | 8.0939 5816 | 8.1176 4397 | 8.1414 0879 | 8.2493 3472 | 8.2856 7056 |
| 9 | 9.1209 3802 | 9.1514 6749 | 9.1821 1583 | 9.3215 1640 | 9.3685 2727 |
| 10 | 10.1513 4114 | 10.1895 9860 | 10.2280 2641 | 10.4030 7967 | 10.4622 1254 |
| 11 | 11.1851 7895 | 11.2320 5526 | 11.2791 6654 | 11.4941 0662 | 11.5668 3467 |
| 12 | 12.2224 6288 | 12.2788 5549 | 12.3355 6237 | 12.5946 8005 | 12.6825 0301 |
| 13 | 13.2632 0442 | 13.3300 1739 | 13.3972 4018 | 13.7048 8350 | 13.8093 2804 |
| 14 | 14.3074 1510 | 14.3855 5913 | 14.4642 2639 | 14.8248 0123 | 14.9474 2132 |
| 15 | 15.3551 0648 | 15.4454 9896 | 15.5365 4752 | 15.9545 1824 | 16.0968 9554 |
| 16 | 16.4062 9017 | 16.5098 5520 | 16.6142 3026 | 17.0941 2028 | 17.2578 6449 |
| 17 | 17.4609 7781 | 17.5786 4627 | 17.6973 0141 | 18.2436 9383 | 18.4304 4314 |
| 18 | 18.5191 8107 | 18.6518 9063 | 18.7857 8791 | 19.4033 2615 | 19.6147 4757 |
| 19 | 19.5809 1167 | 19.7296 0684 | 19.8797 1685 | 20.5731 0526 | 20.8108 9504 |
| 20 | 20.6461 8137 | 20.8118 1353 | 20.9791 1544 | 21.7531 1993 | 22.0190 0399 |
| 21 | 21.7150 0198 | 21.8985 2942 | 22.0840 1101 | 22.9434 5973 | 23.2391 9403 |
| 22 | 22.7873 8532 | 22.9897 7330 | 23.1944 3107 | 24.1442 1500 | 24.4715 8598 |
| 23 | 23.8633 4327 | 24.0855 6402 | 24.3104 0322 | 25.3554 7688 | 25.7163 0183 |
| 24 | 24.9428 8775 | 25.1859 2054 | 25.4319 5524 | 26.5773 3730 | 26.9734 6485 |
| 25 | 26.0260 3071 | 26.2908 6187 | 26.5591 1502 | 27.8098 8900 | 28.2431 9950 |
| 26 | 27.1127 8414 | 27.4004 0713 | 27.6919 1059 | 29.0532 2553 | 29.5256 3150 |
| 27 | 28.2031 6009 | 28.5145 7549 | 28.8303 7015 | 30.3074 4126 | 30.8208 8781 |
| 28 | 29.2971 7062 | 29.6333 8622 | 29.9745 2200 | 31.5726 3137 | 32.1290 9669 |
| 29 | 30.3948 2786 | 30.7568 5867 | 31.1243 9461 | 32.8488 9189 | 33.4503 8766 |
| 30 | 31.4961 4395 | 31.8850 1224 | 32.2800 1658 | 34.1363 1970 | 34.7848 9153 |
| 31 | 32.6011 3110 | 33.0178 6646 | 33.4414 1666 | 35.4350 1249 | 36.1327 4045 |
| 32 | 33.7098 0154 | 34.1554 4090 | 34.6086 2375 | 36.7450 6885 | 37.4940 6785 |
| 33 | 34.8221 6754 | 35.2977 5524 | 35.7816 6686 | 38.0665 8820 | 38.8690 0853 |
| 34 | 35.9382 4143 | 36.4448 2922 | 36.9605 7520 | 39.3996 7085 | 40.2576 9862 |
| 35 | 37.0580 3557 | 37.5966 8268 | 38.1453 7807 | 40.7444 1797 | 41.6602 7560 |
| 36 | 38.1815 6236 | 38.7533 3552 | 39.3361 0496 | 42.1009 3163 | 43.0768 7836 |
| 37 | 39.3088 3423 | 39.9148 0775 | 40.5327 8549 | 43.4693 1478 | 44.5076 4714 |
| 38 | 40.4398 6368 | 41.0811 1945 | 41.7354 4942 | 44.8496 7128 | 45.9527 2361 |
| 39 | 41.5746 6322 | 42.2522 9078 | 42.9441 2666 | 46.2421 0591 | 47.4122 5085 |
| 40 | 42.7132 4543 | 43.4283 4199 | 44.1588 4730 | 47.6467 2433 | 48.8863 7336 |
| 41 | 43.8556 2292 | 44.6092 9342 | 45.3796 4153 | 49.0636 3317 | 50.3752 3709 |
| 42 | 45.0018 0833 | 45.7951 6548 | 46.6065 3974 | 50.4929 3996 | 51.8789 8946 |
| 43 | 46.1518 1436 | 46.9859 7866 | 47.8395 7244 | 51.9347 5319 | 53.3977 7936 |
| 44 | 47.3056 5374 | 48.1817 5358 | 49.0787 7030 | 53.3891 8228 | 54.9317 5715 |
| 45 | 48.4633 3925 | 49.3825 1088 | 50.3241 6415 | 54.8563 3762 | 56.4810 7472 |
| 46 | 49.6248 8371 | 50.5882 7134 | 51.5757 8497 | 56.3363 3058 | 58.0458 8547 |
| 47 | 50.7902 9999 | 51.7990 5581 | 52.8336 6390 | 57.8292 7347 | 59.6263 4432 |
| 48 | 51.9596 0099 | 53.0148 8521 | 54.0978 3222 | 59.3352 7961 | 61.2226 0777 |
| 49 | 53.1327 9966 | 54.2357 8056 | 55.3683 2138 | 60.8544 6331 | 62.8348 3385 |
| 50 | 54.3099 0899 | 55.4617 6298 | 56.6451 6299 | 62.3869 3986 | 64.4631 8218 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | 1 |
|---|---|---|---|---|---|
| 51 | 55.4909 4202 | 56.6928 5366 | 57.9283 8880 | 63.9328 2559 | 66.1078 1401 |
| 52 | 56.6759 1183 | 57.9290 7388 | 59.2180 3075 | 65.4922 3781 | 67.7688 9215 |
| 53 | 57.8648 3154 | 59.1704 4503 | 60.5141 2090 | 67.0652 9489 | 69.4465 8107 |
| 54 | 59.0577 1431 | 60.4169 8855 | 61.8166 9150 | 68.6521 1622 | 71.1410 4688 |
| 55 | 60.2545 7336 | 61.6687 2600 | 63.1257 7496 | 70.2528 2224 | 72.8524 5735 |
| 56 | 61.4554 2194 | 62.9256 7902 | 64.4414 0384 | 71.8675 3443 | 74.5809 8192 |
| 57 | 62.6602 7334 | 64.1878 6935 | 65.7636 1086 | 73.4963 7536 | 76.3267 9174 |
| 58 | 63.8691 4092 | 65.4553 1881 | 67.0924 2891 | 75.1394 6864 | 78.0900 5966 |
| 59 | 65.0820 3806 | 66.7280 4930 | 68.4278 9105 | 76.7969 3900 | 79.8709 6025 |
| 60 | 66.2989 7818 | 68.0060 8284 | 69.7700 3051 | 78.4689 1221 | 81.6696 6986 |
| 61 | 67.5199 7478 | 69.2894 4152 | 71.1188 8066 | 80.1555 1519 | 83.4863 6655 |
| 62 | 68.7450 4136 | 70.5781 4753 | 72.4744 7507 | 81.8568 7595 | 85.3212 3022 |
| 63 | 69.9741 9150 | 71.8722 2314 | 73.8368 4744 | 83.5731 2362 | 87.1744 4252 |
| 64 | 71.2074 3880 | 73.1716 9074 | 75.2060 3168 | 85.3043 8845 | 89.0461 8695 |
| 65 | 72.4447 9693 | 74.4765 7278 | 76.5820 6184 | 87.0508 0185 | 90.9366 4882 |
| 66 | 73.6862 7959 | 75.7868 9184 | 77.9649 7215 | 88.8124 9636 | 92.8460 1531 |
| 67 | 74.9319 0052 | 77.1026 7055 | 79.3547 9701 | 90.5896 0571 | 94.7744 7546 |
| 68 | 76.1816 7352 | 78.4239 3168 | 80.7515 7099 | 92.3822 6476 | 96.7222 2021 |
| 69 | 77.4356 1243 | 79.7506 9806 | 82.1553 2885 | 94.1906 0957 | 98.6894 4242 |
| 70 | 78.6937 3114 | 81.0829 9264 | 83.5661 0549 | 96.0147 7741 | 100.6763 3684 |
| 71 | 79.9560 4358 | 82.4208 3844 | 84.9839 3602 | 97.8549 0671 | 102.6831 0021 |
| 72 | 81.2225 6372 | 83.7642 5860 | 86.4088 5570 | 99.7111 3714 | 104.7099 3121 |
| 73 | 82.4933 0560 | 85.1132 7634 | 87.8408 9998 | 101.5836 0959 | 106.7570 3052 |
| 74 | 83.7682 8329 | 86.4679 1500 | 89.2801 0448 | 103.4724 6618 | 108.8246 0083 |
| 75 | 85.0475 1090 | 87.8281 9797 | 90.7265 0500 | 105.3778 5025 | 110.9128 4684 |
| 76 | 86.3310 0260 | 89.1941 4880 | 92.1801 3752 | 107.2999 0644 | 113.0219 7530 |
| 77 | 87.6187 7261 | 90.5657 9109 | 93.6410 3821 | 109.2387 8063 | 115.1521 9506 |
| 78 | 88.9108 3519 | 91.9431 4855 | 95.1092 4340 | 111.1946 1996 | 117.3037 1701 |
| 79 | 90.2072 0464 | 93.3262 4500 | 96.5847 8962 | 113.1675 7288 | 119.4767 5418 |
| 80 | 91.5078 9532 | 94.7151 0436 | 98.0677 1357 | 115.1577 8914 | 121.6715 2172 |
| 81 | 92.8129 2164 | 96.1097 5062 | 99.5580 5214 | 117.1654 1980 | 123.8882 3694 |
| 82 | 94.1222 9804 | 97.5102 0792 | 101.0558 4240 | 119.1906 1722 | 126.1271 1931 |
| 83 | 95.4360 3904 | 98.9165 0045 | 102.5611 2161 | 121.2335 3512 | 128.3883 9050 |
| 84 | 96.7541 5917 | 100.3286 5254 | 104.0739 2722 | 123.2943 2855 | 130.6722 7440 |
| 85 | 98.0766 7303 | 101.7466 8859 | 105.5942 9685 | 125.3731 5393 | 132.9789 9715 |
| 86 | 99.4035 9527 | 103.1706 3312 | 107.1222 6834 | 127.4701 6903 | 135.3087 8712 |
| 87 | 100.7349 4059 | 104.6005 1076 | 108.6578 7968 | 129.5855 3301 | 137.6618 7499 |
| 88 | 102.0707 2373 | 106.0363 4622 | 110.2011 6908 | 131.7194 0642 | 140.0384 9374 |
| 89 | 103.4109 5947 | 107.4781 6433 | 111.7521 7492 | 133.8719 5123 | 142.4388 7868 |
| 90 | 104.7556 6267 | 108.9259 9002 | 113.3109 3580 | 136.0433 3080 | 144.8632 6746 |
| 91 | 106.1048 4821 | 110.3798 4831 | 114.8774 9048 | 138.2337 0994 | 147.3119 0014 |
| 92 | 107.4585 3104 | 111.8397 6434 | 116.4518 7793 | 140.4432 5491 | 149.7850 1914 |
| 93 | 108.8167 2614 | 113.3057 6336 | 118.0341 3732 | 142.6721 3339 | 152.2828 6933 |
| 94 | 110.1794 4856 | 114.7778 7071 | 119.6243 0800 | 144.9205 1455 | 154.8056 9803 |
| 95 | 111.5467 1339 | 116.2561 1184 | 121.2224 2954 | 147.1885 6906 | 157.3537 5501 |
| 96 | 112.9185 3577 | 117.7405 1230 | 122.8285 4169 | 149.4764 6903 | 159.9272 9256 |
| 97 | 114.2949 3089 | 119.2310 9777 | 124.4426 8440 | 151.7843 8813 | 162.5265 6548 |
| 98 | 115.6759 1399 | 120.7278 9401 | 126.0648 9782 | 154.1125 0153 | 165.1518 3114 |
| 99 | 117.0615 0037 | 122.2309 2690 | 127.6952 2231 | 156.4609 8592 | 167.8033 4945 |
| 100 | 118.4517 0537 | 123.7402 2243 | 129.3336 9842 | 158.8300 1955 | 170.4813 8294 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 1 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 |
| 2 | 2.0112 5000 | 2.0125 0000 | 2.0137 5000 | 2.0150 0000 | 2.0175 0000 |
| 3 | 3.0338 7656 | 3.0376 5625 | 3.0414 3906 | 3.0452 2500 | 3.0528 0625 |
| 4 | 4.0680 0767 | 4.0756 2695 | 4.0832 5885 | 4.0909 0338 | 4.1062 3036 |
| 5 | 5.1137 7276 | 5.1265 7229 | 5.1394 0366 | 5.1522 6693 | 5.1780 8938 |
| 6 | 6.1713 0270 | 6.1906 5444 | 6.2100 7046 | 6.2295 5093 | 6.2687 0596 |
| 7 | 7.2407 2986 | 7.2680 3762 | 7.2954 5893 | 7.3229 9419 | 7.3784 0831 |
| 8 | 8.3221 8807 | 8.3588 8809 | 8.3957 7149 | 8.4328 3911 | 8.5075 3045 |
| 9 | 9.4158 1269 | 9.4633 7420 | 9.5112 1335 | 9.5593 3169 | 9.6564 1224 |
| 10 | 10.5217 4058 | 10.5816 6637 | 10.6419 9253 | 10.7027 2167 | 10.8253 9945 |
| 11 | 11.6401 1016 | 11.7139 3720 | 11.7883 1993 | 11.8632 6249 | 12.0148 4394 |
| 12 | 12.7710 6140 | 12.8603 6142 | 12.9504 0933 | 13.0412 1143 | 13.2251 0371 |
| 13 | 13.9147 3584 | 14.0211 1594 | 14.1284 7745 | 14.2368 2960 | 14.4565 4303 |
| 14 | 15.0712 7662 | 15.1963 7988 | 15.3227 4402 | 15.4503 8205 | 15.7095 3253 |
| 15 | 16.2408 2848 | 16.3863 3463 | 16.5334 3175 | 16.6821 3778 | 16.9844 4935 |
| 16 | 17.4235 3780 | 17.5911 6382 | 17.7607 6644 | 17.9323 6984 | 18.2816 7721 |
| 17 | 18.6195 5260 | 18.8110 5336 | 19.0049 7697 | 19.2013 5539 | 19.6016 0656 |
| 18 | 19.8290 2257 | 20.0461 9153 | 20.2662 9541 | 20.4893 7572 | 20.9446 3468 |
| 19 | 21.0520 9907 | 21.2967 6893 | 21.5449 5697 | 21.7967 1636 | 22.3111 6578 |
| 20 | 22.2889 3519 | 22.5629 7854 | 22.8412 0013 | 23.1236 6710 | 23.7016 1119 |
| 21 | 23.5396 8571 | 23.8450 1577 | 24.1552 6663 | 24.4705 2211 | 25.1163 8938 |
| 22 | 24.8045 0717 | 25.1430 7847 | 25.4874 0155 | 25.8375 7994 | 26.5559 2620 |
| 23 | 26.0835 5788 | 26.4573 6695 | 26.8378 5332 | 27.2251 4364 | 28.0206 5490 |
| 24 | 27.3769 9790 | 27.7880 8403 | 28.2068 7380 | 28.6335 2080 | 29.5110 1637 |
| 25 | 28.6849 8913 | 29.1354 3508 | 29.5947 1832 | 30.0630 2361 | 31.0274 5915 |
| 26 | 30.0076 9526 | 30.4996 2802 | 31.0016 4569 | 31.5139 6896 | 32.5704 3969 |
| 27 | 31.3452 8183 | 31.8808 7337 | 32.4279 1832 | 32.9866 7850 | 34.1404 2238 |
| 28 | 32.6979 1625 | 33.2793 8429 | 33.8738 0220 | 34.4814 7867 | 35.7378 7977 |
| 29 | 34.0657 6781 | 34.6953 7659 | 35.3395 6698 | 35.9987 0085 | 37.3632 9267 |
| 30 | 35.4490 0769 | 36.1290 6880 | 36.8254 8602 | 37.5386 8137 | 39.0171 5029 |
| 31 | 36.8478 0903 | 37.5806 8216 | 38.3318 3646 | 39.1017 6159 | 40.6999 5042 |
| 32 | 38.2623 4688 | 39.0504 4069 | 39.8588 9921 | 40.6882 8801 | 42.4121 9955 |
| 33 | 39.6927 9829 | 40.5385 7120 | 41.4069 5907 | 42.2986 1233 | 44.1544 1305 |
| 34 | 41.1393 4227 | 42.0453 0334 | 42.9763 0476 | 43.9330 9152 | 45.9271 1527 |
| 35 | 42.6021 5987 | 43.5708 6963 | 44.5672 2895 | 45.5920 8789 | 47.7308 3979 |
| 36 | 44.0814 3417 | 45.1155 0550 | 46.1800 2835 | 47.2759 6021 | 49.5661 2949 |
| 37 | 45.5773 5030 | 46.6794 4932 | 47.8150 0374 | 48.9851 0874 | 51.4335 3675 |
| 38 | 47.0900 9549 | 48.2926 4243 | 49.4724 6004 | 50.7198 8538 | 53.3336 2365 |
| 39 | 48.6198 5906 | 49.8862 2921 | 51.1527 0636 | 52.4806 8366 | 55.2669 6206 |
| 40 | 50.1668 3248 | 51.4895 5708 | 52.8560 5608 | 54.2678 9391 | 57.2341 3390 |
| 41 | 51.7312 0934 | 53.1331 7654 | 54.5828 2685 | 56.0819 1232 | 59.2357 3124 |
| 42 | 53.3131 8545 | 54.7973 4125 | 56.3333 4072 | 57.9231 4100 | 61.2723 5654 |
| 43 | 54.9129 5879 | 56.4823 0801 | 58.1079 2415 | 59.7919 8812 | 63.3446 2278 |
| 44 | 56.5307 2957 | 58.1883 3687 | 59.9069 0811 | 61.6888 6794 | 65.4531 5367 |
| 45 | 58.1667 0028 | 59.9156 9108 | 61.7306 2810 | 63.6142 0096 | 67.5985 8386 |
| 46 | 59.8210 7566 | 61.6646 3721 | 63.5794 2423 | 65.5684 1398 | 69.7815 5908 |
| 47 | 61.4940 6276 | 63.4354 4518 | 65.4536 4131 | 67.5519 4018 | 72.0027 3637 |
| 48 | 63.1858 7097 | 65.2283 8824 | 67.3536 2888 | 69.5652 1929 | 74.2627 8425 |
| 49 | 64.8967 1201 | 67.0437 4310 | 69.2797 4128 | 71.6086 9758 | 76.5623 8298 |
| 50 | 66.6268 0002 | 68.8817 8989 | 71.2323 3772 | 73.6828 2804 | 78.9022 2468 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 51 | 68.3763 5152 | 70.7428 1226 | 73.2117 8237 | 75.7880 7046 | 81.2830 1361 |
| 52 | 70.1455 8548 | 72.6270 9741 | 75.2184 4437 | 77.9248 9152 | 83.7054 6635 |
| 53 | 71.9347 2332 | 74.5349 3613 | 77.2526 9798 | 80.0937 6489 | 86.1703 1201 |
| 54 | 73.7439 8895 | 76.4666 2283 | 79.3149 2258 | 82.2951 7136 | 88.6782 9247 |
| 55 | 75.5736 0883 | 78.4224 5562 | 81.4055 0277 | 84.5295 9893 | 91.2301 6259 |
| 56 | 77.4238 1193 | 80.4027 3631 | 83.5248 2843 | 86.7975 4292 | 93.8266 9043 |
| 57 | 79.2948 2981 | 82.4077 7052 | 85.6732 9482 | 89.0995 0606 | 96.4686 5752 |
| 58 | 81.1868 9665 | 84.4378 6765 | 87.8513 0262 | 91.4359 9865 | 99.1568 5902 |
| 59 | 83.1002 4923 | 86.4933 4099 | 90.0592 5804 | 93.8075 3863 | 101.8921 0405 |
| 60 | 85.0351 2704 | 88.5745 0776 | 92.2975 7283 | 96.2146 5171 | 104.6752 1588 |
| 61 | 86.9917 7222 | 90.6816 8910 | 94.5666 6446 | 98.6578 7149 | 107.5070 3215 |
| 62 | 88.9704 2966 | 92.8152 1022 | 96.8669 5610 | 101.1377 3956 | 110.3884 0522 |
| 63 | 90.9713 4699 | 94.9754 0034 | 99.1988 7674 | 103.6548 0565 | 113.3202 0231 |
| 64 | 91.9947 7464 | 97.1625 9285 | 101.5628 6130 | 106.2096 2774 | 116.3033 0585 |
| 65 | 95.0409 6586 | 99.3771 2526 | 103.9593 5064 | 108.8027 7215 | 119.3386 1370 |
| 66 | 97.1101 7672 | 101.6193 3933 | 106.3887 9171 | 111.4348 1374 | 122.4270 3941 |
| 67 | 99.2026 6621 | 103.8895 8107 | 108.8516 3760 | 114.1063 3594 | 125.5695 1263 |
| 68 | 101.3186 9621 | 106.1882 0083 | 111.3483 4761 | 116.8179 3098 | 128.7669 7910 |
| 69 | 103.4585 3154 | 108.5155 5334 | 113.8793 8739 | 119.5701 9995 | 132.0204 0124 |
| 70 | 105.6224 4002 | 110.8719 9776 | 116.4452 2897 | 122.3637 5295 | 135.3307 5826 |
| 71 | 107.8106 9247 | 113.2578 9773 | 119.0463 5087 | 125.1992 0924 | 138.6990 4653 |
| 72 | 110.0235 6276 | 115.6736 2145 | 121.6832 3819 | 128.0771 9738 | 142.1262 7984 |
| 73 | 112.2613 2784 | 118.1195 4172 | 124.3563 8272 | 130.9983 5534 | 145.6134 8974 |
| 74 | 114.5242 6778 | 120.5960 3599 | 127.0662 8298 | 133.9633 3067 | 149.1617 2581 |
| 75 | 116.8126 6579 | 123.1034 8644 | 129.8134 4437 | 136.9727 8063 | 152.7720 5601 |
| 76 | 119.1268 0828 | 125.6422 8002 | 132.5983 7923 | 140.0273 7234 | 156.4455 6699 |
| 77 | 121.4669 8487 | 128.2128 0352 | 135.4216 0695 | 143.1277 8292 | 160.1833 6441 |
| 78 | 123.8334 8845 | 130.8154 6863 | 138.2836 5404 | 146.2746 9967 | 163.9865 7329 |
| 79 | 126.2266 1520 | 133.4506 6199 | 141.1850 5429 | 149.4688 2016 | 167.8563 3872 |
| 80 | 128.6466 6462 | 136.1187 9526 | 144.1263 4878 | 152.7108 5247 | 171.7938 2424 |
| 81 | 131.0939 3960 | 138.8202 8020 | 147.1080 8608 | 156.0015 1525 | 175.8002 1617 |
| 82 | 133.5687 4642 | 141.5555 3370 | 150.1308 2226 | 159.3415 3798 | 179.8767 1995 |
| 83 | 136.0713 9481 | 144.3249 7787 | 153.1951 2107 | 162.7316 6105 | 184.0245 6255 |
| 84 | 138.6021 9801 | 147.1290 4010 | 156.3015 5398 | 166.1726 3597 | 188.2449 9239 |
| 85 | 141.1614 7273 | 149.9681 5310 | 159.4507 0035 | 169.6652 2551 | 192.5392 7976 |
| 86 | 143.7495 3930 | 152.8427 5501 | 162.6431 4748 | 173.2102 0389 | 196.9087 1716 |
| 87 | 146.3667 2162 | 155.7532 8945 | 165.8794 9076 | 176.8083 5695 | 201.3546 1971 |
| 88 | 149.0133 4724 | 158.7002 0557 | 169.1603 3375 | 180.4604 8230 | 205.8783 2555 |
| 89 | 151.6897 4739 | 161.6839 5814 | 172.4862 8834 | 184.1673 8954 | 210.4811 9625 |
| 90 | 154.3962 5705 | 164.7050 0762 | 175.8579 7481 | 187.9299 0038 | 215.1646 1718 |
| 91 | 157.1332 1494 | 167.7638 2021 | 179.2760 2196 | 191.7488 4889 | 219.9299 9798 |
| 92 | 159.9009 6361 | 170.8608 6796 | 182.7410 6726 | 195.6250 8162 | 224.7787 7295 |
| 93 | 162.6998 4945 | 173.9966 2881 | 186.2537 5694 | 199.5594 5784 | 229.7124 0148 |
| 94 | 165.5302 2276 | 177.1715 8667 | 189.8147 4610 | 203.5528 4971 | 234.7323 6850 |
| 95 | 168.3924 3776 | 180.3862 3151 | 193.4246 9886 | 207.6061 4246 | 239.8401 8495 |
| 96 | 171.2868 5269 | 183.6410 5940 | 197.0842 8847 | 211.7202 3459 | 245.0373 8819 |
| 97 | 174.2138 2978 | 186.9365 7264 | 200.7941 9743 | 215.8960 3811 | 250.3255 4248 |
| 98 | 177.1737 3537 | 190.2732 7980 | 204.5551 1765 | 220.1344 7868 | 255.7062 3947 |
| 99 | 180.1669 3989 | 193.6516 9580 | 208.3677 5051 | 224.4364 9586 | 261.1810 9866 |
| 100 | 183.1938 1796 | 197.0723 4200 | 212.2328 0708 | 228.8030 4330 | 266.7517 6789 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | 2% | $2\frac{1}{4}$% | $2\frac{1}{2}$% | $2\frac{3}{4}$% | 3% |
|---|---|---|---|---|---|
| 1 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 |
| 2 | 2.0200 0000 | 2.0225 0000 | 2.0250 0000 | 2.0275 0000 | 2.0300 0000 |
| 3 | 3.0604 0000 | 3.0680 0625 | 3.0756 2500 | 3.0832 5625 | 3.0909 0000 |
| 4 | 4.1216 0800 | 4.1370 3639 | 4.1525 1563 | 4.1680 4580 | 4.1836 2700 |
| 5 | 5.2040 4016 | 5.2301 1971 | 5.2563 2852 | 5.2826 6706 | 5.3091 3581 |
| 6 | 6.3081 2096 | 6.3477 9740 | 6.3877 3673 | 6.4279 4040 | 6.4684 0988 |
| 7 | 7.4342 8338 | 7.4906 2284 | 7.5474 3015 | 7.6047 0876 | 7.6624 6218 |
| 8 | 8.5829 6905 | 8.6591 6186 | 8.7361 1590 | 8.8138 3825 | 8.8923 3605 |
| 9 | 9.7546 2843 | 9.8539 9300 | 9.9545 1880 | 10.0562 1880 | 10.1591 0613 |
| 10 | 10.9497 2100 | 11.0757 0784 | 11.2033 8177 | 11.3327 6482 | 11.4638 7931 |
| 11 | 12.1687 1542 | 12.3249 1127 | 12.4834 6631 | 12.6444 1585 | 12.8077 9569 |
| 12 | 13.4120 8973 | 13.6022 2177 | 13.7955 5297 | 13.9921 3729 | 14.1920 2956 |
| 13 | 14.6803 3152 | 14.9082 7176 | 15.1404 4179 | 15.3769 2107 | 15.6177 9045 |
| 14 | 15.9739 3815 | 16.2437 0788 | 16.5189 5284 | 16.7997 8639 | 17.0863 2416 |
| 15 | 17.2934 1692 | 17.6091 9130 | 17.9319 2666 | 18.2617 8052 | 18.5989 1389 |
| 16 | 18.6392 8525 | 19.0053 9811 | 19.3802 2483 | 19.7639 7948 | 20.1568 8130 |
| 17 | 20.0120 7096 | 20.4330 1957 | 20.8647 3045 | 21.3074 8892 | 21.7615 8774 |
| 18 | 21.4123 1238 | 21.8927 6251 | 22.3863 4871 | 22.8934 4487 | 23.4144 3537 |
| 19 | 22.8405 5863 | 23.3853 4966 | 23.9460 0743 | 24.5230 1460 | 25.1168 6844 |
| 20 | 24.2973 6980 | 24.9115 2003 | 25.5446 5761 | 26.1973 9750 | 26.8703 7449 |
| 21 | 25.7833 1719 | 26.4720 2923 | 27.1832 7405 | 27.9178 2593 | 28.6764 8572 |
| 22 | 27.2989 8354 | 28.0676 4989 | 28.8628 5590 | 29.6855 6615 | 30.5367 8030 |
| 23 | 28.8449 6321 | 29.6991 7201 | 30.5844 2730 | 31.5019 1921 | 32.4528 8370 |
| 24 | 30.4218 6247 | 31.3674 0338 | 32.3490 3798 | 33.3682 2199 | 34.4264 7022 |
| 25 | 32.0302 9972 | 33.0731 6996 | 34.1577 6393 | 35.2858 4810 | 36.4592 6432 |
| 26 | 33.6709 0572 | 34.8173 1628 | 36.0117 0803 | 37.2562 0892 | 38.5530 4225 |
| 27 | 35.3443 2383 | 36.6007 0590 | 37.9120 0073 | 39.2807 5467 | 40.7096 3352 |
| 28 | 37.0512 1031 | 38.4242 2178 | 39.8598 0075 | 41.3609 7542 | 42.9309 2252 |
| 29 | 38.7922 3451 | 40.2887 6677 | 41.8562 9577 | 43.4984 0224 | 45.2188 5020 |
| 30 | 40.5680 7921 | 42.1952 6402 | 43.9027 0316 | 45.6946 0830 | 47.5754 1571 |
| 31 | 42.3794 4079 | 44.1446 5746 | 46.0002 7074 | 47.9512 1003 | 50.0026 7818 |
| 32 | 44.2270 2961 | 46.1379 1226 | 48.1502 7751 | 50.2698 6831 | 52.5027 5852 |
| 33 | 46.1115 7020 | 48.1760 1528 | 50.3540 3445 | 52.6522 8969 | 55.0778 4128 |
| 34 | 48.0338 0160 | 50.2599 7563 | 52.6128 8531 | 55.1002 2765 | 57.7301 7652 |
| 35 | 49.9944 7763 | 52.3908 2508 | 54.9282 0744 | 57.6154 8391 | 60.4620 8181 |
| 36 | 51.9943 6719 | 54.5696 1864 | 57.3014 1263 | 60.1999 0972 | 63.2759 4427 |
| 37 | 54.0342 5453 | 56.7974 3506 | 59.7339 4794 | 62.8554 0724 | 66.1742 2259 |
| 38 | 56.1149 3962 | 59.0753 7735 | 62.2272 9664 | 65.5839 3094 | 69.1594 4927 |
| 39 | 58.2372 3841 | 61.4045 7334 | 64.7829 7906 | 68.3874 8904 | 72.2342 3275 |
| 40 | 60.4019 8318 | 63.7861 7624 | 67.4025 5354 | 71.2681 4499 | 75.4012 5973 |
| 41 | 62.6100 2284 | 66.2213 6521 | 70.0876 1737 | 74.2280 1898 | 78.6632 9753 |
| 42 | 64.8622 2330 | 68.7113 4592 | 72.8398 0781 | 77.2692 8950 | 82.0231 9645 |
| 43 | 67.1594 6777 | 71.2573 5121 | 75.6608 0300 | 80.3941 9496 | 85.4838 9234 |
| 44 | 69.5026 5712 | 73.8606 4161 | 78.5523 2308 | 83.6050 3532 | 89.0484 0911 |
| 45 | 71.8927 1027 | 76.5225 0605 | 81.5161 3116 | 86.9041 7379 | 92.7198 6139 |
| 46 | 74.3305 6447 | 79.2442 6243 | 84.5540 3443 | 90.2940 3857 | 96.5014 5723 |
| 47 | 76.8171 7576 | 82.0272 5834 | 87.6678 8530 | 93.7771 2463 | 100.3965 0095 |
| 48 | 79.3535 1927 | 84.8728 7165 | 90.8595 8243 | 97.3559 9556 | 104.4083 9598 |
| 49 | 81.9405 8966 | 87.7825 1126 | 94.1310 7199 | 101.0332 8544 | 108.5406 4785 |
| 50 | 84.5794 0145 | 90.7576 1776 | 97.4843 4879 | 104.8117 0079 | 112.7968 6729 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | 2% | $2\frac{1}{4}$% | $2\frac{1}{2}$% | $2\frac{3}{4}$% | 3% |
|---|---|---|---|---|---|
| 51 | 87.2709 8948 | 93.7996 6416 | 100.9214 5751 | 108.6940 2256 | 117.1807 7331 |
| 52 | 90.0164 0927 | 96.9101 5661 | 104.4444 9395 | 112.6831 0818 | 121.6961 9651 |
| 53 | 92.8167 3746 | 100.0906 3513 | 108.0556 0629 | 116.7818 9365 | 126.3470 8240 |
| 54 | 95.6730 7221 | 103.3426 7442 | 111.7569 9645 | 120.9933 9573 | 131.1374 9488 |
| 55 | 98.5865 3365 | 106.6678 8460 | 115.5509 2136 | 125.3207 1411 | 136.0716 1972 |
| 56 | 101.5582 6432 | 110.0679 1200 | 119.4396 9440 | 129.7670 3375 | 141.1537 6831 |
| 57 | 104.5894 2961 | 113.5444 4002 | 123.4256 8676 | 134.3356 2718 | 146.3883 8136 |
| 58 | 107.6812 1820 | 117.0991 8992 | 127.5113 2893 | 139.0298 5692 | 151.7800 3280 |
| 59 | 110.8348 4257 | 120.7339 2169 | 131.6991 1215 | 143.8531 7799 | 157.3334 3379 |
| 60 | 114.0515 3942 | 124.4504 3493 | 135.9915 8995 | 148.8091 4038 | 163.0534 3680 |
| 61 | 117.3325 7021 | 128.2505 6972 | 140.3913 7970 | 153.9013 9174 | 168.9450 3991 |
| 62 | 120.6792 2161 | 132.1362 0754 | 144.9011 6419 | 159.1336 8002 | 175.0133 9110 |
| 63 | 124.0928 0604 | 136.1092 7221 | 149.5236 9330 | 164.5098 5622 | 181.2637 9284 |
| 64 | 127.5746 6216 | 140.1717 3083 | 154.2617 8563 | 170.0338 7726 | 187.7017 0662 |
| 65 | 131.1261 5541 | 144.3255 9477 | 159.1183 3027 | 175.7098 0889 | 194.3327 5782 |
| 66 | 134.7486 7852 | 148.5729 2066 | 164.0962 8853 | 181.5418 2863 | 201.1627 4055 |
| 67 | 138.4436 5209 | 152.9158 1137 | 169.1986 9574 | 187.5342 2892 | 208.1976 2277 |
| 68 | 142.2125 2513 | 157.3564 1713 | 174.4286 6314 | 193.6914 2021 | 215.4435 5145 |
| 69 | 146.0567 7563 | 161.8969 3651 | 179.7893 7971 | 200.0179 3427 | 222.9068 5800 |
| 70 | 149.9779 1114 | 166.5396 1758 | 185.2841 1421 | 206.5184 2746 | 230.5940 6374 |
| 71 | 153.9774 6937 | 171.2867 5898 | 190.9162 1706 | 213.1976 8422 | 238.5118 8565 |
| 72 | 158.0570 1875 | 176.1407 1106 | 196.6891 2249 | 220.0606 2054 | 246.6672 4222 |
| 73 | 162.2181 5913 | 181.1038 7705 | 202.6063 5055 | 227.1122 8760 | 255.0672 5949 |
| 74 | 166.4625 2231 | 186.1787 1429 | 208.6715 0931 | 234.3578 7551 | 263.7192 7727 |
| 75 | 170.7917 7276 | 191.3677 3536 | 214.8882 9705 | 241.8027 1709 | 272.6308 5559 |
| 76 | 175.2076 0821 | 196.6735 0941 | 221.2605 0447 | 249.4522 9181 | 281.8097 8126 |
| 77 | 179.7117 6038 | 202.0986 6337 | 227.7920 1709 | 257.3122 2983 | 291.2640 7469 |
| 78 | 184.3059 9558 | 207.6458 8329 | 234.4868 1751 | 265.3883 1615 | 301.0019 9693 |
| 79 | 188.9921 1549 | 213.3179 1567 | 241.3489 8795 | 273.6864 9485 | 311.0320 5684 |
| 80 | 193.7719 5780 | 219.1175 6877 | 248.3827 1265 | 282.2128 7345 | 321.3630 1855 |
| 81 | 198.6473 9696 | 225.0477 1407 | 255.5922 8047 | 290.9737 2747 | 332.0039 0910 |
| 82 | 203.6203 4490 | 231.1112 8763 | 262.9820 8748 | 299.9755 0498 | 342.9640 2638 |
| 83 | 208.6927 5180 | 237.3112 9160 | 270.5566 3966 | 309.2248 3137 | 354.2529 4717 |
| 84 | 213.8666 0683 | 243.6507 9567 | 278.3205 5566 | 318.7285 1423 | 365.8805 3558 |
| 85 | 219.1439 3897 | 250.1329 3857 | 286.2785 6955 | 328.4935 4837 | 377.8569 5165 |
| 86 | 224.5268 1775 | 256.7609 2969 | 294.4355 3379 | 338.5271 2095 | 390.1926 6020 |
| 87 | 230.0173 5411 | 263.5380 5060 | 302.7964 2213 | 348.8366 1678 | 402.8984 4001 |
| 88 | 235.6177 0119 | 270.4676 5674 | 311.3663 3268 | 359.4296 2374 | 415.9853 9321 |
| 89 | 241.3300 5521 | 277.5531 7902 | 320.1504 9100 | 370.3139 3839 | 429.4649 5500 |
| 90 | 247.1566 5632 | 284.7981 2555 | 329.1542 5328 | 381.4975 7170 | 443.3489 0365 |
| 91 | 253.0997 8944 | 292.2060 8337 | 338.3831 0961 | 392.9887 5492 | 457.6493 7076 |
| 92 | 259.1617 8523 | 299.7807 2025 | 347.8426 8735 | 404.7959 4568 | 472.3788 5189 |
| 93 | 265.3450 2094 | 307.5257 8645 | 357.5387 5453 | 416.9278 3418 | 487.5502 1744 |
| 94 | 271.6519 2135 | 315.4451 1665 | 367.4772 2339 | 429.3933 4962 | 503.1767 2397 |
| 95 | 278.0849 5978 | 323.5426 3177 | 377.6641 5398 | 442.2016 6674 | 519.2720 2569 |
| 96 | 284.6466 5898 | 331.8223 4099 | 388.1057 5783 | 455.3622 1257 | 535.8501 8645 |
| 97 | 291.3395 9216 | 340.2883 4366 | 398.8084 0177 | 468.8846 7342 | 552.9256 9205 |
| 98 | 298.1663 8400 | 348.9448 3139 | 409.7786 1182 | 482.7790 0194 | 570.5134 6281 |
| 99 | 305.1297 1168 | 357.7960 9010 | 421.0230 7711 | 497.0554 2449 | 588.6288 6669 |
| 100 | 312.2323 0591 | 366.8465 0213 | 432.5486 5404 | 511.7244 4867 | 607.2877 3270 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 1 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 |
| 2 | 2.0350 0000 | 2.0400 0000 | 2.0450 0000 | 2.0500 0000 | 2.0550 0000 |
| 3 | 3.1062 2500 | 3.1216 0000 | 3.1370 2500 | 3.1525 0000 | 3.1680 2500 |
| 4 | 4.2149 4288 | 4.2464 6400 | 4.2781 9113 | 4.3101 2500 | 4.3422 6638 |
| 5 | 5.3624 6588 | 5.4163 2256 | 5.4707 0973 | 5.5256 3125 | 5.5810 9103 |
| 6 | 6.5501 5218 | 6.6329 7546 | 6.7168 9166 | 6.8019 1281 | 6.8880 5103 |
| 7 | 7.7794 0751 | 7.8992 9448 | 8.0191 5179 | 8.1420 0845 | 8.2668 9384 |
| 8 | 9.0516 8677 | 9.2142 2626 | 9.3800 1362 | 9.5491 0888 | 9.7215 7300 |
| 9 | 10.3684 9581 | 10.5827 9531 | 10.8021 1423 | 11.0265 6432 | 11.2562 5951 |
| 10 | 11.7313 9316 | 12.0061 0712 | 12.2882 0937 | 12.5778 9254 | 12.8753 5379 |
| 11 | 13.1419 9192 | 13.4863 5141 | 13.8411 7879 | 14.2067 8716 | 14.5834 9825 |
| 12 | 14.6019 6164 | 15.0258 0546 | 15.4650 3184 | 15.9171 2652 | 16.3855 9065 |
| 13 | 16.1130 3030 | 16.6268 3768 | 17.1599 1327 | 17.7129 8285 | 18.2867 9814 |
| 14 | 17.6769 8636 | 18.2919 1119 | 18.9321 0937 | 19.5986 3199 | 20.2925 7203 |
| 15 | 19.2956 8088 | 20.0235 8764 | 20.7840 5429 | 21.5785 6359 | 22.4086 6350 |
| 16 | 20.9710 2971 | 21.8245 3114 | 22.7193 3673 | 23.6574 9177 | 24.6411 3999 |
| 17 | 22.7050 1575 | 23.6975 1239 | 24.7417 0689 | 25.8403 6636 | 26.9964 0269 |
| 18 | 24.4996 9130 | 25.6454 1288 | 26.8550 8370 | 28.1323 8467 | 29.4812 0483 |
| 19 | 26.3571 8050 | 27.6712 2940 | 29.0635 6246 | 30.5390 0391 | 32.1026 7110 |
| 20 | 28.2796 8181 | 29.7780 7858 | 31.3714 2277 | 33.0659 5410 | 34.8683 1801 |
| 21 | 30.2694 7068 | 31.9692 0172 | 33.7831 3680 | 35.7192 5181 | 37.7860 7550 |
| 22 | 32.3289 0215 | 34.2479 6979 | 36.3033 7795 | 38.5052 1440 | 40.8643 0965 |
| 23 | 34.4604 1373 | 36.6178 8858 | 38.9370 2996 | 41.4304 7512 | 44.1118 4669 |
| 24 | 36.6665 2821 | 39.0826 0412 | 41.6891 9631 | 44.5019 9887 | 47.5379 9825 |
| 25 | 38.9498 5663 | 41.6459 0829 | 44.5652 1015 | 47.7270 9882 | 51.1525 8816 |
| 26 | 41.3131 0168 | 44.3117 4462 | 47.5706 4460 | 51.1134 5376 | 54.9659 8051 |
| 27 | 43.7590 6024 | 47.0842 1440 | 50.7113 2361 | 54.6691 2645 | 58.9891 0943 |
| 28 | 46.2906 2734 | 49.9675 8298 | 53.9933 3317 | 58.4025 8277 | 63.2335 1045 |
| 29 | 48.9107 9930 | 52.9662 8630 | 57.4230 3316 | 62.3227 1191 | 67.7113 5353 |
| 30 | 51.6226 7728 | 56.0849 3775 | 61.0070 6966 | 66.4388 4750 | 72.4354 7797 |
| 31 | 54.4294 7038 | 59.3283 3526 | 64.7523 8779 | 70.7607 8988 | 77.4194 2926 |
| 32 | 57.3345 0247 | 62.7014 6867 | 68.6662 4524 | 75.2988 2937 | 82.6774 9787 |
| 33 | 60.3412 1005 | 66.2095 2742 | 72.7562 2628 | 80.0637 7084 | 88.2247 6025 |
| 34 | 63.4531 5240 | 69.8579 0851 | 77.0302 5646 | 85.0669 5938 | 94.0771 2207 |
| 35 | 66.6740 1274 | 73.6522 2486 | 81.4966 1800 | 90.3203 0735 | 100.2513 6378 |
| 36 | 70.0076 0318 | 77.5983 1385 | 86.1639 6581 | 95.8363 2272 | 106.7651 8879 |
| 37 | 73.4578 6930 | 81.7022 4640 | 91.0413 4427 | 101.6281 3886 | 113.6372 7417 |
| 38 | 77.0288 9472 | 85.9703 3626 | 96.1382 0476 | 107.7095 4580 | 120.8873 2425 |
| 39 | 80.7249 0604 | 90.4091 4971 | 101.4644 2398 | 114.0950 2309 | 128.5361 2708 |
| 40 | 84.5502 7775 | 95.0255 1570 | 107.0303 2306 | 120.7997 7424 | 136.6056 1407 |
| 41 | 88.5095 3747 | 99.8265 3633 | 112.8466 8760 | 127.8397 6295 | 145.1189 2285 |
| 42 | 92.6073 7128 | 104.8195 9778 | 118.9247 8854 | 135.2317 5110 | 154.1004 6360 |
| 43 | 96.8486 2928 | 110.0123 8169 | 125.2764 0402 | 142.9933 3866 | 163.5759 8910 |
| 44 | 101.2383 3130 | 115.4128 7696 | 131.9138 4220 | 151.1430 0559 | 173.5726 6850 |
| 45 | 105.7816 7290 | 121.0293 9204 | 138.8499 6510 | 159.7001 5587 | 184.1191 6527 |
| 46 | 110.4840 3145 | 126.8705 6772 | 146.0982 1353 | 168.6851 6366 | 195.2457 1936 |
| 47 | 115.3509 7255 | 132.9453 9043 | 153.6726 3314 | 178.1194 2185 | 206.9842 3392 |
| 48 | 120.3882 5659 | 139.2632 0604 | 161.5879 0163 | 188.0253 9294 | 219.3683 6679 |
| 49 | 125.6018 4557 | 145.8337 3429 | 169.8593 5720 | 198.4266 6259 | 232.4336 2696 |
| 50 | 130.9979 1016 | 152.6670 8366 | 178.5030 2828 | 209.3479 9572 | 246.2174 7645 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 51 | 136.5828 3702 | 159.7737 6700 | 187.5356 6455 | 220.8153 9550 | 260.7594 3765 |
| 52 | 142.3632 3631 | 167.1647 1768 | 196.9747 6946 | 232.8561 6528 | 276.1012 0672 |
| 53 | 148.3459 4958 | 174.8513 0639 | 206.8386 3408 | 245.4989 7354 | 292.2867 7309 |
| 54 | 154.5380 5782 | 182.8453 5865 | 217.1463 7262 | 258.7739 2222 | 309.3625 4561 |
| 55 | 160.9468 8984 | 191.1591 7299 | 227.9179 5938 | 272.7126 1833 | 327.3774 8562 |
| 56 | 167.5800 3099 | 199.8055 3991 | 239.1742 6756 | 287.3482 4924 | 346.3832 4733 |
| 57 | 174.4453 3207 | 208.7977 6151 | 250.9371 0960 | 302.7156 6171 | 366.4343 2593 |
| 58 | 181.5509 1869 | 218.1496 7197 | 263.2292 7953 | 318.8514 4479 | 387.5882 1386 |
| 59 | 188.9052 0085 | 227.8756 5885 | 276.0745 9711 | 335.7940 1703 | 409.8055 6562 |
| 60 | 196.5168 8288 | 237.9906 8520 | 289.4979 5398 | 353.5837 1788 | 433.4503 7173 |
| 61 | 204.3949 7378 | 248.5103 1261 | 303.5253 6190 | 372.2629 0378 | 458.2901 4127 |
| 62 | 212.5487 9786 | 259.4507 2511 | 318.1840 0319 | 391.8760 4897 | 484.4960 9999 |
| 63 | 220.9880 0579 | 270.8287 5412 | 333.5022 8333 | 412.4698 5141 | 512.1433 8549 |
| 64 | 229.7225 8599 | 282.6619 0428 | 349.5098 8608 | 434.0933 4398 | 541.3112 7170 |
| 65 | 238.7628 7650 | 294.9683 8045 | 366.2378 3096 | 456.7980 1118 | 572.0833 9164 |
| 66 | 248.1195 7718 | 307.7671 1567 | 383.7185 3335 | 480.6379 1174 | 604.5479 7818 |
| 67 | 257.8037 6238 | 321.0778 0030 | 401.9858 6735 | 505.6698 0733 | 638.7981 1608 |
| 68 | 267.8268 9406 | 334.9209 1231 | 421.0752 3138 | 531.9532 9770 | 674.9320 1341 |
| 69 | 278.2008 3535 | 349.3177 4880 | 441.0236 1679 | 559.5509 6258 | 713.0532 7415 |
| 70 | 288.9378 6459 | 364.2904 5876 | 461.8696 7955 | 588.5285 1071 | 753.2712 0423 |
| 71 | 300.0506 8985 | 379.8620 7711 | 483.6538 1513 | 618.9549 3625 | 795.7011 2046 |
| 72 | 311.5524 6400 | 396.0565 6019 | 506.4182 3681 | 650.9026 8306 | 840.4646 8209 |
| 73 | 323.4568 0024 | 412.8988 2260 | 530.2070 5747 | 684.4478 1721 | 887.6902 3960 |
| 74 | 335.7777 8824 | 430.4147 7550 | 555.0663 7505 | 719.6702 0807 | 937.5132 0278 |
| 75 | 348.5300 1083 | 448.6313 6652 | 581.0443 6193 | 756.6537 1848 | 990.0764 2893 |
| 76 | 361.7285 6121 | 467.5766 2118 | 608.1913 5822 | 795.4864 0440 | 1045.5306 3252 |
| 77 | 375.3890 6085 | 487.2796 8603 | 636.5599 6934 | 836.2607 2462 | 1104.0348 1731 |
| 78 | 389.5276 7798 | 507.7708 7347 | 666.2051 6796 | 879.0737 6085 | 1165.7567 3226 |
| 79 | 404.1611 4671 | 529.0817 0841 | 697.1844 0052 | 924.0274 4889 | 1230.8733 5254 |
| 80 | 419.3067 8685 | 551.2449 7675 | 729.5576 9854 | 971.2288 2134 | 1299.5713 8693 |
| 81 | 434.9825 2439 | 574.2947 7582 | 763.3877 9497 | 1020.7902 6240 | 1372.0478 1321 |
| 82 | 451.2069 1274 | 598.2665 6685 | 798.7402 4575 | 1072.8297 7552 | 1448.5104 4294 |
| 83 | 467.9991 5469 | 623.1972 2952 | 835.6835 5680 | 1127.4712 6430 | 1529.1785 1730 |
| 84 | 485.3791 2510 | 649.1251 1870 | 874.2893 1686 | 1184.8448 2752 | 1614.2833 3575 |
| 85 | 503.3673 9448 | 676.0901 2345 | 914.6323 3612 | 1245.0870 6889 | 1704.0689 1921 |
| 86 | 521.9852 5329 | 704.1337 2839 | 956.7907 9125 | 1308.3414 2234 | 1798.7927 0977 |
| 87 | 541.2547 3715 | 733.2990 7753 | 1000.8463 7685 | 1374.7584 9345 | 1898.7263 0881 |
| 88 | 561.1986 5295 | 763.6310 4063 | 1046.8844 6381 | 1444.4964 1812 | 2004.1562 5579 |
| 89 | 581.8406 0581 | 795.1762 8225 | 1094.9942 6468 | 1517.7212 3903 | 2115.3848 4986 |
| 90 | 603.2050 2701 | 827.9833 3354 | 1145.2690 0659 | 1594.6073 0098 | 2232.7310 1660 |
| 91 | 625.3172 0295 | 862.1026 6688 | 1197.8061 1189 | 1675.3376 6603 | 2356.5312 2252 |
| 92 | 648.2033 0506 | 897.5867 7356 | 1252.7073 8692 | 1760.1045 4933 | 2487.1404 3976 |
| 93 | 671.8904 2073 | 934.4902 4450 | 1310.0792 1933 | 1849.1097 7680 | 2624.9331 6394 |
| 94 | 696.4065 8546 | 972.8698 5428 | 1370.0327 8420 | 1942.5652 6564 | 2770.3044 8796 |
| 95 | 721.7808 1595 | 1012.7846 4845 | 1432.6842 5949 | 2040.6935 2892 | 2923.6712 3480 |
| 96 | 748.0431 4451 | 1054.2960 3439 | 1498.1550 5117 | 2143.7282 0537 | 3085.4731 5271 |
| 97 | 775.2246 5457 | 1097.4678 7577 | 1566.5720 2847 | 2251.9146 1564 | 3256.1741 7611 |
| 98 | 803.3575 1748 | 1142.3665 9080 | 1638.0677 6976 | 2365.5103 4642 | 3436.2637 5580 |
| 99 | 832.4750 3059 | 1189.0612 5443 | 1712.7808 1939 | 2484.7858 6374 | 3626.2582 6237 |
| 100 | 862.6116 5666 | 1237.6237 0461 | 1790.8559 5627 | 2610.0251 5693 | 3826.7024 6680 |

## TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

| $n$ | 6% | $6\frac{1}{2}$% | 7% | $7\frac{1}{2}$% | 8% |
|---|---|---|---|---|---|
| 1 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 | 1.0000 0000 |
| 2 | 2.0600 0000 | 2.0650 0000 | 2.0700 0000 | 2.0750 0000 | 2.0800 0000 |
| 3 | 3.1836 0000 | 3.1992 2500 | 3.2149 0000 | 3.2306 2500 | 3.2464 0000 |
| 4 | 4.3746 1600 | 4.4071 7463 | 4.4399 4300 | 4.4729 2188 | 4.5061 1200 |
| 5 | 5.6370 9296 | 5.6936 4098 | 5.7507 3901 | 5.8083 9102 | 5.8666 0096 |
| 6 | 6.9753 1854 | 7.0637 2764 | 7.1532 9074 | 7.2440 2034 | 7.3359 2904 |
| 7 | 8.3938 3765 | 8.5228 6994 | 8.6540 2109 | 8.7873 2187 | 8.9228 0336 |
| 8 | 9.8974 6791 | 10.0768 5648 | 10.2598 0257 | 10.4463 7101 | 10.6366 2763 |
| 9 | 11.4913 1598 | 11.7318 5215 | 11.9779 8875 | 12.2298 4883 | 12.4875 5784 |
| 10 | 13.1807 9494 | 13.4944 2254 | 13.8164 4796 | 14.1470 8750 | 14.4865 6247 |
| 11 | 14.9716 4264 | 15.3715 6001 | 15.7835 9932 | 16.2081 1906 | 16.6454 8746 |
| 12 | 16.8699 4120 | 17.3707 1141 | 17.8884 5127 | 18.4237 2799 | 18.9771 2646 |
| 13 | 18.8821 3767 | 19.4998 0765 | 20.1406 4286 | 20.8055 0759 | 21.4952 9658 |
| 14 | 21.0150 6593 | 21.7672 9515 | 22.5504 8786 | 23.3659 2066 | 24.2149 2030 |
| 15 | 23.2759 6988 | 24.1821 6933 | 25.1290 2201 | 26.1183 6470 | 27.1521 1393 |
| 16 | 25.6725 2808 | 26.7540 1034 | 27.8880 5355 | 29.0772 4206 | 30.3242 8304 |
| 17 | 28.2128 7976 | 29.4930 2101 | 30.8402 1730 | 32.2580 3521 | 33.7502 2569 |
| 18 | 30.9056 5255 | 32.4100 6738 | 33.9990 3251 | 35.6773 8785 | 37.4502 4374 |
| 19 | 33.7599 9170 | 35.5167 2176 | 37.3789 6479 | 39.3531 9194 | 41.4462 6324 |
| 20 | 36.7855 9120 | 38.8253 0867 | 40.9954 9232 | 43.3046 8134 | 45.7619 6430 |
| 21 | 39.9927 2668 | 42.3489 5373 | 44.8651 7678 | 47.5525 3244 | 50.4229 2144 |
| 22 | 43.3922 9028 | 46.1016 3573 | 49.0057 3916 | 52.1189 7237 | 55.4567 5516 |
| 23 | 46.9958 2769 | 50.0982 4205 | 53.4361 4090 | 57.0278 9530 | 60.8932 9557 |
| 24 | 50.8155 7735 | 54.3546 2778 | 58.1766 7076 | 62.3049 8744 | 66.7647 5922 |
| 25 | 54.8645 1200 | 58.8876 7859 | 63.2490 3772 | 67.9778 6150 | 73.1059 3995 |
| 26 | 59.1563 8272 | 63.7153 7769 | 68.6764 7036 | 74.0762 0112 | 79.9544 1515 |
| 27 | 63.7057 6568 | 68.8568 7725 | 74.4838 2328 | 80.6319 1620 | 87.3507 6836 |
| 28 | 68.5281 1162 | 74.3325 7427 | 80.6976 9091 | 87.6793 0991 | 95.3388 2983 |
| 29 | 73.6397 9832 | 80.1641 9159 | 87.3465 2927 | 95.2552 5816 | 103.9659 3622 |
| 30 | 79.0581 8622 | 86.3748 6405 | 94.4607 8632 | 103.3994 0252 | 113.2832 1111 |
| 31 | 84.8016 7739 | 92.9892 3021 | 102.0730 4137 | 112.1543 5771 | 123.3458 6800 |
| 32 | 90.8897 7803 | 100.0335 3017 | 110.2181 5426 | 121.5659 3454 | 134.2135 3744 |
| 33 | 97.3431 6471 | 107.5357 0963 | 118.9334 2506 | 131.6833 7963 | 145.9506 2044 |
| 34 | 104.1837 5460 | 115.5255 3076 | 128.2587 6481 | 142.5596 3310 | 158.6266 7007 |
| 35 | 111.4347 7987 | 124.0346 9026 | 138.2368 7835 | 154.2516 0558 | 172.3168 0368 |
| 36 | 119.1208 6666 | 133.0969 4513 | 148.9134 5984 | 166.8204 7600 | 187.1021 4797 |
| 37 | 127.2681 1866 | 142.7482 4656 | 160.3374 0202 | 180.3320 1170 | 203.0703 1981 |
| 38 | 135.9042 0578 | 153.0268 8259 | 172.5610 2017 | 194.8569 1258 | 220.3159 4540 |
| 39 | 145.0584 5813 | 163.9736 2995 | 185.6402 9158 | 210.4711 8102 | 238.9412 2103 |
| 40 | 154.7619 6562 | 175.6319 1590 | 199.6351 1199 | 227.2565 1960 | 259.0565 1871 |
| 41 | 165.0476 8356 | 188.0479 9044 | 214.6095 6983 | 245.3007 5857 | 280.7810 4021 |
| 42 | 175.9505 4457 | 201.2711 0981 | 230.6322 3972 | 264.6983 1546 | 304.2435 2342 |
| 43 | 187.5075 7724 | 215.3537 3195 | 247.7764 9650 | 285.5506 8912 | 329.5830 0530 |
| 44 | 199.7580 3188 | 230.3517 2453 | 266.1208 5125 | 307.9669 9080 | 356.9496 4572 |
| 45 | 212.7435 1379 | 246.3245 8662 | 285.7493 1084 | 332.0645 1511 | 386.5056 1738 |
| 46 | 226.5081 2462 | 263.3356 8475 | 306.7517 6260 | 357.9693 5375 | 418.4260 6677 |
| 47 | 241.0986 1210 | 281.4525 0426 | 329.2243 8598 | 385.8170 5528 | 452.9001 5211 |
| 48 | 256.5645 2882 | 300.7469 1704 | 353.2700 9300 | 415.7533 3442 | 490.1321 6428 |
| 49 | 272.9584 0055 | 321.2954 6665 | 378.9989 9951 | 447.9348 3451 | 530.3427 3742 |
| 50 | 290.3359 0458 | 343.1796 7198 | 406.5289 2947 | 482.5299 4709 | 573.7701 5642 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 1 | 0.9966 7774 | 0.9958 5062 | 0.9950 2488 | 0.9913 2590 | 0.9900 9901 |
| 2 | 1.9900 4426 | 1.9875 6908 | 1.9850 9938 | 1.9740 5294 | 1.9703 9506 |
| 3 | 2.9801 1056 | 2.9751 7253 | 2.9702 4814 | 2.9482 5570 | 2.9409 8521 |
| 4 | 3.9668 8760 | 3.9586 7804 | 3.9504 9566 | 3.9140 0813 | 3.9019 6555 |
| 5 | 4.9503 8631 | 4.9381 0261 | 4.9258 6633 | 4.8713 8352 | 4.8534 3124 |
| 6 | 5.9306 1759 | 5.9134 6318 | 5.8963 8441 | 5.8204 5454 | 5.7954 7647 |
| 7 | 6.9075 9228 | 6.8847 7661 | 6.8620 7404 | 6.7612 9323 | 6.7281 9453 |
| 8 | 7.8813 2121 | 7.8520 5969 | 7.8229 5924 | 7.6939 7098 | 7.6516 7775 |
| 9 | 8.8518 1516 | 8.8153 2915 | 8.7790 6392 | 8.6185 5859 | 8.5660 1758 |
| 10 | 9.8190 8487 | 9.7746 0164 | 9.7304 1186 | 9.5351 2624 | 9.4713 0453 |
| 11 | 10.7831 4107 | 10.7298 9374 | 10.6770 2673 | 10.4437 4348 | 10.3676 2825 |
| 12 | 11.7439 9442 | 11.6812 2198 | 11.6189 3207 | 11.3444 7929 | 11.2550 7747 |
| 13 | 12.7016 5557 | 12.6286 0280 | 12.5561 5131 | 12.2374 0202 | 12.1337 4007 |
| 14 | 13.6561 3512 | 13.5720 5257 | 13.4887 0777 | 13.1225 7945 | 13.0037 0304 |
| 15 | 14.6074 4364 | 14.5115 8762 | 14.4166 2465 | 14.0000 7876 | 13.8650 5252 |
| 16 | 15.5555 9167 | 15.4472 2418 | 15.3399 2502 | 14.8699 6656 | 14.7178 7378 |
| 17 | 16.5005 8970 | 16.3789 7843 | 16.2586 3186 | 15.7323 0885 | 15.5622 5127 |
| 18 | 17.4424 4821 | 17.3068 6648 | 17.1727 6802 | 16.5871 7111 | 16.3982 6858 |
| 19 | 18.3811 7762 | 18.2309 0438 | 18.0823 5624 | 17.4346 1820 | 17.2260 0850 |
| 20 | 19.3167 8832 | 19.1511 0809 | 18.9874 1915 | 18.2747 1445 | 18.0455 5297 |
| 21 | 20.2492 9069 | 20.0674 9352 | 19.8879 7925 | 19.1075 2361 | 18.8569 8313 |
| 22 | 21.1786 9504 | 20.9800 7653 | 20.7840 5896 | 19.9331 0891 | 19.6603 7934 |
| 23 | 22.1050 1167 | 21.8888 7289 | 21.6756 8055 | 20.7515 3300 | 20.4558 2113 |
| 24 | 23.0282 5083 | 22.7938 9831 | 22.5628 6622 | 21.5628 5799 | 21.2433 8726 |
| 25 | 23.9484 2275 | 23.6951 6843 | 23.4456 3803 | 22.3671 4547 | 22.0231 5570 |
| 26 | 24.8655 3763 | 24.5926 9884 | 24.3240 1794 | 23.1644 5647 | 22.7952 0366 |
| 27 | 25.7796 0561 | 25.4865 0506 | 25.1980 2780 | 23.8548 5152 | 23.5596 0759 |
| 28 | 26.6906 3682 | 26.3766 0254 | 26.0676 8936 | 24.7383 9060 | 24.3164 4316 |
| 29 | 27.5986 4135 | 27.2630 0668 | 26.9330 2423 | 25.5151 3319 | 25.0657 8530 |
| 30 | 28.5036 2925 | 28.1457 3278 | 27.7940 5397 | 26.2851 3823 | 25.8077 0822 |
| 31 | 29.4056 1055 | 29.0247 9612 | 28.6507 9997 | 27.0484 6417 | 26.5422 8537 |
| 32 | 30.3045 9523 | 29.9002 1189 | 29.5032 8355 | 27.8051 6894 | 27.2695 8947 |
| 33 | 31.2005 9325 | 30.7719 9524 | 30.3515 2592 | 28.5553 0998 | 27.9896 9255 |
| 34 | 32.0936 1454 | 31.6401 6122 | 31.1955 4818 | 29.2989 4422 | 28.7026 6589 |
| 35 | 32.9836 6898 | 32.5047 2486 | 32.0353 7132 | 30.0361 2809 | 29.4085 8009 |
| 36 | 33.8707 6642 | 33.3657 0109 | 32.8710 1624 | 30.7669 1757 | 30.1075 0504 |
| 37 | 34.7549 1670 | 34.2231 0481 | 33.7025 0372 | 31.4913 6810 | 30.7995 0994 |
| 38 | 35.6361 2960 | 35.0769 5084 | 34.5298 5445 | 32.2095 3467 | 31.4846 6330 |
| 39 | 36.5144 1488 | 35.9272 5394 | 35.3530 8900 | 32.9214 7179 | 32.1630 3298 |
| 40 | 37.3897 8228 | 36.7740 2881 | 36.1722 2786 | 33.6272 3350 | 32.8346 8611 |
| 41 | 38.2622 4147 | 37.6172 9009 | 36.9872 9141 | 34.3268 7335 | 33.4996 8922 |
| 42 | 39.1318 0213 | 38.4570 5236 | 37.7982 9991 | 35.0204 4446 | 34.1581 0814 |
| 43 | 39.9984 7388 | 39.2933 3013 | 38.6052 7354 | 35.7079 9947 | 34.8100 0806 |
| 44 | 40.8622 6633 | 40.1261 3788 | 39.4082 3238 | 36.3895 9055 | 35.4554 5352 |
| 45 | 41.7231 8903 | 40.9554 8999 | 40.2071 9640 | 37.0652 6944 | 36.0945 0844 |
| 46 | 42.5812 5153 | 41.7814 0081 | 41.0021 8547 | 37.7350 8743 | 36.7272 3608 |
| 47 | 43.4364 6332 | 42.6038 8461 | 41.7932 1937 | 38.3990 9535 | 37.3536 9909 |
| 48 | 44.2888 3387 | 43.4229 5562 | 42.5803 1778 | 39.0573 4359 | 37.9739 5949 |
| 49 | 45.1383 7263 | 44.2386 2799 | 43.3635 0028 | 39.7098 8212 | 38.5880 7871 |
| 50 | 45.9850 2900 | 45.0509 1582 | 44.1427 8635 | 40.3567 6047 | 39.1961 1753 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 51 | 46.8289 9236 | 45.8598 3317 | 44.9181 9537 | 40.9980 2772 | 39.7981 3617 |
| 52 | 47.6700 9205 | 46.6653 9401 | 45.6897 4664 | 41.6337 3256 | 40.3941 9423 |
| 53 | 48.5083 9739 | 47.4676 1228 | 46.4574 5934 | 42.2639 2324 | 40.9843 5072 |
| 54 | 49.3439 1767 | 48.2665 0184 | 47.2213 5258 | 42.8886 4757 | 41.5686 6408 |
| 55 | 50.1766 6213 | 49.0620 7651 | 47.9814 4535 | 43.5079 5298 | 42.1471 9216 |
| 56 | 51.0066 3999 | 49.8543 5003 | 48.7377 5657 | 44.1218 8647 | 42.7199 9224 |
| 57 | 51.8338 6046 | 50.6433 3612 | 49.4903 0505 | 44.7304 9465 | 43.2871 2102 |
| 58 | 52.6583 3268 | 51.4290 4840 | 50.2391 0950 | 45.3338 2369 | 43.8486 3468 |
| 59 | 53.4800 6580 | 52.2115 0046 | 50.9841 8855 | 45.9519 1939 | 44.4045 8879 |
| 60 | 54.2990 6890 | 52.9907 0584 | 51.7255 6075 | 46.5248 2716 | 44.9550 3841 |
| 61 | 55.1153 5106 | 53.7666 7800 | 52.4632 4453 | 47.1125 9198 | 45.5000 3803 |
| 62 | 55.9289 2133 | 54.5394 3035 | 53.1972 5824 | 47.6952 5847 | 46.0396 4161 |
| 63 | 56.7397 8870 | 55.3089 7627 | 53.9276 2014 | 48.2728 7085 | 46.5739 0258 |
| 64 | 57.5479 6216 | 56.0753 2905 | 54.6543 4839 | 48.8454 7296 | 47.1028 7385 |
| 65 | 58.3534 5065 | 56.8385 0194 | 55.3774 6109 | 49.4131 0826 | 47.6266 0777 |
| 66 | 59.1562 6311 | 57.5985 0814 | 56.0969 7621 | 49.9758 1984 | 48.1451 5621 |
| 67 | 59.9564 0842 | 58.3553 6078 | 56.8129 1165 | 50.5336 5040 | 48.6585 7050 |
| 68 | 60.7538 9543 | 59.1090 7296 | 57.5252 8522 | 51.0866 4228 | 49.1669 0149 |
| 69 | 61.5487 3299 | 59.8596 5770 | 58.2341 1465 | 51.6348 3745 | 49.6701 9949 |
| 70 | 62.3409 2989 | 60.6071 2798 | 58.9394 1756 | 52.1782 7752 | 50.1685 1435 |
| 71 | 63.1304 9490 | 61.3514 9672 | 59.6412 1151 | 52.7170 0374 | 50.6618 9539 |
| 72 | 63.9174 3678 | 62.0927 7680 | 60.3395 1394 | 53.2510 5699 | 51.1503 9148 |
| 73 | 64.7017 6423 | 62.8309 8103 | 61.0343 4222 | 53.7804 7781 | 51.6340 5097 |
| 74 | 65.4834 8595 | 63.5661 2216 | 61.7257 1366 | 54.3053 0638 | 52.1129 2175 |
| 75 | 66.2626 1058 | 64.2982 1292 | 62.4136 4543 | 54.8255 8253 | 52.5870 5124 |
| 76 | 67.0391 4676 | 65.0272 6596 | 63.0981 5466 | 55.3413 4575 | 53.0564 8637 |
| 77 | 67.8131 0308 | 65.7532 9388 | 63.7792 5836 | 55.8526 3520 | 53.5212 7364 |
| 78 | 68.5844 8812 | 66.4763 0924 | 64.4569 7350 | 56.3594 8966 | 53.9814 5905 |
| 79 | 69.3533 1042 | 67.1963 2453 | 65.1313 1691 | 56.8619 4762 | 54.4370 8817 |
| 80 | 70.1195 7849 | 67.9133 5221 | 65.8023 0538 | 57.3600 4721 | 54.8882 0611 |
| 81 | 70.8833 0082 | 68.6274 0467 | 66.4699 5561 | 57.8538 2623 | 55.3348 5753 |
| 82 | 71.6444 8587 | 69.3384 9426 | 67.1342 8419 | 58.3433 2216 | 55.7770 8666 |
| 83 | 72.4031 4206 | 70.0466 3326 | 67.7953 0765 | 58.8285 7215 | 56.2149 3729 |
| 84 | 73.1592 7780 | 70.7518 3393 | 68.4530 4244 | 59.3096 1304 | 56.6484 5276 |
| 85 | 73.9129 0146 | 71.4541 0846 | 69.1075 0491 | 59.7864 8133 | 57.0776 7600 |
| 86 | 74.6640 2139 | 72.1534 6898 | 69.7587 1135 | 60.2592 1321 | 57.5026 4951 |
| 87 | 75.4126 4591 | 72.8499 2759 | 70.4066 7796 | 60.7278 4457 | 57.9234 1535 |
| 88 | 76.1587 8329 | 73.5434 9633 | 71.0514 2086 | 61.1924 1097 | 58.3400 1520 |
| 89 | 76.9024 4182 | 74.2341 8720 | 71.6929 5608 | 61.6529 4768 | 58.7524 9030 |
| 90 | 77.6436 2972 | 74.9220 1212 | 72.3312 9958 | 62.1094 8965 | 59.1608 8148 |
| 91 | 78.3823 5520 | 75.6069 8300 | 72.9664 6725 | 62.5620 7152 | 59.5652 2919 |
| 92 | 79.1186 2645 | 76.2891 1168 | 73.5984 7487 | 63.0107 2765 | 59.9655 7346 |
| 93 | 79.8524 5161 | 76.9684 0995 | 74.2273 3818 | 63.4554 9210 | 60.3619 5392 |
| 94 | 80.5838 3882 | 77.6448 8955 | 74.8530 7282 | 63.8963 9861 | 60.7544 0982 |
| 95 | 81.3127 9616 | 78.3185 6218 | 75.4756 9434 | 64.3334 8065 | 61.1429 8002 |
| 96 | 82.0393 3172 | 78.9894 3950 | 76.0952 1825 | 64.7667 7140 | 61.5277 0299 |
| 97 | 82.7634 5354 | 79.6575 3308 | 76.7116 5995 | 65.1963 0375 | 61.9086 1682 |
| 98 | 83.4851 6964 | 80.3228 5450 | 77.3250 3478 | 65.6221 1028 | 62.2857 5923 |
| 99 | 84.2044 8802 | 80.9854 1524 | 77.9353 5799 | 66.0442 2333 | 62.6591 6755 |
| 100 | 84.9214 1663 | 81.6452 2677 | 78.5426 4477 | 66.4626 7492 | 63.0288 7877 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 1 | 0.9888 7515 | 0.9876 5432 | 0.9864 3650 | 0.9852 2167 | 0.9828 0098 |
| 2 | 1.9667 4923 | 1.9631 1538 | 1.9594 9346 | 1.9558 8342 | 1.9486 9875 |
| 3 | 2.9337 4460 | 2.9265 3371 | 2.9193 5237 | 2.9122 0042 | 2.8979 8403 |
| 4 | 3.8899 8230 | 3.8780 5798 | 3.8661 9222 | 3.8543 8465 | 3.8309 4254 |
| 5 | 4.8355 8200 | 4.8178 3504 | 4.8001 8962 | 4.7826 4497 | 4.7478 5508 |
| 6 | 5.7706 6205 | 5.7460 0992 | 5.7215 1874 | 5.6971 8717 | 5.6489 9762 |
| 7 | 6.6953 3948 | 6.6627 2585 | 6.6303 5140 | 6.5982 1396 | 6.5346 4139 |
| 8 | 7.6097 3002 | 7.5681 2429 | 7.5268 5712 | 7.4859 2508 | 7.4050 5297 |
| 9 | 8.5139 4810 | 8.4623 4498 | 8.4112 0308 | 8.3605 1732 | 8.2604 9432 |
| 10 | 9.4081 0690 | 9.3455 2591 | 9.2835 5421 | 9.2221 8455 | 9.1012 2291 |
| 11 | 10.2923 1832 | 10.2178 0337 | 10.1440 7320 | 10.0711 1779 | 9.9274 9181 |
| 12 | 11.1666 9302 | 11.0793 1197 | 10.9929 2054 | 10.9075 0521 | 10.7395 4969 |
| 13 | 12.0313 4044 | 11.9301 8466 | 11.8302 5454 | 11.7315 3222 | 11.5376 4097 |
| 14 | 12.8863 6880 | 12.7705 5275 | 12.6562 3136 | 12.5433 8150 | 12.3220 0587 |
| 15 | 13.7318 8509 | 13.6005 4592 | 13.4710 0504 | 13.3432 3301 | 13.0928 8046 |
| 16 | 14.5679 9514 | 14.4202 9227 | 14.2747 2754 | 14.1312 6405 | 13.8504 9677 |
| 17 | 15.3948 0360 | 15.2299 1829 | 15.0675 4874 | 14.9076 4931 | 14.5950 8282 |
| 18 | 16.2124 1395 | 16.0295 4893 | 15.8696 1651 | 15.6725 6089 | 15.3268 6272 |
| 19 | 17.0209 2850 | 16.8193 0759 | 16.6210 7671 | 16.4261 6837 | 16.0460 5673 |
| 20 | 17.8204 4845 | 17.5993 1613 | 17.3820 7320 | 17.1686 3879 | 16.7528 8130 |
| 21 | 18.6110 7387 | 18.3696 9495 | 18.1327 4792 | 17.9001 3673 | 17.4475 4919 |
| 22 | 19.3929 0371 | 19.1305 6291 | 18.8732 4086 | 18.6208 2437 | 18.1302 6948 |
| 23 | 20.1660 3580 | 19.8820 3744 | 19.6036 9012 | 19.3308 6145 | 18.8012 4764 |
| 24 | 20.9305 6693 | 20.6242 3451 | 20.3242 3193 | 20.0304 0537 | 19.4606 8565 |
| 25 | 21.6865 9276 | 21.3572 6865 | 21.0350 0067 | 20.7196 1120 | 20.1087 8196 |
| 26 | 22.4342 0792 | 22.0812 5299 | 21.7361 2890 | 21.3986 3172 | 20.7457 3166 |
| 27 | 23.1735 0598 | 22.7962 9925 | 22.4277 4737 | 22.0676 1746 | 21.3717 2644 |
| 28 | 23.9045 7946 | 23.5025 1778 | 23.1099 8508 | 22.7267 1671 | 21.9869 5474 |
| 29 | 24.6275 1986 | 24.2000 1756 | 23.7829 6925 | 23.3760 7558 | 22.5916 0171 |
| 30 | 25.3424 1766 | 24.8889 0623 | 24.4468 2540 | 24.0158 3801 | 23.1858 4934 |
| 31 | 26.0493 6233 | 25.5692 9010 | 25.1016 7734 | 24.6461 4582 | 23.7698 7650 |
| 32 | 26.7484 4236 | 26.2412 7418 | 25.7476 4719 | 25.2671 3874 | 24.3438 5897 |
| 33 | 27.4397 4522 | 26.9049 6215 | 26.3848 5543 | 25.8789 5442 | 24.9079 6951 |
| 34 | 28.1233 5745 | 27.5604 5644 | 27.0134 2089 | 26.4817 2849 | 25.4623 7789 |
| 35 | 28.7993 6460 | 28.2078 5822 | 27.6334 6080 | 27.0755 9458 | 26.0072 5100 |
| 36 | 29.4678 5127 | 28.8472 6737 | 28.2450 9080 | 27.6606 8431 | 26.5427 5283 |
| 37 | 30.1289 0114 | 29.4787 8259 | 28.8484 2496 | 28.2371 2740 | 27.0690 4455 |
| 38 | 30.7825 9692 | 30.1025 0133 | 29.4435 7579 | 28.8050 5163 | 27.5862 8457 |
| 39 | 31.4290 2044 | 30.7185 1983 | 30.0306 5430 | 29.3645 8288 | 28.0946 2857 |
| 40 | 32.0682 5260 | 31.3269 3316 | 30.6097 6996 | 29.9158 4520 | 28.5942 2955 |
| 41 | 32.7903 7340 | 31.9278 3522 | 31.1810 3079 | 30.4589 6079 | 29.0852 3789 |
| 42 | 33.3254 6195 | 32.5213 1874 | 31.7445 4332 | 30.9940 5004 | 29.5678 0135 |
| 43 | 33.9435 9649 | 33.1074 7530 | 32.3004 1264 | 31.5212 3157 | 30.0420 6522 |
| 44 | 34.5548 5438 | 33.6863 9536 | 32.8487 4243 | 32.0406 2223 | 30.5081 7221 |
| 45 | 35.1593 1212 | 34.2581 6825 | 33.3896 3495 | 32.5523 3718 | 30.9662 6261 |
| 46 | 35.7570 4536 | 34.8228 8222 | 33.9231 9108 | 33.0564 8983 | 31.4164 7431 |
| 47 | 36.3481 2891 | 35.3806 2442 | 34.4495 1031 | 33.5531 9195 | 31.8589 4281 |
| 48 | 36.9326 3674 | 35.9314 8091 | 34.9686 9081 | 34.0425 5365 | 32.2938 0129 |
| 49 | 37.5106 4202 | 36.4755 3670 | 35.4808 2941 | 34.5246 8339 | 32.7211 8063 |
| 50 | 38.0822 1708 | 37.0128 7574 | 35.9860 2161 | 34.9996 8807 | 33.1412 0946 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 51 | 38.6474 3345 | 37.5435 8099 | 36.4843 6164 | 35.4676 7298 | 33.5540 1421 |
| 52 | 39.2063 6188 | 38.0677 3431 | 36.9759 4243 | 35.9287 4185 | 33.9597 1913 |
| 53 | 39.7590 7232 | 38.5854 1660 | 37.4608 5566 | 36.3829 9690 | 34.3584 4633 |
| 54 | 40.3056 3394 | 39.0967 0776 | 37.9391 9178 | 36.8305 3882 | 34.7503 1579 |
| 55 | 40.8461 1514 | 39.6016 8667 | 38.4110 3998 | 37.2714 6681 | 35.1354 4550 |
| 56 | 41.3805 8358 | 40.1004 3128 | 38.8764 8826 | 37.7058 7863 | 35.5139 5135 |
| 57 | 41.9091 0613 | 40.5930 1855 | 39.3356 2344 | 38.1338 7058 | 35.8859 4727 |
| 58 | 42.4317 4896 | 41.0795 2449 | 39.7885 3114 | 38.5555 3751 | 36.2515 4523 |
| 59 | 42.9485 7746 | 41.5600 2419 | 40.2352 9582 | 38.9709 7292 | 36.6108 5526 |
| 60 | 43.4596 5633 | 42.0345 9179 | 40.6760 0081 | 39.3802 6889 | 36.9639 8552 |
| 61 | 43.9650 4952 | 42.5033 0054 | 41.1107 2829 | 39.7835 1614 | 37.3110 4228 |
| 62 | 44.4648 2023 | 42.9662 2275 | 41.5395 5935 | 40.1808 0408 | 37.6521 3000 |
| 63 | 44.9590 3119 | 43.4234 2988 | 41.9625 7396 | 40.5722 2077 | 37.9873 5135 |
| 64 | 45.4477 4407 | 43.8749 9247 | 42.3798 5101 | 40.9578 5298 | 38.3168 0723 |
| 65 | 45.9310 2009 | 44.3209 8022 | 42.7914 6832 | 41.3377 8618 | 38.6405 9678 |
| 66 | 46.4089 1975 | 44.7614 6195 | 43.1975 0266 | 41.7121 0461 | 38.9588 1748 |
| 67 | 46.8815 0284 | 45.1965 0563 | 43.5980 2075 | 42.0808 9125 | 39.2715 6509 |
| 68 | 47.3488 2852 | 45.6261 7840 | 43.9931 2429 | 42.4442 2783 | 39.5789 3375 |
| 69 | 47.8109 5527 | 46.0505 4656 | 44.3828 5997 | 42.8021 9490 | 39.8810 1597 |
| 70 | 48.2679 4094 | 46.4696 7562 | 44.7673 0946 | 43.1548 7183 | 40.1779 0267 |
| 71 | 48.7198 4270 | 46.8836 3024 | 45.1465 4448 | 43.5023 3678 | 40.4696 8321 |
| 72 | 49.1667 1714 | 47.2924 7431 | 45.5206 3573 | 43.8446 6677 | 40.7564 4542 |
| 73 | 49.6086 2016 | 47.6962 7093 | 45.8896 5300 | 44.1819 3771 | 41.0382 7560 |
| 74 | 50.0456 0708 | 48.0950 8240 | 46.2536 6511 | 44.5142 2434 | 41.3152 5857 |
| 75 | 50.4777 3259 | 48.4889 7027 | 46.6127 3994 | 44.8416 0034 | 41.5874 7771 |
| 76 | 50.9050 5077 | 48.8779 9533 | 46.9669 4445 | 45.1641 3826 | 41.8550 1495 |
| 77 | 51.3276 1510 | 49.2622 1761 | 47.3163 4471 | 45.4819 0962 | 42.1179 5081 |
| 78 | 51.7454 7847 | 49.6416 9640 | 47.6610 0588 | 45.7949 8485 | 42.3763 6443 |
| 79 | 52.1586 9317 | 50.0164 9027 | 48.0009 9224 | 46.1034 3335 | 42.6303 3359 |
| 80 | 52.5673 1092 | 50.3866 5706 | 48.3363 6719 | 46.4073 2349 | 42.8799 3474 |
| 81 | 52.9713 8286 | 50.7522 5389 | 48.6671 9328 | 46.7067 2265 | 43.1252 4298 |
| 82 | 53.3709 5957 | 51.1133 3717 | 48.9935 3221 | 47.0016 9720 | 43.3663 3217 |
| 83 | 53.7660 9104 | 51.4699 6264 | 49.3154 4484 | 47.2923 1251 | 43.6032 7486 |
| 84 | 54.1568 2674 | 51.8221 8532 | 49.6329 9122 | 47.5786 3301 | 43.8361 4237 |
| 85 | 54.5432 1557 | 52.1700 5958 | 49.9462 3055 | 47.8607 2218 | 44.0650 0479 |
| 86 | 54.9253 0588 | 52.5136 3909 | 50.2552 2125 | 48.1386 4254 | 44.2899 3099 |
| 87 | 55.3031 4549 | 52.8529 7688 | 50.5600 2096 | 48.4124 5571 | 44.5109 8869 |
| 88 | 55.6767 8169 | 53.1881 2531 | 50.8606 8653 | 48.6822 2237 | 44.7282 4441 |
| 89 | 56.0462 6126 | 53.5191 3611 | 51.1572 7401 | 48.9480 0234 | 44.9417 6355 |
| 90 | 56.4116 3041 | 53.8460 6035 | 51.4498 3873 | 49.2098 5452 | 45.1516 1037 |
| 91 | 56.7729 3490 | 54.1689 4850 | 51.7384 3524 | 49.4678 3696 | 45.3578 4803 |
| 92 | 57.1302 1992 | 54.4878 5037 | 52.0231 1738 | 49.7220 0686 | 45.5605 3860 |
| 93 | 57.4835 3021 | 54.8028 1518 | 52.3039 3823 | 49.9724 2055 | 45.7597 4310 |
| 94 | 57.8329 0997 | 55.1138 9154 | 52.5809 5016 | 50.2191 3355 | 45.9555 2147 |
| 95 | 58.1784 0294 | 55.4211 2744 | 52.8542 0484 | 50.4622 0054 | 46.1479 3265 |
| 96 | 58.5200 5235 | 55.7245 7031 | 53.1237 5324 | 50.7016 7541 | 46.3370 3455 |
| 97 | 58.8579 0096 | 56.0242 6698 | 53.3896 4561 | 50.9376 1124 | 46.5228 8408 |
| 98 | 59.1919 9106 | 56.3202 6368 | 53.6519 3155 | 51.1700 6034 | 46.7055 3718 |
| 99 | 59.5223 6446 | 56.6126 0610 | 53.9106 5998 | 51.3990 7422 | 46.8850 4882 |
| 100 | 59.8490 6251 | 56.9013 3936 | 54.1658 7914 | 51.6247 0367 | 47.0614 7304 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | 2% | $2\frac{1}{4}$% | $2\frac{1}{2}$% | $2\frac{3}{4}$% | 3% |
|---|---|---|---|---|---|
| 1 | 0.9803 9216 | 0.9779 9511 | 0.9756 0976 | 0.9732 3601 | 0.9708 7379 |
| 2 | 1.9415 6094 | 1.9344 6955 | 1.9274 2415 | 1.9204 2434 | 1.9134 6970 |
| 3 | 2.8838 8327 | 2.8698 9687 | 2.8560 2356 | 2.8422 6213 | 2.8286 1135 |
| 4 | 3.8077 2870 | 3.7847 4021 | 3.7619 7421 | 3.7394 2787 | 3.7170 9840 |
| 5 | 4.7134 5951 | 4.6794 5253 | 4.6458 2850 | 4.6125 8186 | 4.5797 0719 |
| 6 | 5.6014 3089 | 5.5544 7680 | 5.5081 2536 | 5.4623 6678 | 5.4171 9144 |
| 7 | 6.4719 9107 | 6.4102 4626 | 6.3493 9060 | 6.2894 0806 | 6.2302 8296 |
| 8 | 7.3254 8144 | 7.2471 8461 | 7.1701 3717 | 7.0943 1441 | 7.0196 9219 |
| 9 | 8.1622 3671 | 8.0657 0622 | 7.9708 6553 | 7.8776 7826 | 7.7861 0892 |
| 10 | 8.9825 8501 | 8.8662 1635 | 8.7520 6393 | 8.6400 7616 | 8.5302 0284 |
| 11 | 9.7868 4805 | 9.6491 1134 | 9.5142 0871 | 9.3820 6926 | 9.2526 2411 |
| 12 | 10.5753 4122 | 10.4147 7882 | 10.2577 6460 | 10.1042 0366 | 9.9540 0399 |
| 13 | 11.3483 7375 | 11.1635 9787 | 10.9831 8497 | 10.8070 1086 | 10.6349 5533 |
| 14 | 12.1062 4877 | 11.8959 3924 | 11.6909 1217 | 11.4910 0814 | 11.2960 7314 |
| 15 | 12.8492 6350 | 12.6121 6551 | 12.3813 7773 | 12.1566 9892 | 11.9379 3509 |
| 16 | 13.5777 0931 | 13.3126 3131 | 13.0550 0266 | 12.8045 7315 | 12.5611 0203 |
| 17 | 14.2918 7188 | 13.9976 8343 | 13.7121 9772 | 13.4351 0769 | 13.1661 1847 |
| 18 | 14.9920 3125 | 14.6676 6106 | 14.3533 6363 | 14.0487 6661 | 13.7535 1308 |
| 19 | 15.6784 6201 | 15.3228 9590 | 14.9788 9134 | 14.6460 0157 | 14.3237 9911 |
| 20 | 16.3514 3334 | 15.9637 1237 | 15.5891 6229 | 15.2272 5213 | 14.8774 7486 |
| 21 | 17.0112 0916 | 16.5904 2775 | 16.1845 4857 | 15.7929 4612 | 15.4150 2414 |
| 22 | 17.6580 4820 | 17.2033 5232 | 16.7654 1324 | 16.3434 9987 | 15.9369 1664 |
| 23 | 18.2922 0412 | 17.8027 8955 | 17.3321 1048 | 16.8793 1861 | 16.4436 0839 |
| 24 | 18.9139 2560 | 18.3890 3624 | 17.8849 8583 | 17.4007 9670 | 16.9355 4212 |
| 25 | 19.5234 5647 | 18.9623 8263 | 18.4243 7642 | 17.9083 1795 | 17.4131 4769 |
| 26 | 20.1210 3576 | 19.5231 1260 | 18.9506 1114 | 18.4022 5592 | 17.8768 4242 |
| 27 | 20.7068 9780 | 20.0715 0376 | 19.4640 1087 | 18.8829 7413 | 18.3270 3147 |
| 28 | 21.2812 7236 | 20.6078 2764 | 19.9648 8866 | 19.3508 2640 | 18.7641 0823 |
| 29 | 21.8443 8466 | 21.1323 4977 | 20.4535 4991 | 19.8061 5708 | 19.1884 5459 |
| 30 | 22.3964 5555 | 21.6453 2985 | 20.9302 9259 | 20.2493 0130 | 19.6004 4135 |
| 31 | 22.9377 0152 | 22.1470 2186 | 21.3954 0741 | 20.6805 8520 | 20.0004 2849 |
| 32 | 23.4683 3482 | 22.6376 7419 | 21.8491 7796 | 21.1003 2623 | 20.3887 6553 |
| 33 | 23.9885 6355 | 23.1175 2977 | 22.2918 8094 | 21.5088 3332 | 20.7657 9178 |
| 34 | 24.4985 9172 | 23.5868 2618 | 22.7237 8628 | 21.9064 0712 | 21.1318 3668 |
| 35 | 24.9986 1933 | 24.0457 9577 | 23.1451 5734 | 22.2933 4026 | 21.4872 2007 |
| 36 | 25.4888 4248 | 24.4946 6579 | 23.5562 5107 | 22.6699 1753 | 21.8322 5250 |
| 37 | 25.9694 5341 | 24.9336 5848 | 23.9573 1812 | 23.0364 1609 | 22.1672 3544 |
| 38 | 26.4406 4060 | 25.3629 9118 | 24.3486 0304 | 23.3931 0568 | 22.4924 6159 |
| 39 | 26.9025 8883 | 25.7828 7646 | 24.7303 4443 | 23.7402 4884 | 22.8082 1513 |
| 40 | 27.3554 7924 | 26.1935 2221 | 25.1027 7505 | 24.0781 0106 | 23.1147 7197 |
| 41 | 27.7994 8945 | 26.5951 3174 | 25.4661 2200 | 24.4069 1101 | 23.4123 9997 |
| 42 | 28.2347 9358 | 26.9879 0390 | 25.8206 0683 | 24.7269 2069 | 23.7013 5920 |
| 43 | 28.6615 6233 | 27.3720 3316 | 26.1664 4569 | 25.0383 6563 | 23.9819 0213 |
| 44 | 29.0799 6307 | 27.7477 0969 | 26.5038 4945 | 25.3414 7507 | 24.2542 7392 |
| 45 | 29.4901 5987 | 28.1151 1950 | 26.8330 2386 | 25.6364 7209 | 24.5187 1254 |
| 46 | 29.8923 1360 | 28.4744 4450 | 27.1541 6962 | 25.9235 7381 | 24.7754 4907 |
| 47 | 30.2865 8196 | 28.8258 6259 | 27.4674 8255 | 26.2029 9154 | 25.0247 0783 |
| 48 | 30.6731 1957 | 29.1695 4777 | 27.7731 5371 | 26.4749 3094 | 25.2667 0664 |
| 49 | 31.0520 7801 | 29.5056 7019 | 28.0713 6947 | 26.7395 9215 | 25.5016 5693 |
| 50 | 31.4236 0589 | 29.8343 9627 | 28.3623 1168 | 26.9971 6998 | 25.7297 6401 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | 2% | $2\frac{1}{4}$% | $2\frac{1}{2}$% | $2\frac{3}{4}$% | 3% |
|---|---|---|---|---|---|
| 51 | 31.7878 4892 | 30.1558 8877 | 28.6461 5774 | 27.2478 5400 | 25.9512 2719 |
| 52 | 32.1449 4992 | 30.4703 0687 | 28.9230 8072 | 27.4918 2871 | 26.1662 3999 |
| 53 | 32.4950 4894 | 30.7778 0623 | 29.1932 4948 | 27.7292 7368 | 26.3749 9028 |
| 54 | 32.8382 8327 | 31.0785 3910 | 29.4568 2876 | 27.9603 6368 | 26.5776 6047 |
| 55 | 33.1747 8752 | 31.3726 5438 | 29.7139 7928 | 28.1852 6879 | 26.7744 2764 |
| 56 | 33.5046 9365 | 31.6602 9768 | 29.9648 5784 | 28.4041 5454 | 26.9654 6373 |
| 57 | 33.8281 3103 | 31.9416 1142 | 30.2096 1740 | 28.6171 8203 | 27.1509 3566 |
| 58 | 34.1452 2650 | 32.2167 3489 | 30.4484 0722 | 28.8245 0806 | 27.3310 0549 |
| 59 | 34.4561 0441 | 32.4858 0429 | 30.6813 7290 | 29.0262 8522 | 27.5058 3058 |
| 60 | 34.7608 8668 | 32.7489 5285 | 30.9086 5649 | 29.2226 6201 | 27.6755 6367 |
| 61 | 35.0596 9282 | 33.0063 1086 | 31.1303 9657 | 29.4137 8298 | 27.8403 5307 |
| 62 | 35.3526 4002 | 33.2580 0573 | 31.3467 2836 | 29.5997 8879 | 28.0003 4279 |
| 63 | 35.6398 4316 | 33.5041 6208 | 31.5577 8377 | 29.7808 1634 | 28.1556 7261 |
| 64 | 35.9214 1486 | 33.7449 0179 | 31.7636 9148 | 29.9569 9887 | 28.3064 7826 |
| 65 | 36.1974 6555 | 33.9803 4405 | 31.9645 7705 | 30.1284 6605 | 28.4528 9152 |
| 66 | 36.4681 0348 | 34.2106 0543 | 32.1605 6298 | 30.2953 4409 | 28.5950 4031 |
| 67 | 36.7334 3478 | 34.4357 9993 | 32.3517 6876 | 30.4577 5581 | 28.7330 4884 |
| 68 | 36.9935 6351 | 34.6560 3905 | 32.5383 1099 | 30.6158 2074 | 28.8670 3771 |
| 69 | 37.2485 9168 | 34.8714 3183 | 32.7203 0340 | 30.7696 5522 | 28.9971 2399 |
| 70 | 37.4986 1920 | 35.0820 8492 | 32.8978 5698 | 30.9193 7247 | 29.1234 2135 |
| 71 | 37.7437 4441 | 35.2881 0261 | 33.0710 7998 | 31.0650 8270 | 29.2460 4015 |
| 72 | 37.9840 6314 | 35.4895 8691 | 33.2400 7803 | 31.2068 9314 | 29.3650 8752 |
| 73 | 38.2196 6975 | 35.6866 3756 | 33.4049 5417 | 31.3449 0816 | 29.4806 6750 |
| 74 | 38.4506 5662 | 35.8793 5214 | 33.5658 0895 | 31.4792 2936 | 29.5928 8106 |
| 75 | 38.6771 1433 | 36.0678 2605 | 33.7227 4044 | 31.6099 5558 | 29.7018 2628 |
| 76 | 38.8991 3170 | 36.2521 5262 | 33.8758 4433 | 31.7371 8304 | 29.8075 9833 |
| 77 | 39.1167 9578 | 36.4324 2310 | 34.0252 1398 | 31.8610 0540 | 29.9102 8964 |
| 78 | 39.3301 9194 | 36.6087 2675 | 34.1709 4047 | 31.9815 1377 | 30.0099 8994 |
| 79 | 39.5394 0386 | 36.7811 5085 | 34.3131 1265 | 32.0987 9685 | 30.1067 8635 |
| 80 | 39.7445 1359 | 36.9497 8079 | 34.4518 1722 | 32.2129 4098 | 30.2007 6345 |
| 81 | 39.9456 0156 | 37.1147 0004 | 34.5871 3875 | 32.3240 3015 | 30.2920 0335 |
| 82 | 40.1427 4663 | 37.2759 9026 | 34.7191 5976 | 32.4321 4613 | 30.3805 8577 |
| 83 | 40.3360 2611 | 37.4337 3130 | 34.8479 6074 | 32.5373 6850 | 30.4665 8813 |
| 84 | 40.5255 1579 | 37.5880 0127 | 34.9736 2023 | 32.6397 7469 | 30.5500 8556 |
| 85 | 40.7112 8999 | 37.7388 7655 | 35.0962 1486 | 32.7394 4009 | 30.6311 5103 |
| 86 | 40.8934 2156 | 37.8864 3183 | 35.2158 1938 | 32.8364 3804 | 30.7098 5537 |
| 87 | 41.0719 8192 | 38.0307 4018 | 35.3325 0671 | 32.9308 3994 | 30.7862 6735 |
| 88 | 41.2470 4110 | 38.1718 7304 | 35.4463 4801 | 33.0227 1527 | 30.8604 5374 |
| 89 | 41.4186 6774 | 38.3099 0028 | 35.5574 1269 | 33.1121 3165 | 30.9324 7936 |
| 90 | 41.5869 2916 | 38.4448 9025 | 35.6657 6848 | 33.1991 5489 | 31.0024 0714 |
| 91 | 41.7518 9133 | 38.5769 0978 | 35.7714 8144 | 33.2838 4905 | 31.0702 9820 |
| 92 | 41.9136 1985 | 38.7060 2423 | 35.8746 1604 | 33.3662 7644 | 31.1362 1184 |
| 93 | 42.0721 7545 | 38.8322 9754 | 35.9752 3516 | 33.4464 9776 | 31.2002 0567 |
| 94 | 42.2276 2299 | 38.9557 9221 | 36.0734 0016 | 33.5245 7202 | 31.2623 3560 |
| 95 | 42.3800 2254 | 39.0765 6940 | 36.1691 7089 | 33.6005 5671 | 31.3226 5592 |
| 96 | 42.5294 3386 | 39.1946 8890 | 36.2626 0574 | 33.6745 0775 | 31.3812 1934 |
| 97 | 42.6759 1555 | 39.3102 0920 | 36.3537 6170 | 33.7464 7956 | 31.4380 7703 |
| 98 | 42.8195 2505 | 39.4231 8748 | 36.4426 9434 | 33.8165 2512 | 31.4932 7867 |
| 99 | 42.9603 1867 | 39.5336 7968 | 36.5294 5790 | 33.8846 9598 | 31.5468 7250 |
| 100 | 43.0983 5164 | 39.6417 4052 | 36.6141 0526 | 33.9510 4232 | 31.5989 0534 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 1 | 0.9661 8357 | 0.9615 3846 | 0.9569 3780 | 0.9523 8095 | 0.9478 6730 |
| 2 | 1.8996 9428 | 1.8860 9467 | 1.8726 6775 | 1.8594 1043 | 1.8463 1971 |
| 3 | 2.8016 3698 | 2.7750 9103 | 2.7489 6435 | 2.7232 4803 | 2.6979 3338 |
| 4 | 3.6730 7921 | 3.6298 9522 | 3.5875 2570 | 3.5459 5050 | 3.5051 5012 |
| 5 | 4.5150 5238 | 4.4518 2233 | 4.3899 7674 | 4.3294 7667 | 4.2702 8448 |
| 6 | 5.3285 5302 | 5.2421 3686 | 5.1578 7248 | 5.0756 9206 | 4.9955 3031 |
| 7 | 6.1145 4298 | 6.0020 5467 | 5.8927 0094 | 5.7863 7340 | 5.6829 6712 |
| 8 | 6.8739 5554 | 6.7327 4487 | 6.5958 8607 | 6.4632 1276 | 6.3345 6599 |
| 9 | 7.6076 8651 | 7.4353 3161 | 7.2687 9050 | 7.1078 2168 | 6.9521 9525 |
| 10 | 8.3166 0532 | 8.1108 9578 | 7.9127 1818 | 7.7217 3493 | 7.5376 2583 |
| 11 | 9.0015 5104 | 8.7604 7671 | 8.5289 1692 | 8.3064 1422 | 8.0925 3633 |
| 12 | 9.6633 3433 | 9.3850 7376 | 9.1185 8078 | 8.8632 5164 | 8.6185 1785 |
| 13 | 10.3027 3849 | 9.9856 4785 | 9.6828 5242 | 9.3935 7299 | 9.1170 7853 |
| 14 | 10.9205 2028 | 10.5631 2293 | 10.2228 2528 | 9.8986 4094 | 9.5896 4790 |
| 15 | 11.5174 1090 | 11.1183 8743 | 10.7395 4573 | 10.3796 5804 | 10.0375 8094 |
| 16 | 12.0941 1681 | 11.6522 9561 | 11.2340 1505 | 10.8377 6956 | 10.4621 6203 |
| 17 | 12.6513 2059 | 12.1656 6885 | 11.7071 9143 | 11.2740 6625 | 10.8646 0856 |
| 18 | 13.1896 8173 | 12.6592 9697 | 12.1599 9180 | 11.6895 8690 | 11.2460 7447 |
| 19 | 13.7098 3742 | 13.1339 3940 | 12.5932 9359 | 12.0853 2086 | 11.6076 5352 |
| 20 | 14.2124 0330 | 13.5903 2634 | 13.0079 3645 | 12.4622 1034 | 11.9503 8249 |
| 21 | 14.6979 7420 | 14.0291 5995 | 13.4047 2388 | 12.8211 5271 | 12.2752 4406 |
| 22 | 15.1671 2484 | 14.4511 1533 | 13.7844 2476 | 13.1630 0258 | 12.5831 6973 |
| 23 | 15.6204 1047 | 14.8568 4167 | 14.1477 7489 | 13.4885 7388 | 12.8750 4240 |
| 24 | 16.0583 6760 | 15.2469 6314 | 14.4954 7837 | 13.7986 4179 | 13.1516 9805 |
| 25 | 16.4815 1459 | 15.6220 7994 | 14.8282 0896 | 14.0939 4457 | 13.4139 3266 |
| 26 | 16.8903 5226 | 15.9827 6918 | 15.1466 1145 | 14.3751 8530 | 13.6624 9541 |
| 27 | 17.2853 6451 | 16.3295 8575 | 15.4513 0282 | 14.6430 3362 | 13.8980 9991 |
| 28 | 17.6670 1885 | 16.6630 6322 | 15.7428 7351 | 14.8981 2726 | 14.1214 2172 |
| 29 | 18.0357 6700 | 16.9837 1463 | 16.0218 8853 | 15.1410 7358 | 14.3331 0116 |
| 30 | 18.3920 4541 | 17.2920 3330 | 16.2888 8854 | 15.3724 5103 | 14.5337 4517 |
| 31 | 18.7362 7576 | 17.5884 9356 | 16.5443 9095 | 15.5928 1050 | 14.7239 2907 |
| 32 | 19.0688 6547 | 17.8735 5150 | 16.7888 9086 | 15.8026 7667 | 14.9041 9817 |
| 33 | 19.3902 0818 | 18.1467 4567 | 17.0228 6207 | 16.0025 4921 | 15.0750 6936 |
| 34 | 19.7006 8423 | 18.4111 9776 | 17.2467 5796 | 16.1929 0401 | 15.2370 3257 |
| 35 | 20.0006 6110 | 18.6646 1323 | 17.4610 1240 | 16.3741 9429 | 15.3905 5220 |
| 36 | 20.2904 9381 | 18.9082 8195 | 17.6660 4058 | 16.5468 5171 | 15.5360 6843 |
| 37 | 20.5705 2542 | 19.1425 7880 | 17.8622 3979 | 16.7112 8734 | 15.6739 9851 |
| 38 | 20.8410 8736 | 19.3678 6423 | 18.0499 9023 | 16.8678 9271 | 15.8047 3793 |
| 39 | 21.1024 9987 | 19.5844 8484 | 18.2296 5572 | 17.0170 4067 | 15.9286 6154 |
| 40 | 21.3550 7234 | 19.7927 7388 | 18.4015 8442 | 17.1590 8635 | 16.0461 2469 |
| 41 | 21.5991 0371 | 19.9930 5181 | 18.5661 0949 | 17.2943 6796 | 16.1574 6416 |
| 42 | 21.8348 8281 | 20.1856 2674 | 18.7235 4975 | 17.4232 0758 | 16.2629 9920 |
| 43 | 22.0626 8870 | 20.3707 9494 | 18.8742 1029 | 17.5459 1198 | 16.3630 3242 |
| 44 | 22.2827 9102 | 20.5488 4129 | 19.0183 8305 | 17.6627 7331 | 16.4578 5063 |
| 45 | 22.4954 5026 | 20.7200 3970 | 19.1563 4742 | 17.7740 6982 | 16.5477 2572 |
| 46 | 22.7009 1813 | 20.8846 5356 | 19.2883 7074 | 17.8800 6650 | 16.6329 1537 |
| 47 | 22.8994 3780 | 21.0429 3612 | 19.4147 0884 | 17.9810 1571 | 16.7136 6386 |
| 48 | 23.0912 4425 | 21.1951 3088 | 19.5356 0654 | 18.0771 5782 | 16.7902 0271 |
| 49 | 23.2765 6450 | 21.3414 7200 | 19.6512 9813 | 18.1687 2173 | 16.8627 5139 |
| 50 | 23.4556 1757 | 21.4821 8462 | 19.7620 0778 | 18.2559 2546 | 16.9315 1790 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | $3\frac{1}{2}\%$ | 4% | $4\frac{1}{2}\%$ | 5% | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 51 | 23.6286 1630 | 21.6174 8521 | 19.8679 5003 | 18.3389 7663 | 16.9966 9943 |
| 52 | 23.7957 6454 | 21.7475 8193 | 19.9693 3017 | 18.4180 7298 | 17.0584 8287 |
| 53 | 23.9572 6043 | 21.8726 7493 | 20.0663 4466 | 18.4934 0284 | 17.1170 4538 |
| 54 | 24.1132 9510 | 21.9929 5667 | 20.1591 8149 | 18.5651 4556 | 17.1725 5486 |
| 55 | 24.2640 5323 | 22.1086 1218 | 20.2480 2057 | 18.6334 7196 | 17.2251 7048 |
| 56 | 24.4097 1327 | 22.2189 1940 | 20.3330 3404 | 18.6985 4473 | 17.2750 4311 |
| 57 | 24.5504 4760 | 22.3267 4943 | 20.4143 8664 | 18.7605 1879 | 17.3223 1575 |
| 58 | 24.6864 2281 | 22.4295 6676 | 20.4922 3602 | 18.8195 4170 | 17.3671 2393 |
| 59 | 24.8177 9981 | 22.5284 2957 | 20.5667 3303 | 18.8757 5400 | 17.4095 9614 |
| 60 | 24.9447 3412 | 22.6234 8997 | 20.6380 2204 | 18.9292 8952 | 17.4498 5416 |
| 61 | 25.0673 7596 | 22.7148 9421 | 20.7062 4118 | 18.9802 7574 | 17.4880 1343 |
| 62 | 25.1858 7049 | 22.8027 8289 | 20.7715 2266 | 19.0288 3404 | 17.5241 8334 |
| 63 | 25.3003 5796 | 22.8872 9124 | 20.8339 9298 | 19.0750 8003 | 17.5584 6762 |
| 64 | 25.4109 7388 | 22.9685 4927 | 20.8937 7319 | 19.1191 2384 | 17.5909 6457 |
| 65 | 25.5178 4916 | 23.0466 8199 | 20.9509 7913 | 19.1610 7033 | 17.6217 6737 |
| 66 | 25.6211 1030 | 23.1218 0961 | 21.0057 2165 | 19.2010 1936 | 17.6509 6433 |
| 67 | 25.7208 7951 | 23.1940 4770 | 21.0581 0684 | 19.2390 6606 | 17.6786 3917 |
| 68 | 25.8172 7489 | 23.2635 0740 | 21.1082 3621 | 19.2753 0101 | 17.7048 7125 |
| 69 | 25.9104 1052 | 23.3302 9558 | 21.1562 0690 | 19.3098 1048 | 17.7297 3579 |
| 70 | 26.0003 9664 | 23.3945 1498 | 21.2021 1187 | 19.3426 7665 | 17.7533 0406 |
| 71 | 26.0873 3975 | 23.4562 6440 | 21.2460 4007 | 19.3739 7776 | 17.7756 4366 |
| 72 | 26.1713 4275 | 23.5156 3885 | 21.2880 7662 | 19.4037 8834 | 17.7968 1864 |
| 73 | 26.2525 0508 | 23.5727 2966 | 21.3283 0298 | 19.4321 7937 | 17.8168 8970 |
| 74 | 26.3309 2278 | 23.6276 2468 | 21.3667 9711 | 19.4592 1845 | 17.8359 1441 |
| 75 | 26.4066 8868 | 23.6804 0834 | 21.4036 3360 | 19.4849 6995 | 17.8539 4731 |
| 76 | 26.4798 9244 | 23.7311 6187 | 21.4388 8383 | 19.5094 9519 | 17.8710 4010 |
| 77 | 26.5506 2072 | 23.7799 6333 | 21.4726 1611 | 19.5328 5257 | 17.8872 4180 |
| 78 | 26.6189 5721 | 23.8268 8782 | 21.5048 9579 | 19.5550 9768 | 17.9025 9887 |
| 79 | 26.6849 8281 | 23.8720 0752 | 21.5357 8545 | 19.5762 8351 | 17.9171 5532 |
| 80 | 26.7487 7567 | 23.9153 9185 | 21.5653 4493 | 19.5964 6048 | 17.9309 5291 |
| 81 | 26.8104 1127 | 23.9571 0754 | 21.5936 3151 | 19.6156 7665 | 17.9440 3120 |
| 82 | 26.8699 6258 | 23.9972 1879 | 21.6207 0001 | 19.6339 7776 | 17.8564 2768 |
| 83 | 26.9275 0008 | 24.0357 8730 | 21.6466 0288 | 19.6514 0739 | 17.9681 7789 |
| 84 | 26.9830 9186 | 24.0728 7240 | 21.6713 9032 | 19.6680 0704 | 17.9793 1554 |
| 85 | 27.0368 0373 | 24.1085 3116 | 21.6951 1035 | 19.6838 1623 | 17.9898 7255 |
| 86 | 27.0886 9926 | 24.1428 1842 | 21.7178 0895 | 19.6988 7260 | 17.9998 7919 |
| 87 | 27.1388 3986 | 24.1757 8694 | 21.7395 3009 | 19.7132 1200 | 18.0093 6416 |
| 88 | 27.1872 8489 | 24.2074 8745 | 21.7603 1588 | 19.7268 6857 | 18.0183 5466 |
| 89 | 27.2340 9168 | 24.2379 6870 | 21.7802 0658 | 19.7398 7483 | 18.0268 7645 |
| 90 | 27.2793 1564 | 24.2672 7759 | 21.7992 4075 | 19.7522 6174 | 18.0349 5398 |
| 91 | 27.3230 1028 | 24.2954 5923 | 21.8174 5526 | 19.7640 5880 | 18.0426 1041 |
| 92 | 27.3652 2732 | 24.3225 5695 | 21.8348 8542 | 19.7752 9410 | 18.0498 6769 |
| 93 | 27.4060 1673 | 24.3486 1245 | 21.8515 6499 | 19.7859 9438 | 18.0567 4662 |
| 94 | 27.4454 2680 | 24.3736 6582 | 21.8675 2631 | 19.7961 8512 | 18.0632 6694 |
| 95 | 27.4835 0415 | 24.3977 5559 | 21.8828 0030 | 19.8058 9059 | 18.0694 4734 |
| 96 | 27.5202 9387 | 24.4209 1884 | 21.8974 1655 | 19.8151 3390 | 18.0753 0553 |
| 97 | 27.5558 3948 | 24.4431 9119 | 21.9114 0340 | 19.8239 3705 | 18.0808 5833 |
| 98 | 27.5901 8308 | 24.4646 0692 | 21.9247 8794 | 19.8323 2100 | 18.0861 2164 |
| 99 | 27.6233 6529 | 24.4851 9896 | 21.9375 9612 | 19.8403 0571 | 18.0911 1055 |
| 100 | 27.6554 2540 | 24.5049 9900 | 21.9498 5274 | 19.8479 1020 | 18.0958 3939 |

## TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

| $n$ | 6% | $6\frac{1}{2}$% | 7% | $7\frac{1}{2}$% | 8% |
|---|---|---|---|---|---|
| 1 | 0.9433 9623 | 0.9389 6714 | 0.9345 7944 | 0.9302 3256 | 0.9259 2593 |
| 2 | 1.8333 9267 | 1.8206 2642 | 1.8080 1817 | 1.7955 6517 | 1.7832 6475 |
| 3 | 2.6730 1195 | 2.6484 7551 | 2.6243 1604 | 2.6005 2574 | 2.5770 9699 |
| 4 | 3.4651 0561 | 3.4257 9860 | 3.3872 1126 | 3.3493 2627 | 3.3121 2684 |
| 5 | 4.2123 6379 | 4.1556 7944 | 4.1001 9744 | 4.0458 8490 | 3.9927 1004 |
| 6 | 4.9173 2433 | 4.8410 1356 | 4.7665 3966 | 4.6938 4642 | 4.6228 7966 |
| 7 | 5.5823 8144 | 5.4845 1977 | 5.3892 8940 | 5.2966 0132 | 5.2063 7006 |
| 8 | 6.2097 9381 | 6.0887 5096 | 5.9712 9851 | 5.8573 0355 | 5.7466 3894 |
| 9 | 6.8016 9227 | 6.6561 0419 | 6.5152 3225 | 6.3788 8703 | 6.2468 8791 |
| 10 | 7.3600 8705 | 7.1888 3022 | 7.0235 8154 | 6.8640 8096 | 6.7100 8140 |
| 11 | 7.8868 7458 | 7.6890 4246 | 7.4986 7434 | 7.3154 2415 | 7.1389 6426 |
| 12 | 8.3838 4394 | 8.1587 2532 | 7.9426 8630 | 7.7352 7827 | 7.5360 7802 |
| 13 | 8.8526 8296 | 8.5997 4208 | 8.3576 5074 | 8.1258 4026 | 7.9037 7594 |
| 14 | 9.2949 8393 | 9.0138 4233 | 8.7454 6799 | 8.4891 5373 | 8.2442 3698 |
| 15 | 9.7122 4899 | 9.4026 6885 | 9.1079 1401 | 8.8271 1974 | 8.5594 7869 |
| 16 | 10.1058 9527 | 9.7677 6418 | 9.4466 4860 | 9.1415 0674 | 8.8513 6916 |
| 17 | 10.4772 5969 | 10.1105 7670 | 9.7632 2299 | 9.4339 5976 | 9.1216 3811 |
| 18 | 10.8276 0348 | 10.4324 6638 | 10.0590 8691 | 9.7060 0908 | 9.3718 8714 |
| 19 | 11.1581 1649 | 10.7347 1022 | 10.3355 9524 | 9.9590 7821 | 9.6035 9920 |
| 20 | 11.4699 2122 | 11.0185 0725 | 10.5940 1425 | 10.1944 9136 | 9.8181 4741 |
| 21 | 11.7640 7662 | 11.2849 8333 | 10.8355 2733 | 10.4134 8033 | 10.0168 0316 |
| 22 | 12.0415 8172 | 11.5351 9562 | 11.0612 4050 | 10.6171 9101 | 10.2007 4366 |
| 23 | 12.3033 7898 | 11.7701 3673 | 11.2721 8738 | 10.8066 8931 | 10.3710 5895 |
| 24 | 12.5503 5753 | 11.9907 3871 | 11.4693 3400 | 10.9829 6680 | 10.5287 5828 |
| 25 | 12.7833 5616 | 12.1978 7672 | 11.6535 8318 | 11.1469 4586 | 10.6747 7619 |
| 26 | 13.0031 6619 | 12.3923 7251 | 11.8257 7867 | 11.2994 8452 | 10.8099 7795 |
| 27 | 13.2105 3414 | 12.5749 9766 | 11.9867 0904 | 11.4413 8095 | 10.9351 6477 |
| 28 | 13.4061 6428 | 12.7464 7668 | 12.1371 1125 | 11.5733 7763 | 11.0510 7849 |
| 29 | 13.5907 2102 | 12.9074 8984 | 12.2776 7407 | 11.6961 6524 | 11.1584 0601 |
| 30 | 13.7648 3115 | 13.0586 7591 | 12.4090 4118 | 11.8103 8627 | 11.2577 8334 |
| 31 | 13.9290 8599 | 13.2006 3465 | 12.5318 1419 | 11.9166 3839 | 11.3497 9939 |
| 32 | 14.0840 4339 | 13.3339 2925 | 12.6465 5532 | 12.0154 7757 | 11.4349 9944 |
| 33 | 14.2302 2961 | 13.4590 8850 | 12.7537 9002 | 12.1074 2099 | 11.5138 8837 |
| 34 | 14.3681 4114 | 13.5766 0832 | 12.8540 0936 | 12.1929 4976 | 11.5869 3367 |
| 35 | 14.4982 4636 | 13.6869 5673 | 12.9476 7230 | 12.2725 1141 | 11.6545 6822 |
| 36 | 14.6209 8713 | 13.7905 6970 | 13.0352 0776 | 12.3465 2224 | 11.7171 9279 |
| 37 | 14.7367 8031 | 13.8878 5887 | 13.1170 1660 | 12.4153 6953 | 11.7751 7851 |
| 38 | 14.8460 1916 | 13.9792 1021 | 13.1934 7345 | 12.4794 1351 | 11.8288 6899 |
| 39 | 14.9490 7468 | 14.0649 8611 | 13.2649 2846 | 12.5389 8931 | 11.8785 8240 |
| 40 | 15.0462 9687 | 14.1455 2687 | 13.3317 0884 | 12.5944 0866 | 11.9246 1333 |
| 41 | 15.1380 1592 | 14.2211 5199 | 13.3941 2041 | 12.6459 6155 | 11.9672 3457 |
| 42 | 15.2245 4332 | 14.2921 6149 | 13.4524 4898 | 12.6939 1772 | 12.0066 9867 |
| 43 | 15.3061 7294 | 14.3588 3708 | 13.5069 6167 | 12.7885 2811 | 12.0432 3951 |
| 44 | 15.3831 8202 | 14.4214 4327 | 13.5579 0810 | 12.7800 2615 | 12.0770 7362 |
| 45 | 15.4558 3209 | 14.4802 2842 | 13.6055 2159 | 12.8186 2898 | 12.1084 0150 |
| 46 | 15.5243 6990 | 14.5354 2575 | 13.6500 2018 | 12.8545 3858 | 12.1374 0880 |
| 47 | 15.5890 2821 | 14.5872 5422 | 13.6916 0764 | 12.8879 4287 | 12.1642 6741 |
| 48 | 15.6500 2661 | 14.6359 1946 | 13.7304 7443 | 12.9190 1662 | 12.1891 3649 |
| 49 | 15.7075 7227 | 14.6816 1451 | 13.7667 9853 | 12.9479 2244 | 12.2121 6341 |
| 50 | 15.7618 6064 | 14.7245 2067 | 13.8007 4629 | 12.9748 1157 | 12.2334 8464 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 1 | 1.0033 3333 | 1.0041 6667 | 1.0050 0000 | 1.0087 5000 | 1.0100 0000 |
| 2 | 0.5025 0139 | 0.5031 2717 | 0.5037 5312 | 0.5065 7203 | 0.5075 1244 |
| 3 | 0.3355 5802 | 0.3361 1496 | 0.3366 7221 | 0.3391 8361 | 0.3400 2211 |
| 4 | 0.2520 8680 | 0.2526 0958 | 0.2531 3279 | 0.2554 9257 | 0.2562 8109 |
| 5 | 0.2020 0444 | 0.2025 0693 | 0.2030 0997 | 0.2052 8049 | 0.2060 3980 |
| 6 | 0.1686 1650 | 0.1691 0564 | 0.1695 9546 | 0.1718 0789 | 0.1725 4837 |
| 7 | 0.1447 6824 | 0.1452 4800 | 0.1457 2854 | 0.1479 0070 | 0.1486 2828 |
| 8 | 0.1268 8228 | 0.1273 5512 | 0.1278 2886 | 0.1299 7190 | 0.1306 9029 |
| 9 | 0.1129 7118 | 0.1134 3876 | 0.1139 0736 | 0.1160 2868 | 0.1167 4037 |
| 10 | 0.1018 4248 | 0.1023 0596 | 0.1027 7057 | 0.1048 7538 | 0.1055 8208 |
| 11 | 0.0927 3736 | 0.0931 9757 | 0.0936 5903 | 0.0957 5111 | 0.0964 5408 |
| 12 | 0.0851 4990 | 0.0856 0748 | 0.0860 6643 | 0.0881 4860 | 0.0888 4879 |
| 13 | 0.0787 2989 | 0.0791 8532 | 0.0796 4224 | 0.0817 1669 | 0.0824 1482 |
| 14 | 0.0732 2716 | 0.0736 8082 | 0.0741 3609 | 0.0762 0453 | 0.0769 0117 |
| 15 | 0.0684 0777 | 0.0689 1045 | 0.0693 6436 | 0.0714 2817 | 0.0721 2378 |
| 16 | 0.0642 8557 | 0.0647 3655 | 0.0651 8937 | 0.0672 4965 | 0.0679 4460 |
| 17 | 0.0606 0389 | 0.0610 5387 | 0.0615 0579 | 0.0635 6346 | 0.0642 5806 |
| 18 | 0.0573 3140 | 0.0577 8053 | 0.0582 3173 | 0.0602 8756 | 0.0609 8205 |
| 19 | 0.0544 0348 | 0.0548 5191 | 0.0553 0253 | 0.0573 5715 | 0.0580 5175 |
| 20 | 0.0517 6844 | 0.0522 1630 | 0.0526 6645 | 0.0547 2042 | 0.0554 1532 |
| 21 | 0.0493 8445 | 0.0498 3183 | 0.0502 8163 | 0.0523 3541 | 0.0530 3075 |
| 22 | 0.0472 1726 | 0.0476 6427 | 0.0481 1380 | 0.0501 6779 | 0.0508 6371 |
| 23 | 0.0452 3861 | 0.0456 8531 | 0.0461 3465 | 0.0481 8921 | 0.0488 8584 |
| 24 | 0.0434 2492 | 0.0438 7139 | 0.0443 2061 | 0.0463 7604 | 0.0470 7347 |
| 25 | 0.0417 5640 | 0.0422 0270 | 0.0426 5186 | 0.0447 0843 | 0.0454 0675 |
| 26 | 0.0402 1630 | 0.0406 6247 | 0.0411 1163 | 0.0431 6959 | 0.0438 6888 |
| 27 | 0.0387 9035 | 0.0392 3645 | 0.0396 8565 | 0.0417 4520 | 0.0424 4553 |
| 28 | 0.0374 6632 | 0.0379 1239 | 0.0383 6167 | 0.0404 2300 | 0.0411 2444 |
| 29 | 0.0362 3367 | 0.0366 7974 | 0.0371 2914 | 0.0391 9243 | 0.0398 9502 |
| 30 | 0.0350 8325 | 0.0355 2936 | 0.0359 7892 | 0.0380 4431 | 0.0387 4811 |
| 31 | 0.0340 0712 | 0.0344 5330 | 0.0349 0304 | 0.0369 7068 | 0.0376 7573 |
| 32 | 0.0329 9830 | 0.0334 4458 | 0.0338 9453 | 0.0359 6454 | 0.0366 7089 |
| 33 | 0.0320 5067 | 0.0324 9708 | 0.0329 4727 | 0.0350 1976 | 0.0357 2744 |
| 34 | 0.0311 5885 | 0.0316 0540 | 0.0320 5586 | 0.0341 3092 | 0.0348 3997 |
| 35 | 0.0303 1803 | 0.0307 6476 | 0.0312 1550 | 0.0332 9324 | 0.0340 0368 |
| 36 | 0.0295 2399 | 0.0299 7090 | 0.0304 2194 | 0.0325 0244 | 0.0332 1431 |
| 37 | 0.0287 7291 | 0.0292 2003 | 0.0296 7139 | 0.0317 5473 | 0.0324 6805 |
| 38 | 0.0280 6141 | 0.0285 0875 | 0.0289 6045 | 0.0310 4671 | 0.0317 6150 |
| 39 | 0.0273 8644 | 0.0278 3402 | 0.0282 8607 | 0.0303 7531 | 0.0310 9160 |
| 40 | 0.0267 4527 | 0.0271 9310 | 0.0276 4552 | 0.0297 3780 | 0.0304 5560 |
| 41 | 0.0261 3543 | 0.0263 8352 | 0.0270 3631 | 0.0291 3169 | 0.0298 5102 |
| 42 | 0.0255 5466 | 0.0260 0303 | 0.0264 5622 | 0.0285 5475 | 0.0292 7563 |
| 43 | 0.0250 0095 | 0.0254 4961 | 0.0259 0320 | 0.0280 0493 | 0.0287 2737 |
| 44 | 0.0244 7246 | 0.0249 2141 | 0.0253 7541 | 0.0274 8039 | 0.0282 0441 |
| 45 | 0.0239 6749 | 0.0244 1675 | 0.0248 7117 | 0.0269 7943 | 0.0277 0505 |
| 46 | 0.0234 8451 | 0.0239 3409 | 0.0243 8894 | 0.0265 0053 | 0.0272 2775 |
| 47 | 0.0230 2213 | 0.0234 7204 | 0.0239 2733 | 0.0260 4228 | 0.0267 7111 |
| 48 | 0.0225 7905 | 0.0230 2929 | 0.0234 8503 | 0.0256 0338 | 0.0263 3384 |
| 49 | 0.0221 5410 | 0.0226 0468 | 0.0230 6087 | 0.0251 0265 | 0.0259 1474 |
| 50 | 0.0217 4618 | 0.0221 9711 | 0.0226 5376 | 0.0247 7900 | 0.0255 1273 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 51 | 0.0213 5429 | 0.0218 0557 | 0.0222 6269 | 0.0243 9142 | 0.0251 2680 |
| 52 | 0.0209 7751 | 0.0214 2916 | 0.0218 8675 | 0.0240 1899 | 0.0247 5603 |
| 53 | 0.0206 1499 | 0.0210 6700 | 0.0215 2507 | 0.0236 6084 | 0.0243 9956 |
| 54 | 0.0202 6592 | 0.0207 1830 | 0.0211 7686 | 0.0233 1619 | 0.0240 5658 |
| 55 | 0.0199 2958 | 0.0203 8234 | 0.0208 4139 | 0.0229 8430 | 0.0237 2637 |
| 56 | 0.0196 0529 | 0.0200 5843 | 0.0205 1797 | 0.0226 6449 | 0.0234 0823 |
| 57 | 0.0192 9241 | 0.0197 4593 | 0.0202 0598 | 0.0223 5611 | 0.0231 0156 |
| 58 | 0.0189 9035 | 0.0194 4426 | 0.0199 0481 | 0.0220 5858 | 0.0228 0573 |
| 59 | 0.0186 9856 | 0.0191 5287 | 0.0196 1392 | 0.0217 7135 | 0.0225 2020 |
| 60 | 0.0184 1652 | 0.0188 7123 | 0.0193 3280 | 0.0214 9390 | 0.0222 4445 |
| 61 | 0.0181 4377 | 0.0185 9888 | 0.0190 6096 | 0.0212 2575 | 0.0219 7800 |
| 62 | 0.0178 7984 | 0.0183 3536 | 0.0187 9796 | 0.0209 6644 | 0.0217 2041 |
| 63 | 0.0176 2432 | 0.0180 8025 | 0.0185 4337 | 0.0207 1557 | 0.0214 7125 |
| 64 | 0.0173 7681 | 0.0178 3315 | 0.0182 9681 | 0.0204 7273 | 0.0212 3013 |
| 65 | 0.0171 3695 | 0.0175 9371 | 0.0180 5789 | 0.0202 3754 | 0.0209 9667 |
| 66 | 0.0169 0438 | 0.0173 6156 | 0.0178 2627 | 0.0200 0968 | 0.0207 7052 |
| 67 | 0.0166 7878 | 0.0171 3639 | 0.0176 0163 | 0.0197 8879 | 0.0205 5136 |
| 68 | 0.0164 5985 | 0.0169 1788 | 0.0173 8366 | 0.0195 7459 | 0.0203 3888 |
| 69 | 0.0162 4729 | 0.0167 0574 | 0.0171 7206 | 0.0193 6677 | 0.0201 3280 |
| 70 | 0.0160 4083 | 0.0164 9971 | 0.0169 6657 | 0.0191 6506 | 0.0199 3282 |
| 71 | 0.0158 4021 | 0.0162 9952 | 0.0167 6693 | 0.0189 6921 | 0.0197 3870 |
| 72 | 0.0156 4518 | 0.0161 0493 | 0.0165 7289 | 0.0187 7897 | 0.0195 5019 |
| 73 | 0.0154 5553 | 0.0159 1572 | 0.0163 8422 | 0.0185 9411 | 0.0193 6706 |
| 74 | 0.0152 7103 | 0.0157 3165 | 0.0162 0070 | 0.0184 1441 | 0.0191 8910 |
| 75 | 0.0150 9147 | 0.0155 5253 | 0.0160 2214 | 0.0182 3966 | 0.0190 1609 |
| 76 | 0.0149 1666 | 0.0153 7816 | 0.0158 4832 | 0.0180 6967 | 0.0188 4784 |
| 77 | 0.0147 4641 | 0.0152 0836 | 0.0156 7908 | 0.0179 0426 | 0.0186 8416 |
| 78 | 0.0145 8056 | 0.0150 4295 | 0.0155 1423 | 0.0177 4324 | 0.0185 2488 |
| 79 | 0.0144 1892 | 0.0148 8177 | 0.0153 5360 | 0.0175 8645 | 0.0183 6984 |
| 80 | 0.0142 6135 | 0.0147 2464 | 0.0151 9704 | 0.0174 3374 | 0.0182 1885 |
| 81 | 0.0141 0770 | 0.0145 7144 | 0.0150 4439 | 0.0172 8494 | 0.0180 7180 |
| 82 | 0.0139 5781 | 0.0144 2200 | 0.0148 9552 | 0.0171 3992 | 0.0179 2851 |
| 83 | 0.0138 1156 | 0.0142 7620 | 0.0147 5028 | 0.0169 9854 | 0.0177 8886 |
| 84 | 0.0136 6881 | 0.0141 3391 | 0.0146 0855 | 0.0168 6067 | 0.0176 5273 |
| 85 | 0.0135 2944 | 0.0139 9500 | 0.0144 7021 | 0.0167 2619 | 0.0175 1998 |
| 86 | 0.0133 9333 | 0.0138 5935 | 0.0143 3513 | 0.0165 9497 | 0.0173 9050 |
| 87 | 0.0132 6038 | 0.0137 2685 | 0.0142 0320 | 0.0164 6691 | 0.0172 6417 |
| 89 | 0.0131 3046 | 0.0135 9740 | 0.0140 7431 | 0.0163 4190 | 0.0171 4089 |
| 89 | 0.0130 0349 | 0.0134 7088 | 0.0139 4837 | 0.0162 1982 | 0.0170 2056 |
| 90 | 0.0128 7936 | 0.0133 4721 | 0.0138 2527 | 0.0161 0060 | 0.0169 0306 |
| 91 | 0.0127 5797 | 0.0132 2629 | 0.0137 0493 | 0.0159 8413 | 0.0167 8832 |
| 92 | 0.0126 3925 | 0.0131 0803 | 0.0135 8724 | 0.0158 7031 | 0.0166 7624 |
| 93 | 0.0125 2310 | 0.0129 9234 | 0.0134 7213 | 0.0157 5908 | 0.0165 6673 |
| 94 | 0.0124 0944 | 0.0128 7915 | 0.0133 5950 | 0.0156 5033 | 0.0164 5971 |
| 95 | 0.0122 9819 | 0.0127 6837 | 0.0132 4930 | 0.0155 4401 | 0.0163 5511 |
| 96 | 0.0121 8928 | 0.0126 5992 | 0.0131 4143 | 0.0154 4002 | 0.0162 5284 |
| 97 | 0.0120 8263 | 0.0125 5374 | 0.0130 3583 | 0.0153 3829 | 0.0161 5284 |
| 98 | 0.0119 7818 | 0.0124 4976 | 0.0129 3242 | 0.0152 3877 | 0.0160 5503 |
| 99 | 0.0118 7585 | 0.0123 4790 | 0.0128 3115 | 0.0151 4137 | 0.0159 5936 |
| 100 | 0.0117 7559 | 0.0122 4811 | 0.0127 3194 | 0.0150 4604 | 0.0158 6574 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 1 | 1.0112 5000 | 1.0125 0000 | 1.0137 5000 | 1.0150 0000 | 1.0175 0000 |
| 2 | 0.5084 5323 | 0.5093 9441 | 0.5103 3597 | 0.5112 7792 | 0.5131 6295 |
| 3 | 0.3408 6130 | 0.3417 0117 | 0.3425 4173 | 0.3433 8296 | 0.3450 6746 |
| 4 | 0.2570 7058 | 0.2578 6102 | 0.2586 5243 | 0.2594 4478 | 0.2610 3237 |
| 5 | 0.2068 0034 | 0.2075 6211 | 0.2083 2510 | 0.2090 8932 | 0.2106 2142 |
| 6 | 0.1732 9034 | 0.1740 3381 | 0.1747 7877 | 0.1755 2521 | 0.1770 2256 |
| 7 | 0.1493 5762 | 0.1500 8872 | 0.1508 2157 | 0.1515 5616 | 0.1530 3059 |
| 8 | 0.1314 1071 | 0.1321 3314 | 0.1328 5758 | 0.1335 8402 | 0.1350 4292 |
| 9 | 0.1174 5432 | 0.1181 7055 | 0.1188 8806 | 0.1196 0982 | 0.1210 5813 |
| 10 | 0.1062 9131 | 0.1070 0307 | 0.1077 1737 | 0.1084 3418 | 0.1098 7534 |
| 11 | 0.0971 5984 | 0.0978 6839 | 0.0985 7973 | 0.0992 9384 | 0.1007 3038 |
| 12 | 0.0895 5203 | 0.0902 5831 | 0.0909 6764 | 0.0916 7999 | 0.0931 1377 |
| 13 | 0.0831 1626 | 0.0838 2100 | 0.0845 2303 | 0.0852 4036 | 0.0866 7283 |
| 14 | 0.0776 0138 | 0.0783 0515 | 0.0790 1246 | 0.0797 2332 | 0.0811 5562 |
| 15 | 0.0728 2321 | 0.0735 2646 | 0.0742 3351 | 0.0749 4436 | 0.0763 7739 |
| 16 | 0.0686 4363 | 0.0693 4672 | 0.0700 5388 | 0.0707 6508 | 0.0721 9958 |
| 17 | 0.0649 5698 | 0.0656 6023 | 0.0663 6780 | 0.0670 7966 | 0.0685 1623 |
| 18 | 0.0616 8113 | 0.0623 8479 | 0.0630 9301 | 0.0638 0578 | 0.0652 4492 |
| 19 | 0.0587 5120 | 0.0594 5548 | 0.0601 6457 | 0.0608 7847 | 0.0623 2061 |
| 20 | 0.0561 1531 | 0.0568 2039 | 0.0575 3054 | 0.0582 4574 | 0.0596 9122 |
| 21 | 0.0537 3145 | 0.0544 3748 | 0.0551 4884 | 0.0558 6550 | 0.0573 1464 |
| 22 | 0.0515 6525 | 0.0522 7238 | 0.0529 8507 | 0.0537 0331 | 0.0551 5638 |
| 23 | 0.0495 8833 | 0.0502 9666 | 0.0510 1080 | 0.0517 3075 | 0.0531 8796 |
| 24 | 0.0477 7701 | 0.0484 8665 | 0.0492 0235 | 0.0499 2410 | 0.0513 8565 |
| 25 | 0.0461 1144 | 0.0468 2247 | 0.0475 3981 | 0.0482 6345 | 0.0497 2952 |
| 26 | 0.0445 7479 | 0.0452 8729 | 0.0460 0635 | 0.0467 3196 | 0.0482 0269 |
| 27 | 0.0431 5273 | 0.0438 6677 | 0.0445 8763 | 0.0453 1527 | 0.0467 9079 |
| 28 | 0.0418 3299 | 0.0425 4863 | 0.0432 7134 | 0.0440 0108 | 0.0454 8151 |
| 29 | 0.0406 0498 | 0.0413 2228 | 0.0420 4689 | 0.0427 7878 | 0.0442 6424 |
| 30 | 0.0394 5953 | 0.0401 7854 | 0.0409 0511 | 0.0416 3919 | 0.0431 2975 |
| 31 | 0.0383 8866 | 0.0391 0942 | 0.0398 3798 | 0.0405 7430 | 0.0420 7005 |
| 32 | 0.0373 8535 | 0.0381 0791 | 0.0388 3850 | 0.0395 7710 | 0.0410 7812 |
| 33 | 0.0364 4349 | 0.0371 6786 | 0.0379 0053 | 0.0386 4144 | 0.0401 4779 |
| 34 | 0.0355 5763 | 0.0362 8387 | 0.0370 1864 | 0.0377 6189 | 0.0392 7363 |
| 35 | 0.0347 2299 | 0.0354 5111 | 0.0361 8801 | 0.0369 3363 | 0.0384 5082 |
| 36 | 0.0339 3529 | 0.0346 6533 | 0.0354 0438 | 0.0361 5240 | 0.0376 7507 |
| 37 | 0.0331 9072 | 0.0339 2270 | 0.0346 6394 | 0.0354 1437 | 0.0369 4257 |
| 38 | 0.0324 8589 | 0.0332 1983 | 0.0339 6327 | 0.0347 1613 | 0.0362 4990 |
| 39 | 0.0318 1773 | 0.0325 5365 | 0.0332 9931 | 0.0340 5463 | 0.0355 9399 |
| 40 | 0.0311 8349 | 0.0319 2141 | 0.0326 6931 | 0.0334 2710 | 0.0349 7209 |
| 41 | 0.0305 8069 | 0.0313 2063 | 0.0320 7078 | 0.0328 3106 | 0.0343 8170 |
| 42 | 0.0300 0709 | 0.0307 4906 | 0.0315 0148 | 0.0322 6426 | 0.0338 2057 |
| 43 | 0.0294 6064 | 0.0302 0466 | 0.0309 5936 | 0.0317 2465 | 0.0332 8666 |
| 44 | 0.0289 3949 | 0.0296 8557 | 0.0304 4257 | 0.0312 1038 | 0.0327 7810 |
| 45 | 0.0284 4197 | 0.0291 9012 | 0.0299 4941 | 0.0307 1976 | 0.0322 9321 |
| 46 | 0.0279 6652 | 0.0287 1675 | 0.0294 7836 | 0.0302 5125 | 0.0318 3043 |
| 47 | 0.0275 1173 | 0.0282 6406 | 0.0290 2799 | 0.0298 0342 | 0.0313 8836 |
| 48 | 0.0270 7632 | 0.0278 3075 | 0.0285 9701 | 0.0293 7500 | 0.0309 6569 |
| 49 | 0.0266 5910 | 0.0274 1563 | 0.0281 8424 | 0.0289 6478 | 0.0305 6124 |
| 50 | 0.0262 5898 | 0.0270 1763 | 0.0277 8857 | 0.0285 7168 | 0.0301 7391 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
|---|---|---|---|---|---|
| 51 | 0.0258 7494 | 0.0266 3571 | 0.0274 0900 | 0.0281 9469 | 0.0298 0269 |
| 52 | 0.0255 0606 | 0.0262 6897 | 0.0270 4461 | 0.0278 3287 | 0.0294 4665 |
| 53 | 0.0251 5149 | 0.0259 1653 | 0.0266 9453 | 0.0274 8537 | 0.0291 0492 |
| 54 | 0.0248 1043 | 0.0255 7760 | 0.0263 5797 | 0.0271 5138 | 0.0287 7672 |
| 55 | 0.0244 8213 | 0.0252 5145 | 0.0260 3418 | 0.0268 3018 | 0.0284 6129 |
| 56 | 0.0241 6592 | 0.0249 3739 | 0.0257 2249 | 0.0265 2106 | 0.0281 5795 |
| 57 | 0.0238 6116 | 0.0246 3478 | 0.0254 2225 | 0.0262 2341 | 0.0278 6606 |
| 58 | 0.0235 6726 | 0.0243 4303 | 0.0251 3287 | 0.0259 3661 | 0.0275 8503 |
| 59 | 0.0232 8366 | 0.0240 6158 | 0.0248 5380 | 0.0256 6012 | 0.0273 1430 |
| 60 | 0.0230 0985 | 0.0237 8993 | 0.0245 8452 | 0.0253 9343 | 0.0270 5336 |
| 61 | 0.0227 4534 | 0.0235 2758 | 0.0243 2455 | 0.0251 3604 | 0.0268 0172 |
| 62 | 0.0224 8969 | 0.0232 7410 | 0.0240 7344 | 0.0248 8751 | 0.0265 5892 |
| 63 | 0.0222 4247 | 0.0230 2904 | 0.0238 3076 | 0.0246 4741 | 0.0263 2455 |
| 44 | 0.0220 0329 | 0.0227 9203 | 0.0235 9612 | 0.0244 1534 | 0.0260 9821 |
| 65 | 0.0217 7178 | 0.0225 6268 | 0.0233 6914 | 0.0241 9094 | 0.0258 7952 |
| 66 | 0.0215 4758 | 0.0223 4065 | 0.0231 4949 | 0.0239 7386 | 0.0256 6813 |
| 67 | 0.0213 3037 | 0.0221 2560 | 0.0229 3682 | 0.0237 6376 | 0.0254 6372 |
| 68 | 0.0211 1985 | 0.0219 1724 | 0.0227 3082 | 0.0235 6033 | 0.0252 6596 |
| 69 | 0.0209 1571 | 0.0217 1527 | 0.0225 3122 | 0.0233 6329 | 0.0250 7459 |
| 70 | 0.0207 1769 | 0.0215 1941 | 0.0223 3773 | 0.0231 7235 | 0.0248 8930 |
| 71 | 0.0205 2552 | 0.0213 2941 | 0.0221 5009 | 0.0229 8727 | 0.0247 0985 |
| 72 | 0.0203 3896 | 0.0211 4501 | 0.0219 6806 | 0.0228 0779 | 0.0245 3600 |
| 73 | 0.0201 5779 | 0.0209 6600 | 0.0217 9140 | 0.0226 3368 | 0.0243 6750 |
| 74 | 0.0199 8177 | 0.0207 9215 | 0.0216 1991 | 0.0224 6473 | 0.0242 0413 |
| 75 | 0.0198 1072 | 0.0206 2325 | 0.0214 5336 | 0.0223 0072 | 0.0240 4570 |
| 76 | 0.0196 4442 | 0.0204 5910 | 0.0212 9157 | 0.0221 4146 | 0.0238 9200 |
| 77 | 0.0194 8269 | 0.0202 9953 | 0.0211 3435 | 0.0219 8676 | 0.0237 4284 |
| 78 | 0.0193 2536 | 0.0201 4435 | 0.0209 8151 | 0.0218 3645 | 0.0235 9806 |
| 79 | 0.0191 7226 | 0.0199 9341 | 0.0208 3290 | 0.0216 9036 | 0.0234 5748 |
| 80 | 0.0190 2323 | 0.0198 4652 | 0.0206 8836 | 0.0215 4832 | 0.0233 2093 |
| 81 | 0.0188 7812 | 0.0197 0356 | 0.0205 4772 | 0.0214 1019 | 0.0231 8828 |
| 82 | 0.0187 3678 | 0.0195 6437 | 0.0204 1086 | 0.0212 7583 | 0.0230 5936 |
| 83 | 0.0185 9908 | 0.0194 2881 | 0.0202 7762 | 0.0211 4509 | 0.0229 3406 |
| 84 | 0.0184 6489 | 0.0192 9675 | 0.0201 4789 | 0.0210 1784 | 0.0228 1223 |
| 85 | 0.0183 3409 | 0.0191 6808 | 0.0200 2153 | 0.0208 9396 | 0.0226 9375 |
| 86 | 0.0182 0654 | 0.0190 4267 | 0.0198 9843 | 0.0207 7333 | 0.0225 7850 |
| 87 | 0.0180 8215 | 0.0189 2041 | 0.0197 7847 | 0.0206 5584 | 0.0224 6636 |
| 88 | 0.0179 6081 | 0.0188 0119 | 0.0196 6155 | 0.0205 4138 | 0.0223 5724 |
| 89 | 0.0178 4240 | 0.0186 8490 | 0.0195 4756 | 0.0204 2984 | 0.0222 5102 |
| 90 | 0.0177 2684 | 0.0185 7146 | 0.0194 3641 | 0.0203 2113 | 0.0221 4760 |
| 91 | 0.0176 1403 | 0.0184 6076 | 0.0193 2799 | 0.0202 1516 | 0.0220 4690 |
| 92 | 0.0175 0387 | 0.0983 5271 | 0.0192 2222 | 0.0201 1182 | 0.0219 4882 |
| 93 | 0.0173 9629 | 0.0182 4724 | 0.0191 1902 | 0.0200 1104 | 0.0218 5327 |
| 94 | 0.0172 9119 | 0.0181 4425 | 0.0190 1829 | 0.0199 1273 | 0.0217 6017 |
| 95 | 0.0171 8851 | 0.0180 4366 | 0.0189 1997 | 0.0198 1681 | 0.0216 6944 |
| 96 | 0.0170 8816 | 0.0179 4540 | 0.0188 2397 | 0.0197 2321 | 0.0215 8101 |
| 97 | 0.0169 9007 | 0.0178 4941 | 0.0187 3022 | 0.0196 3186 | 0.0214 9480 |
| 98 | 0.0168 9418 | 0.0177 5560 | 0.0186 3866 | 0.0195 4268 | 0.0214 1074 |
| 99 | 0.0168 0041 | 0.0176 6391 | 0.0185 4921 | 0.0194 5560 | 0.0213 2876 |
| 100 | 0.0167 0870 | 0.0175 7428 | 0.0184 6181 | 0.0193 7057 | 0.0212 4880 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | 2% | $2\frac{1}{4}\%$ | $2\frac{1}{2}\%$ | $2\frac{3}{4}\%$ | 3% |
|---|---|---|---|---|---|
| 1 | 1.0200 0000 | 1.0225 0000 | 1.0250 0000 | 1.0275 0000 | 1.0300 0000 |
| 2 | 0.5150 4950 | 0.5169 3758 | 0.5188 2716 | 0.5207 1825 | 0.5226 1084 |
| 3 | 0.3467 5467 | 0.3484 4458 | 0.3501 3717 | 0.3518 3243 | 0.3535 3036 |
| 4 | 0.2626 2375 | 0.2642 1893 | 0.2658 1788 | 0.2674 2059 | 0.2690 2705 |
| 5 | 0.2121 5839 | 0.2137 0021 | 0.2152 4686 | 0.2167 9832 | 0.2183 5457 |
| 6 | 0.1785 2581 | 0.1800 3496 | 0.1815 4997 | 0.1830 7083 | 0.1845 9750 |
| 7 | 0.1545 1196 | 0.1560 0025 | 0.1574 9543 | 0.1589 9747 | 0.1605 0635 |
| 8 | 0.1365 0980 | 0.1379 8462 | 0.1394 6735 | 0.1409 5795 | 0.1424 5639 |
| 9 | 0.1225 1544 | 0.1239 8170 | 0.1254 5689 | 0.1269 4095 | 0.1284 3386 |
| 10 | 0.1113 2653 | 0.1127 8768 | 0.1142 5876 | 0.1157 3972 | 0.1172 3051 |
| 11 | 0.1021 7794 | 0.1036 3649 | 0.1051 0596 | 0.1065 8629 | 0.1080 7745 |
| 12 | 0.0945 5960 | 0.0960 1740 | 0.0974 8713 | 0.0989 6871 | 0.1004 6209 |
| 13 | 0.0881 1835 | 0.0895 7686 | 0.0910 4827 | 0.0925 3252 | 0.0940 2954 |
| 14 | 0.0826 0197 | 0.0840 6230 | 0.0855 3653 | 0.0870 2457 | 0.0885 2634 |
| 15 | 0.0778 2547 | 0.0792 8852 | 0.0807 6646 | 0.0822 5917 | 0.0837 6658 |
| 16 | 0.0736 5013 | 0.0751 1663 | 0.0765 9899 | 0.0780 9710 | 0.0796 1085 |
| 17 | 0.0699 6984 | 0.0714 4039 | 0.0729 2777 | 0.0744 3186 | 0.0759 5253 |
| 18 | 0.0667 0210 | 0.0681 7720 | 0.0696 7008 | 0.0711 8063 | 0.0727 0870 |
| 19 | 0.0637 8177 | 0.0652 6182 | 0.0667 6062 | 0.0682 7802 | 0.0698 1388 |
| 20 | 0.0611 5672 | 0.0626 4207 | 0.0641 4713 | 0.0656 7173 | 0.0672 1571 |
| 21 | 0.0587 8477 | 0.0602 7572 | 0.0617 8733 | 0.0633 1941 | 0.0648 7178 |
| 22 | 0.0566 3140 | 0.0581 2821 | 0.0596 4661 | 0.0611 8640 | 0.0627 4730 |
| 23 | 0.0546 6810 | 0.0561 7097 | 0.0576 9638 | 0.0592 4410 | 0.0608 1380 |
| 24 | 0.0528 7110 | 0.0543 8023 | 0.0559 1282 | 0.0574 6863 | 0.0590 4742 |
| 25 | 0.0512 2044 | 0.0527 3599 | 0.0542 7592 | 0.0558 3997 | 0.0574 2787 |
| 26 | 0.0496 9923 | 0.0512 2134 | 0.0527 6875 | 0.0543 4116 | 0.0559 3829 |
| 27 | 0.0482 9309 | 0.0498 2188 | 0.0513 7687 | 0.0529 5776 | 0.0545 6421 |
| 28 | 0.0469 8967 | 0.0485 2525 | 0.0500 8793 | 0.0516 7738 | 0.0532 9323 |
| 29 | 0.0457 7836 | 0.0473 2081 | 0.0488 9127 | 0.0504 8935 | 0.0521 1467 |
| 30 | 0.0446 4992 | 0.0461 9934 | 0.0477 7764 | 0.0493 8442 | 0.0510 1926 |
| 31 | 0.0435 9635 | 0.0451 5280 | 0.0467 3300 | 0.0483 5453 | 0.0499 9893 |
| 32 | 0.0426 1061 | 0.0441 7415 | 0.0457 6831 | 0.0473 9263 | 0.0490 4662 |
| 33 | 0.0416 8653 | 0.0432 5722 | 0.0448 5938 | 0.0464 9253 | 0.0481 5612 |
| 34 | 0.0408 1867 | 0.0423 9655 | 0.0440 0675 | 0.0456 4875 | 0.0473 2196 |
| 35 | 0.0400 0221 | 0.0415 8731 | 0.0432 0558 | 0.0448 5645 | 0.0465 3929 |
| 36 | 0.0392 3285 | 0.0408 2522 | 0.0424 5158 | 0.0441 1132 | 0.0458 0379 |
| 37 | 0.0385 0678 | 0.0401 0643 | 0.0417 4090 | 0.0434 0953 | 0.0451 1162 |
| 38 | 0.0378 5207 | 0.0394 2753 | 0.0410 7012 | 0.0427 4764 | 0.0444 5934 |
| 39 | 0.0371 7114 | 0.0387 8543 | 0.0404 3615 | 0.0421 2256 | 0.0438 4385 |
| 40 | 0.0365 5575 | 0.0381 7738 | 0.0398 3623 | 0.0415 3151 | 0.0432 6238 |
| 41 | 0.0359 7188 | 0.0376 0087 | 0.0392 6786 | 0.0409 7200 | 0.0427 1241 |
| 42 | 0.0354 1729 | 0.0370 5364 | 0.0387 2876 | 0.0404 4175 | 0.0421 9167 |
| 43 | 0.0348 8993 | 0.0365 3364 | 0.0382 1688 | 0.0399 3871 | 0.0416 9811 |
| 44 | 0.0343 8794 | 0.0360 3901 | 0.0377 3037 | 0.0394 6100 | 0.0412 2985 |
| 45 | 0.0339 0962 | 0.0355 6805 | 0.0372 6752 | 0.0390 0693 | 0.0407 8518 |
| 46 | 0.0334 5342 | 0.0351 1921 | 0.0368 2676 | 0.0385 7493 | 0.0403 6254 |
| 47 | 0.0330 1792 | 0.0346 9107 | 0.0364 0669 | 0.0381 6358 | 0.0399 6051 |
| 48 | 0.0326 0184 | 0.0342 8233 | 0.0360 0599 | 0.0377 7158 | 0.0395 7777 |
| 49 | 0.0322 0396 | 0.0338 9179 | 0.0356 2348 | 0.0373 9773 | 0.0392 1314 |
| 50 | 0.0318 2321 | 0.0335 1836 | 0.0352 5806 | 0.0370 4092 | 0.0388 6550 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | 2% | $2\frac{1}{4}\%$ | $2\frac{1}{2}\%$ | $2\frac{3}{4}\%$ | 3% |
|---|---|---|---|---|---|
| 51 | 0.0314 5856 | 0.0331 6102 | 0.0349 0870 | 0.0367 0014 | 0.0385 3382 |
| 52 | 0.0311 0909 | 0.0328 1884 | 0.0345 7446 | 0.0363 7444 | 0.0382 1718 |
| 53 | 0.0307 7392 | 0.0324 9094 | 0.0342 5449 | 0.0360 6297 | 0.0379 1471 |
| 54 | 0.0304 5226 | 0.0321 7654 | 0.0339 4799 | 0.0357 6491 | 0.0376 2558 |
| 55 | 0.0301 4337 | 0.0318 7489 | 0.0336 5419 | 0.0354 7953 | 0.0373 4907 |
| 56 | 0.0298 4656 | 0.0315 8530 | 0.0333 7243 | 0.0352 0612 | 0.0370 8447 |
| 57 | 0.0295 6120 | 0.0313 0712 | 0.0331 0204 | 0.0349 4404 | 0.0368 3114 |
| 58 | 0.0292 8667 | 0.0310 3977 | 0.0328 4244 | 0.0346 9270 | 0.0365 8848 |
| 59 | 0.0290 2243 | 0.0307 8268 | 0.0325 9307 | 0.0344 5153 | 0.0363 5593 |
| 60 | 0.0287 6797 | 0.0305 3533 | 0.0323 5340 | 0.0342 2002 | 0.0361 3296 |
| 61 | 0.0285 2278 | 0.0302 9724 | 0.0321 2294 | 0.0339 9767 | 0.0359 1908 |
| 62 | 0.0282 8643 | 0.0300 6795 | 0.0319 0126 | 0.0337 8402 | 0.0357 1385 |
| 63 | 0.0280 5848 | 0.0298 4704 | 0.0316 8790 | 0.0335 7866 | 0.0355 1682 |
| 64 | 0.0278 3855 | 0.0296 3411 | 0.0314 8249 | 0.0333 8118 | 0.0353 2760 |
| 65 | 0.0276 2624 | 0.0294 2878 | 0.0312 8463 | 0.0331 9120 | 0.0351 4581 |
| 66 | 0.0274 2122 | 0.0292 3070 | 0.0310 9398 | 0.0330 0837 | 0.0349 7110 |
| 67 | 0.0272 2316 | 0.0290 3955 | 0.0309 1021 | 0.0328 3236 | 0.0348 0313 |
| 68 | 0.0270 3173 | 0.0288 5500 | 0.0307 3300 | 0.0326 6285 | 0.0346 4159 |
| 69 | 0.0268 4665 | 0.0286 7677 | 0.0305 6206 | 0.0324 9955 | 0.0344 8618 |
| 70 | 0.0266 6765 | 0.0285 0458 | 0.0303 9712 | 0.0323 4218 | 0.0343 3663 |
| 71 | 0.0264 9446 | 0.0283 3816 | 0.0302 3790 | 0.0321 9048 | 0.0341 9266 |
| 72 | 0.0263 2683 | 0.0281 7728 | 0.0300 8417 | 0.0320 4420 | 0.0340 5404 |
| 73 | 0.0261 6454 | 0.0280 2169 | 0.0299 3568 | 0.0319 0311 | 0.0339 2053 |
| 74 | 0.0260 0736 | 0.0278 7118 | 0.0297 9222 | 0.0317 6698 | 0.0337 9191 |
| 75 | 0.0258 5508 | 0.0277 2554 | 0.0296 5358 | 0.0316 3560 | 0.0336 6796 |
| 76 | 0.0257 0751 | 0.0275 8457 | 0.0295 1956 | 0.0315 0878 | 0.0335 4849 |
| 77 | 0.0255 6447 | 0.0274 4808 | 0.0293 8997 | 0.0313 8633 | 0.0334 3331 |
| 78 | 0.0254 2576 | 0.0273 1589 | 0.0292 6463 | 0.0312 6806 | 0.0333 2224 |
| 79 | 0.0252 9123 | 0.0271 8784 | 0.0291 4338 | 0.0311 5382 | 0.0332 1510 |
| 80 | 0.0251 6071 | 0.0270 6376 | 0.0290 2605 | 0.0310 4342 | 0.0331 1175 |
| 81 | 0.0250 3405 | 0.0269 4350 | 0.0289 1248 | 0.0309 3674 | 0.0330 1201 |
| 82 | 0.0249 1110 | 0.0268 2692 | 0.0288 0254 | 0.0308 3361 | 0.0329 1576 |
| 83 | 0.0247 9173 | 0.0267 1387 | 0.0286 9608 | 0.0307 3389 | 0.0328 2284 |
| 84 | 0.0246 7581 | 0.0266 0423 | 0.0285 9298 | 0.0306 3747 | 0.0327 3313 |
| 85 | 0.0245 6321 | 0.0264 9787 | 0.0284 9310 | 0.0305 4420 | 0.0326 4650 |
| 86 | 0.0244 5381 | 0.0263 9467 | 0.0283 9633 | 0.0304 5397 | 0.0325 6284 |
| 87 | 0.0243 4750 | 0.0262 9452 | 0.0283 0255 | 0.0303 6667 | 0.0324 8202 |
| 88 | 0.0242 4416 | 0.0261 9730 | 0.0282 1165 | 0.0302 8219 | 0.0324 0393 |
| 89 | 0.0241 4370 | 0.0261 0291 | 0.0281 2353 | 0.0302 0041 | 0.0323 2848 |
| 90 | 0.0240 4602 | 0.0260 1126 | 0.0280 3809 | 0.0301 2125 | 0.0322 5556 |
| 91 | 0.0239 5101 | 0.0259 2224 | 0.0279 5523 | 0.0300 4460 | 0.0321 8508 |
| 92 | 0.0238 5859 | 0.0258 3577 | 0.0278 7486 | 0.0299 7038 | 0.0321 1694 |
| 93 | 0.0237 6868 | 0.0257 5176 | 0.0277 9690 | 0.0298 9850 | 0.0320 5107 |
| 94 | 0.0236 8118 | 0.0256 7012 | 0.0277 2126 | 0.0298 2887 | 0.0319 8737 |
| 95 | 0.0235 9602 | 0.0255 9078 | 0.0276 4786 | 0.0297 6141 | 0.0319 2577 |
| 96 | 0.0235 1313 | 0.0255 1366 | 0.0275 7662 | 0.0296 9605 | 0.0318 6619 |
| 97 | 0.0234 3242 | 0.0254 3868 | 0.0275 0747 | 0.0296 3272 | 0.0318 0856 |
| 98 | 0.0233 5383 | 0.0253 6578 | 0.0274 4034 | 0.0295 7134 | 0.0317 5281 |
| 99 | 0.0232 7729 | 0.0252 9489 | 0.0273 7517 | 0.0295 1185 | 0.0316 9886 |
| 100 | 0.0232 0274 | 0.0252 2594 | 0.0273 1188 | 0.0294 5418 | 0.0316 4667 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 1 | 1.0350 0000 | 1.0400 0000 | 1.0450 0000 | 1.0500 0000 | 1.0550 0000 |
| 2 | 0.5264 0049 | 0.5301 9608 | 0.5339 9756 | 0.5378 0488 | 0.5416 1800 |
| 3 | 0.3569 3418 | 0.3603 4854 | 0.3637 7336 | 0.3672 0856 | 0.3706 5407 |
| 4 | 0.2722 5114 | 0.2754 9005 | 0.2787 4365 | 0.2820 1183 | 0.2852 9449 |
| 5 | 0.2214 8137 | 0.2246 2711 | 0.2277 9164 | 0.2309 7480 | 0.2341 7644 |
| 6 | 0.1876 6821 | 0.1907 6190 | 0.1938 7839 | 0.1970 1747 | 0.2001 7895 |
| 7 | 0.1635 4449 | 0.1666 0961 | 0.1697 0147 | 0.1728 1982 | 0.1759 6442 |
| 8 | 0.1454 7665 | 0.1485 2783 | 0.1516 0965 | 0.1547 2181 | 0.1578 6401 |
| 9 | 0.1314 4601 | 0.1344 9299 | 0.1375 7447 | 0.1406 9008 | 0.1438 3946 |
| 10 | 0.1202 4137 | 0.1232 9094 | 0.1263 7882 | 0.1295 0458 | 0.1326 6777 |
| 11 | 0.1110 9197 | 0.1141 4904 | 0.1172 4818 | 0.1203 8889 | 0.1235 7065 |
| 12 | 0.1034 8395 | 0.1065 5217 | 0.1096 6619 | 0.1128 2541 | 0.1160 2923 |
| 13 | 0.0970 6157 | 0.1001 4373 | 0.1032 7535 | 0.1064 5577 | 0.1096 8426 |
| 14 | 0.0915 7073 | 0.0946 6897 | 0.0978 2032 | 0.1010 2397 | 0.1042 7912 |
| 15 | 0.0868 2507 | 0.0899 4110 | 0.0931 1381 | 0.0963 4229 | 0.0996 2560 |
| 16 | 0.0826 8483 | 0.0858 2000 | 0.0890 1537 | 0.0922 6991 | 0.0955 8254 |
| 17 | 0.0790 4313 | 0.0821 9852 | 0.0854 1758 | 0.0886 9914 | 0.0920 4197 |
| 18 | 0.0758 1684 | 0.0789 9333 | 0.0822 3690 | 0.0855 4622 | 0.0889 1992 |
| 19 | 0.0729 4033 | 0.0761 3862 | 0.0794 0734 | 0.0827 4501 | 0.0861 5006 |
| 20 | 0.0703 6108 | 0.0735 8175 | 0.0768 7614 | 0.0802 4259 | 0.0836 7933 |
| 21 | 0.0680 3659 | 0.0712 8011 | 0.0746 0057 | 0.0779 9611 | 0.0814 6478 |
| 22 | 0.0659 3207 | 0.0691 9881 | 0.0725 4565 | 0.0759 7051 | 0.0794 7123 |
| 23 | 0.0640 1880 | 0.0673 0906 | 0.0706 8249 | 0.0741 3682 | 0.0776 6965 |
| 24 | 0.0622 7283 | 0.0655 8683 | 0.0689 8703 | 0.0724 7090 | 0.0760 3580 |
| 25 | 0.0606 7404 | 0.0640 1196 | 0.0674 3903 | 0.0709 5246 | 0.0745 4935 |
| 26 | 0.0592 0540 | 0.0625 6738 | 0.0660 2137 | 0.0695 6432 | 0.0731 9307 |
| 27 | 0.0578 5241 | 0.0612 3854 | 0.0647 1946 | 0.0682 9186 | 0.0719 5228 |
| 28 | 0.0566 0265 | 0.0600 1298 | 0.0635 2081 | 0.0671 2253 | 0.0708 1440 |
| 29 | 0.0554 4538 | 0.0588 7993 | 0.0624 1461 | 0.0660 4551 | 0.0697 6857 |
| 30 | 0.0543 7133 | 0.0578 3010 | 0.0613 9154 | 0.0650 5144 | 0.0688 0539 |
| 31 | 0.0533 7240 | 0.0568 5535 | 0.0604 4345 | 0.0641 3212 | 0.0679 1665 |
| 32 | 0.0524 4150 | 0.0559 4859 | 0.0595 6320 | 0.0632 8042 | 0.0670 9519 |
| 33 | 0.0515 7242 | 0.0551 0357 | 0.0587 4453 | 0.0624 9004 | 0.0663 3469 |
| 34 | 0.0507 5966 | 0.0543 1477 | 0.0579 8191 | 0.0617 5545 | 0.0656 2958 |
| 35 | 0.0499 9835 | 0.0535 7732 | 0.0572 7045 | 0.0610 7171 | 0.0649 7493 |
| 36 | 0.0492 8416 | 0.0528 8688 | 0.0566 0578 | 0.0604 3446 | 0.0643 6635 |
| 37 | 0.0486 1325 | 0.0522 3957 | 0.0559 8402 | 0.0598 3979 | 0.0637 9993 |
| 38 | 0.0479 8214 | 0.0516 3192 | 0.0554 0169 | 0.0592 8423 | 0.0632 7217 |
| 39 | 0.0473 8775 | 0.0510 6083 | 0.0548 5567 | 0.0587 6462 | 0.0627 7991 |
| 40 | 0.0468 2728 | 0.0505 2349 | 0.0543 4315 | 0.0582 7816 | 0.0623 2034 |
| 41 | 0.0462 9822 | 0.0500 1738 | 0.0538 6158 | 0.0578 2229 | 0.0618 9090 |
| 42 | 0.0457 9828 | 0.0495 4020 | 0.0534 0868 | 0.0573 9471 | 0.0614 8927 |
| 43 | 0.0453 2539 | 0.0490 8989 | 0.0529 8235 | 0.0569 9333 | 0.0611 1337 |
| 44 | 0.0448 7768 | 0.0486 6454 | 0.0525 8071 | 0.0566 1625 | 0.0607 6128 |
| 45 | 0.0444 5343 | 0.0482 6246 | 0.0522 0202 | 0.0562 6173 | 0.0604 3127 |
| 46 | 0.0440 5108 | 0.0478 8205 | 0.0518 4471 | 0.0559 2820 | 0.0601 2175 |
| 47 | 0.0436 6919 | 0.0475 2189 | 0.0515 0734 | 0.0556 1421 | 0.0598 3129 |
| 48 | 0.0433 0646 | 0.0471 8065 | 0.0511 8858 | 0.0553 1843 | 0.0595 5854 |
| 49 | 0.0429 6167 | 0.0468 5712 | 0.0508 8722 | 0.0550 3965 | 0.0593 0230 |
| 50 | 0.0426 3371 | 0.0465 5020 | 0.0506 0215 | 0.0547 7674 | 0.0590 6145 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
|---|---|---|---|---|---|
| 51 | 0.0423 2156 | 0.0462 5885 | 0.0503 3232 | 0.0545 2867 | 0.0588 3495 |
| 52 | 0.0420 2429 | 0.0459 8212 | 0.0500 7679 | 0.0542 9450 | 0.0586 2186 |
| 53 | 0.0417 4100 | 0.0457 1915 | 0.0498 3469 | 0.0540 7334 | 0.0584 2130 |
| 54 | 0.0414 7090 | 0.0454 6910 | 0.0496 0519 | 0.0538 6438 | 0.0582 3245 |
| 55 | 0.0412 1323 | 0.0452 3124 | 0.0493 8754 | 0.0536 6686 | 0.0580 5458 |
| 56 | 0.0409 6730 | 0.0450 0487 | 0.0491 8105 | 0.0534 8010 | 0.0578 8698 |
| 57 | 0.0407 3245 | 0.0447 8932 | 0.0489 8506 | 0.0533 0343 | 0.0577 2900 |
| 58 | 0.0405 0810 | 0.0445 8401 | 0.0487 9897 | 0.0531 3626 | 0.0575 8006 |
| 59 | 0.0402 9366 | 0.0443 8836 | 0.0486 2221 | 0.0529 7802 | 0.0574 3959 |
| 60 | 0.0400 8862 | 0.0442 0185 | 0.0484 5426 | 0.0528 2818 | 0.0573 0707 |
| 61 | 0.0398 9249 | 0.0440 2398 | 0.0482 9462 | 0.0526 8627 | 0.0571 8202 |
| 62 | 0.0397 0480 | 0.0438 5430 | 0.0481 4284 | 0.0525 5183 | 0.0570 6400 |
| 63 | 0.0395 2513 | 0.0436 9237 | 0.0479 9848 | 0.0524 2442 | 0.0569 5258 |
| 64 | 0.0393 5308 | 0.0435 3780 | 0.0478 6115 | 0.0523 0365 | 0.0568 4737 |
| 65 | 0.0391 8826 | 0.0433 9019 | 0.0477 3047 | 0.0521 8915 | 0.0567 4800 |
| 66 | 0.0390 3031 | 0.0432 4921 | 0.0476 0608 | 0.0520 8057 | 0.0566 5413 |
| 67 | 0.0388 7892 | 0.0431 1451 | 0.0474 8765 | 0.0519 7757 | 0.0565 6544 |
| 68 | 0.0387 3375 | 0.0429 8578 | 0.0473 7487 | 0.0518 7986 | 0.0564 8163 |
| 69 | 0.0385 9453 | 0.0428 6272 | 0.0472 6745 | 0.0517 8715 | 0.0564 0242 |
| 70 | 0.0384 6095 | 0.0427 4506 | 0.0471 6511 | 0.0516 9915 | 0.0563 2754 |
| 71 | 0.0383 3277 | 0.0426 3253 | 0.0470 6759 | 0.0516 1563 | 0.0562 5675 |
| 72 | 0.0382 0973 | 0.0425 2489 | 0.0469 7465 | 0.0515 3633 | 0.0561 8982 |
| 73 | 0.0380 9160 | 0.0424 2190 | 0.0468 8606 | 0.0514 6103 | 0.0561 2652 |
| 74 | 0.0379 7816 | 0.0423 2334 | 0.0468 0159 | 0.0513 8953 | 0.0560 6665 |
| 75 | 0.0378 6919 | 0.0422 2900 | 0.0467 2104 | 0.0513 2161 | 0.0560 1002 |
| 76 | 0.0377 6450 | 0.0421 3869 | 0.0466 4422 | 0.0512 5709 | 0.0559 5645 |
| 77 | 0.0376 6390 | 0.0420 5221 | 0.0465 7094 | 0.0511 9580 | 0.0559 0577 |
| 78 | 0.0375 6721 | 0.0419 6939 | 0.0465 0104 | 0.0511 3756 | 0.0558 5781 |
| 79 | 0.0374 7426 | 0.0418 9007 | 0.0464 3434 | 0.0510 8222 | 0.0558 1243 |
| 80 | 0.0373 8489 | 0.0418 1408 | 0.0463 7069 | 0.0510 2962 | 0.0557 6948 |
| 81 | 0.0372 9894 | 0.0417 4127 | 0.0463 0995 | 0.0509 7963 | 0.0557 2884 |
| 82 | 0.0372 1628 | 0.0416 7150 | 0.0462 5197 | 0.0509 3211 | 0.0556 9036 |
| 83 | 0.0371 3676 | 0.0416 0463 | 0.0461 9663 | 0.0508 8694 | 0.0556 5395 |
| 84 | 0.0370 6025 | 0.0415 4054 | 0.0461 4379 | 0.0508 4399 | 0.0556 1947 |
| 85 | 0.0369 8662 | 0.0414 7909 | 0.0460 9334 | 0.0508 0316 | 0.0555 8683 |
| 86 | 0.0369 1576 | 0.0414 2018 | 0.0460 4516 | 0.0507 6433 | 0.0555 5593 |
| 87 | 0.0368 4756 | 0.0413 6370 | 0.0459 9915 | 0.0507 2740 | 0.0555 2667 |
| 88 | 0.0367 8190 | 0.0413 0953 | 0.0459 5522 | 0.0506 9228 | 0.0554 9896 |
| 89 | 0.0367 1868 | 0.0412 5758 | 0.0459 1325 | 0.0506 5888 | 0.0554 7273 |
| 90 | 0.0366 5781 | 0.0412 0775 | 0.0458 7316 | 0.0506 2711 | 0.0554 4788 |
| 91 | 0.0365 9919 | 0.0411 5995 | 0.0458 3486 | 0.0505 9689 | 0.0554 2435 |
| 92 | 0.0365 4273 | 0.0411 1410 | 0.0457 9827 | 0.0505 6815 | 0.0554 0207 |
| 93 | 0.0364 8834 | 0.0410 7010 | 0.0457 6331 | 0.0505 4080 | 0.0553 8096 |
| 94 | 0.0364 3594 | 0.0410 2789 | 0.0457 2991 | 0.0505 1478 | 0.0553 6097 |
| 95 | 0.0363 8546 | 0.0409 8738 | 0.0456 9799 | 0.0504 9003 | 0.0553 4204 |
| 96 | 0.0363 3682 | 0.0409 4850 | 0.0456 6749 | 0.0504 6648 | 0.0553 2410 |
| 97 | 0.0362 8995 | 0.0409 1119 | 0.0456 3834 | 0.0504 4407 | 0.0553 0711 |
| 98 | 0.0362 4478 | 0.0408 7538 | 0.0456 1048 | 0.0504 2274 | 0.0552 9101 |
| 99 | 0.0362 0124 | 0.0408 4100 | 0.0455 8385 | 0.0504 0245 | 0.0552 7577 |
| 100 | 0.0361 5927 | 0.0408 0800 | 0.0455 5839 | 0.0503 8314 | 0.0552 6132 |

**TABLE VII. The Annuity Whose Present Value at Compound Interest is 1**

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

| $n$ | 6% | $6\frac{1}{2}$% | 7% | $7\frac{1}{2}$% | 8% |
|---|---|---|---|---|---|
| 1 | 1.0600 0000 | 1.0650 0000 | 1.0700 0000 | 1.0750 0000 | 1.0800 0000 |
| 2 | 0.5454 3689 | 0.5492 6150 | 0.5530 9179 | 0.5569 2771 | 0.5607 6923 |
| 3 | 0.3741 0981 | 0.3775 7570 | 0.3810 5166 | 0.3845 3763 | 0.3880 3351 |
| 4 | 0.2885 9149 | 0.2919 0274 | 0.2952 2812 | 0.2985 6751 | 0.3019 2080 |
| 5 | 0.2373 9640 | 0.2406 3454 | 0.2438 9069 | 0.2471 6472 | 0.2504 5645 |
| 6 | 0.2033 6263 | 0.2065 6831 | 0.2097 9580 | 0.2130 4489 | 0.2163 1539 |
| 7 | 0.1791 3502 | 0.1823 3137 | 0.1855 5322 | 0.1888 0032 | 0.1920 7240 |
| 8 | 0.1610 3594 | 0.1642 3730 | 0.1674 6776 | 0.1707 2702 | 0.1740 1476 |
| 9 | 0.1470 2224 | 0.1502 3803 | 0.1534 8647 | 0.1567 6716 | 0.1600 7971 |
| 10 | 0.1358 6796 | 0.1391 0469 | 0.1423 7750 | 0.1456 8593 | 0.1490 2949 |
| 11 | 0.1267 9294 | 0.1300. 5521 | 0.1333 5690 | 0.1366 9747 | 0.1400 7634 |
| 12 | 0.1192 7703 | 0.1225 6817 | 0.1259 0199 | 0.1292 7783 | 0.1326 9502 |
| 13 | 0.1129 6011 | 0.1162 8256 | 0.1196 5085 | 0.1230 6420 | 0.1265 2181 |
| 14 | 0.1075 8491 | 0.1109 4048 | 0.1143 4494 | 0.1177 9737 | 0.1212 9685 |
| 15 | 0.1029 6276 | 0.1063 5278 | 0.1097 9462 | 0.1132 8724 | 0.1168 2954 |
| 16 | 0.0989 5214 | 0.1023 7757 | 0.1058 5765 | 0.1093 9116 | 0.1129 7687 |
| 17 | 0.0954 4480 | 0.0989 0633 | 0.1024 2519 | 0.1060 0003 | 0.1096 2943 |
| 18 | 0.0923 5654 | 0.0958 5461 | 0.0994 1260 | 0.1030 2896 | 0.1067 0210 |
| 19 | 0.0896 2086 | 0.0931 5575 | 0.0967 5301 | 0.1004 1090 | 0.1041 2763 |
| 20 | 0.0871 8456 | 0.0907 5640 | 0.0943 9293 | 0.0980 9219 | 0.1018 5221 |
| 21 | 0.0850 0455 | 0.0886 1333 | 0.0922 8900 | 0.0960 2937 | 0.0998 3225 |
| 22 | 0.0830 4557 | 0.0866 9120 | 0.0904 0577 | 0.0941 8687 | 0.0980 3207 |
| 23 | 0.0812 7848 | 0.0849 6078 | 0.0887 1393 | 0.0925 3528 | 0.0964 2217 |
| 24 | 0.0796 7900 | 0.0833 9770 | 0.0871 8902 | 0.0910 5008 | 0.0949 7796 |
| 25 | 0.0782 2672 | 0.0819 8148 | 0.0858 1052 | 0.0897 1067 | 0.0936 7878 |
| 26 | 0.0769 0435 | 0.0806 9480 | 0.0845 6103 | 0.0884 9961 | 0.0925 0713 |
| 27 | 0.0756 9717 | 0.0795 2288 | 0.0834 2573 | 0.0874 0204 | 0.0914 4809 |
| 28 | 0.0745 9255 | 0.0784 5305 | 0.0823 9193 | 0.0864 0520 | 0.0904 8891 |
| 29 | 0.0735 7961 | 0.0774 7440 | 0.0814 4865 | 0.0854 9811 | 0.0896 1854 |
| 30 | 0.0726 4891 | 0.0765 7744 | 0.0805 8640 | 0.0846 7124 | 0.0888 2743 |
| 31 | 0.0717 9222 | 0.0757 5393 | 0.0797 9691 | 0.0839 1628 | 0.0881 0728 |
| 32 | 0.0710 0234 | 0.0749 9665 | 0.0790 7292 | 0.0832 2599 | 0.0874 5081 |
| 33 | 0.0702 7293 | 0.0742 9924 | 0.0784 0807 | 0.0825 9397 | 0.0868 5163 |
| 34 | 0.0695 9843 | 0.0736 5610 | 0.0777 9674 | 0.0820 1461 | 0.0863 0411 |
| 35 | 0.0689 7386 | 0.0730 6226 | 0.0772 3396 | 0.0814 8291 | 0.0858 0326 |
| 36 | 0.0683 9483 | 0.0725 1332 | 0.0767 1531 | 0.0809 9447 | 0.0853 4467 |
| 37 | 0.0678 5743 | 0.0720 0534 | 0.0762 3685 | 0.0805 4533 | 0.0849 2440 |
| 38 | 0.0673 5812 | 0.0715 3480 | 0.0757 9505 | 0.0801 3197 | 0.0845 3894 |
| 39 | 0.0668 9377 | 0.0710 9854 | 0.0753 8676 | 0.0797 5124 | 0.0841 8513 |
| 40 | 0.0664 6154 | 0.0706 9373 | 0.0750 0914 | 0.0794 0031 | 0.0838 6016 |
| 41 | 0.0660 5886 | 0.0703 1779 | 0.0746 5962 | 0.0790 7663 | 0.0835 6149 |
| 42 | 0.0656 8342 | 0.0699 6842 | 0.0743 3591 | 0.0787 7789 | 0.0832 8684 |
| 43 | 0.0653 3312 | 0.0696 4352 | 0.0740 3590 | 0.0785 0201 | 0.0830 3414 |
| 44 | 0.0650 0606 | 0.0693 4119 | 0.0737 5769 | 0.0782 4710 | 0.0828 0152 |
| 45 | 0.0647 0050 | 0.0690 5968 | 0.0734 9957 | 0.0780 1146 | 0.0825 8728 |
| 46 | 0.0644 1485 | 0.0687 9743 | 0.0732 5996 | 0.0777 9353 | 0.0823 8991 |
| 47 | 0.0641 4768 | 0.0685 5300 | 0.0730 3744 | 0.0775 9190 | 0.0822 0799 |
| 48 | 0.0638 9766 | 0.0683 2506 | 0.0728 3070 | 0.0774 0527 | 0.0820 4027 |
| 49 | 0.0636 6356 | 0.0681 1240 | 0.0726 3853 | 0.0772 3247 | 0.0818 8557 |
| 50 | 0.0634 4429 | 0.0679 1393 | 0.0724 5985 | 0.0770 7241 | 0.0817 4286 |

**TABLE VIII. The Amount of 1 at Compound Interest for Fractional Periods**

$$(1+i)^{\frac{1}{p}}$$

| $p$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 2 | 1.0016 6528 | 1.0020 8117 | 1.0024 9688 | 1.0043 6547 | 1.0049 8756 |
| 3 | 1.0011 0988 | 1.0013 8696 | 1.0016 6390 | 1.0029 0820 | 1.0033 2228 |
| 4 | 1.0008 3229 | 1.0010 4004 | 1.0012 4766 | 1.0021 8036 | 1.0024 9068 |
| 6 | 1.0005 5479 | 1.0006 9324 | 1.0008 3160 | 1.0014 5304 | 1.0016 5977 |
| 12 | 1.0002 7735 | 1.0003 4656 | 1.0004 1571 | 1.0007 2626 | 1.0008 2954 |
| 13 | 1.0002 5602 | 1.0003 1990 | 1.0003 8373 | 1.0006 7037 | 1.0007 6570 |
| 26 | 1.0001 2800 | 1.0001 5994 | 1.0001 9185 | 1.0003 3513 | 1.0003 8276 |
| 52 | 1.0000 6400 | 1.0000 7996 | 1.0000 9592 | 1.0001 6755 | 1.0001 9137 |
| $p$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
| 2 | 1.0056 0927 | 1.0062 3059 | 1.0068 5153 | 1.0074 7208 | 1.0087 1205 |
| 3 | 1.0037 3602 | 1.0041 4943 | 1.0045 6249 | 1.0049 7521 | 1.0057 9963 |
| 4 | 1.0028 0081 | 1.0031 1046 | 1.0034 1992 | 1.0037 2909 | 1.0043 4658 |
| 6 | 1.0018 6627 | 1.0020 7257 | 1.0022 7865 | 1.0024 8452 | 1.0028 9562 |
| 12 | 1.0009 3270 | 1.0010 3575 | 1.0011 3868 | 1.0012 4149 | 1.0014 4677 |
| 13 | 1.0008 6092 | 1.0009 5604 | 1.0010 5104 | 1.0011 4594 | 1.0013 3540 |
| 26 | 1.0004 3037 | 1.0004 7790 | 1.0005 2538 | 1.0005 7280 | 1.0006 6748 |
| 52 | 1.0002 1516 | 1.0002 3892 | 1.0002 6266 | 1.0002 8636 | 1.0003 3368 |
| $p$ | $2\%$ | $2\frac{1}{4}\%$ | $2\frac{1}{2}\%$ | $2\frac{3}{4}\%$ | $3\%$ |
| 2 | 1.0099 5050 | 1.0111 8742 | 1.0124 2284 | 1.0136 5675 | 1.0148 8916 |
| 3 | 1.0066 2271 | 1.0074 4444 | 1.0082 6484 | 1.0090 8390 | 1.0099 0163 |
| 4 | 1.0049 6293 | 1.0055 7815 | 1.0061 9225 | 1.0068 0522 | 1.0074 1707 |
| 6 | 1.0033 0589 | 1.0037 1532 | 1.0041 2392 | 1.0045 3168 | 1.0049 3862 |
| 12 | 1.0016 5158 | 1.0018 5594 | 1.0020 5984 | 1.0022 6328 | 1.0024 6627 |
| 13 | 1.0015 2444 | 1.0017 1305 | 1.0019 0124 | 1.0020 8900 | 1.0022 7634 |
| 26 | 1.0007 6193 | 1.0008 5616 | 1.0009 5017 | 1.0010 4396 | 1.0011 3752 |
| 52 | 1.0003 8089 | 1.0004 2799 | 1.0004 7497 | 1.0005 2184 | 1.0005 6860 |
| $p$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
| 2 | 1.0173 4950 | 1.0198 0390 | 1.0222 5242 | 1.0246 9508 | 1.0271 3193 |
| 3 | 1.0115 3314 | 1.0131 5941 | 1.0147 8046 | 1.0163 9636 | 1.0180 0713 |
| 4 | 1.0086 3745 | 1.0098 5341 | 1.0110 6499 | 1.0122 7224 | 1.0134 7518 |
| 6 | 1.0057 5004 | 1.0065 5820 | 1.0073 6312 | 1.0081 6485 | 1.0089 6340 |
| 12 | 1.0028 7090 | 1.0032 7374 | 1.0036 7481 | 1.0040 7412 | 1.0044 7170 |
| 13 | 1.0026 4977 | 1.0030 2153 | 1.0033 9165 | 1.0037 6014 | 1.0041 2701 |
| 26 | 1.0013 2401 | 1.0015 0963 | 1.0016 9439 | 1.0018 7831 | 1.0020 6138 |
| 52 | 1.0006 6179 | 1.0007 5453 | 1.0008 4684 | 1.0009 3871 | 1.0010 3016 |
| $p$ | $6\%$ | $6\frac{1}{2}\%$ | $7\%$ | $7\frac{1}{2}\%$ | $8\%$ |
| 2 | 1.0295 6302 | 1.0319 8837 | 1.0344 0804 | 1.0368 2207 | 1.0392 3048 |
| 3 | 1.0196 1282 | 1.0212 1347 | 1.0228 0912 | 1.0243 9981 | 1.0259 8557 |
| 4 | 1.0146 7385 | 1.0158 6828 | 1.0170 5853 | 1.0182 4460 | 1.0194 2655 |
| 6 | 1.0097 5880 | 1.0105 5107 | 1.0113 4026 | 1.0121 2638 | 1.0129 0946 |
| 12 | 1.0048 6755 | 1.0052 6169 | 1.0056 5415 | 1.0060 4492 | 1.0064 3403 |
| 13 | 1.0044 9228 | 1.0048 5597 | 1.0052 1808 | 1.0055 7863 | 1.0059 3764 |
| 26 | 1.0022 4363 | 1.0024 2504 | 1.0026 0564 | 1.0027 8544 | 1.0029 6443 |
| 52 | 1.0011 2118 | 1.0012 1179 | 1.0013 0197 | 1.0013 9175 | 1.0014 8112 |

**TABLE IX. Nominal Rate of Interest $j$ with Frequency of Conversion $p$ Corresponding to Effective Rate of Interest $i$**

$$j_{(p)} = p[(1+i)^{\frac{1}{p}} - 1]$$

| $p$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 2 | .0033 3056 | .0041 6234 | .0049 9377 | .0087 3094 | .0099 7512 |
| 3 | .0033 2964 | .0041 6089 | .0049 9169 | .0087 2460 | .0099 6685 |
| 4 | .0033 2917 | .0041 6017 | .0049 9065 | .0087 2143 | .0099 6272 |
| 6 | .0033 2871 | .0041 5945 | .0049 8962 | .0087 1827 | .0099 5859 |
| 12 | .0033 2825 | .0041 5873 | .0049 8858 | .0087 1510 | .0099 5446 |
| 13 | .0033 2822 | .0041 5868 | .0049 8850 | .0087 1486 | .0099 5414 |
| 26 | .0033 2800 | .0041 5834 | .0049 8802 | .0087 1340 | .0099 5224 |
| 52 | .0033 2790 | .0041 5818 | .0048 8778 | .0087 1267 | .0099 5128 |
| $p$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
| 2 | .0112 1854 | .0124 6118 | .0137 0360 | .0149 4417 | .0174 2410 |
| 3 | .0112 0807 | .0124 4828 | .0136 8746 | .0149 2562 | .0173 9890 |
| 4 | .0112 0285 | .0124 4183 | .0136 7966 | .0149 1636 | .0173 8631 |
| 6 | .0111 9763 | .0124 3539 | .0136 7188 | .0149 0710 | .0173 7374 |
| 12 | .0111 9241 | .0124 2895 | .0136 6410 | .0148 9785 | .0173 6119 |
| 13 | .0111 9200 | .0124 2846 | .0136 6350 | .0148 9714 | .0173 6022 |
| 26 | .0111 8960 | .0124 2549 | .0136 5991 | .0148 9288 | .0173 5443 |
| 52 | .0111 8839 | .0124 2400 | .0136 5812 | .0148 9074 | .0173 5153 |
| $p$ | $2\%$ | $2\frac{1}{4}\%$ | $2\frac{1}{2}\%$ | $2\frac{3}{4}\%$ | $3\%$ |
| 2 | .0199 0099 | .0223 7484 | .0248 4567 | .0273 1349 | .0297 7831 |
| 3 | .0198 6813 | .0223 3333 | .0247 9451 | .0272 5170 | .0297 0490 |
| 4 | .0198 5173 | .0223 1261 | .0247 6899 | .0272 2087 | .0296 6829 |
| 6 | .0198 3534 | .0222 9192 | .0247 4349 | .0271 9009 | .0296 3173 |
| 12 | .0198 1898 | .0222 7125 | .0247 1804 | .0271 5936 | .0295 9524 |
| 13 | .0198 1772 | .0222 6966 | .0247 1608 | .0271 5699 | .0295 9243 |
| 26 | .0198 1017 | .0222 6013 | .0247 0434 | .0271 4283 | .0295 7561 |
| 52 | .0198 0640 | .0222 5537 | .0246 9848 | .0271 3575 | .0295 6721 |
| $p$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
| 2 | .0346 9899 | .0396 0781 | .0445 0483 | .0493 9015 | .0542 6386 |
| 3 | .0345 9943 | .0394 7821 | .0443 4138 | .0491 8907 | .0540 2139 |
| 4 | .0345 4978 | .0394 1363 | .0442 5996 | .0490 8894 | .0539 0070 |
| 6 | .0345 0024 | .0393 4918 | .0441 7874 | .0489 8908 | .0537 8036 |
| 12 | .0344 5078 | .0392 8488 | .0440 9771 | .0488 8949 | .0536 6039 |
| 13 | .0344 4698 | .0392 7994 | .0440 9149 | .0488 8184 | .0536 5117 |
| 26 | .0344 2420 | .0392 5031 | .0440 5417 | .0488 3597 | .0535 9593 |
| 52 | .0344 1281 | .0392 3551 | .0440 3552 | .0488 1306 | .0535 6834 |
| $p$ | $6\%$ | $6\frac{1}{2}\%$ | $7\%$ | $7\frac{1}{2}\%$ | $8\%$ |
| 2 | .0591 2603 | .0639 7674 | .0688 1609 | .0736 4414 | .0784 6097 |
| 3 | .0588 3847 | .0636 4042 | .0684 2737 | .0731 9942 | .0779 5670 |
| 4 | .0586 9538 | .0634 7314 | .0682 3410 | .0729 7840 | .0777 0619 |
| 6 | .0585 5277 | .0633 0644 | .0680 4156 | .0727 5827 | .0774 5674 |
| 12 | .0584 1061 | .0631 4033 | .0678 4974 | .0725 3903 | .0772 0836 |
| 13 | .0583 9969 | .0631 2758 | .0678 3502 | .0725 2220 | .0771 8930 |
| 26 | .0583 3425 | .0630 5113 | .0677 4676 | .0724 2134 | .0770 7506 |
| 52 | .0583 0157 | .0630 1295 | .0677 0268 | .0723 7098 | .0770 1802 |

**TABLE X. Amount at End of Year at Compound Interest of $p$ Instalments Each of $\frac{1}{p}$ Deposited at End of Each $p$th Part of a Year**

$$\frac{i}{j_{(p)}} = s^{(p)}_{\overline{1}|i} = \frac{1}{p s_{\overline{\frac{1}{p}}|i}} = \frac{i}{p\left[(1+i)^{\frac{1}{p}} - 1\right]}$$

| $p$ | $\frac{1}{3}\%$ | $\frac{5}{12}\%$ | $\frac{1}{2}\%$ | $\frac{7}{8}\%$ | $1\%$ |
|---|---|---|---|---|---|
| 2 | 1.0008 3264 | 1.0010 4058 | 1.0012 4844 | 1.0021 8274 | 1.0024 9378 |
| 3 | 1.0011 1029 | 1.0013 8761 | 1.0016 6482 | 1.0029 1102 | 1.0033 2596 |
| 4 | 1.0012 4913 | 1.0015 6115 | 1.0018 7305 | 1.0032 7529 | 1.0037 4223 |
| 6 | 1.0013 8799 | 1.0017 3471 | 1.0020 8131 | 1.0036 3967 | 1.0041 5861 |
| 12 | 1.0015 2686 | 1.0019 0829 | 1.0022 8960 | 1.0040 0411 | 1.0045 7510 |
| 13 | 1.0015 3754 | 1.0019 2164 | 1.0023 0563 | 1.0040 3215 | 1.0046 0714 |
| 26 | 1.0016 0164 | 1.0020 0176 | 1.0024 2182 | 1.0042 0039 | 1.0047 9941 |
| 52 | 1.0016 3369 | 1.0020 4183 | 1.0024 4985 | 1.0042 8452 | 4.0048 9556 |
| $p$ | $1\frac{1}{8}\%$ | $1\frac{1}{4}\%$ | $1\frac{3}{8}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ |
| 2 | 1.0028 0463 | 1.0031 1529 | 1.0034 2576 | 1.0037 3604 | 1.0043 6176 |
| 3 | 1.0037 4068 | 1.0041 5516 | 1.0045 6942 | 1.0049 8346 | 1.0058 1084 |
| 4 | 1.0042 0892 | 1.0046 7537 | 1.0051 4158 | 1.0056 0755 | 1.0065 3878 |
| 6 | 1.0046 7730 | 1.0051 9575 | 1.0057 1395 | 1.0062 3191 | 1.0072 6707 |
| 12 | 1.0051 4583 | 1.0057 1632 | 1.0062 8654 | 1.0068 5652 | 1.0079 9571 |
| 13 | 1.0051 8188 | 1.0057 5637 | 1.0063 3060 | 1.0069 0458 | 1.0080 5177 |
| 26 | 1.0053 9818 | 1.0059 9669 | 1.0065 9495 | 1.0071 9296 | 1.0083 8820 |
| 52 | 1.0055 0634 | 1.0061 1687 | 1.0067 2715 | 1.0073 3717 | 1.0085 5644 |
| $p$ | $2\%$ | $2\frac{1}{4}\%$ | $2\frac{1}{2}\%$ | $2\frac{3}{4}\%$ | $3\%$ |
| 2 | 1.0049 7525 | 1.0055 9371 | 1.0062 1142 | 1.0068 2837 | 1.0074 4458 |
| 3 | 1.0066 3733 | 1.0074 6292 | 1.0082 8761 | 1.0091 1141 | 1.0099 3431 |
| 4 | 1.0074 6856 | 1.0083 9839 | 1.0093 2677 | 1.0102 5422 | 1.0111 8072 |
| 6 | 1.0083 0125 | 1.0093 3444 | 1.0103 6665 | 1.0113 9789 | 1.0124 2816 |
| 12 | 1.0091 3389 | 1.0102 7107 | 1.0114 0725 | 1.0125 4243 | 1.0136 7662 |
| 13 | 1.0091 9796 | 1.0103 4314 | 1.0114 8732 | 1.0126 3051 | 1.0137 7270 |
| 26 | 1.0095 8243 | 1.0107 7565 | 1.0119 6786 | 1.0131 5908 | 1.0143 4929 |
| 52 | 1.0097 7470 | 1.0109 9195 | 1.0122 0819 | 1.0134 2343 | 1.0146 3757 |
| $p$ | $3\frac{1}{2}\%$ | $4\%$ | $4\frac{1}{2}\%$ | $5\%$ | $5\frac{1}{2}\%$ |
| 2 | 1.0086 7475 | 1.0099 0195 | 1.0111 2621 | 1.0123 4754 | 1.0135 6596 |
| 3 | 1.0115 7748 | 1.0132 1713 | 1.0148 5328 | 1.0164 8597 | 1.0181 1522 |
| 4 | 1.0130 3094 | 1.0148 7744 | 1.0167 2026 | 1.0185 5942 | 1.0203 9495 |
| 6 | 1.0144 8578 | 1.0165 3957 | 1.0185 8953 | 1.0206 3570 | 1.0226 7810 |
| 12 | 1.0159 4203 | 1.0182 0351 | 1.0204 6109 | 1.0227 1479 | 1.0249 6465 |
| 13 | 1.0160 5410 | 1.0183 3158 | 1.0206 0515 | 1.0228 7484 | 1.0251 4068 |
| 26 | 1.0167 2674 | 1.0191 0023 | 1.0214 6980 | 1.0238 3548 | 1.0261 9729 |
| 52 | 1.0170 6316 | 1.0194 8470 | 1.0219 6231 | 1.0243 1602 | 1.0267 2586 |
| $p$ | $6\%$ | $6\frac{1}{2}\%$ | $7\%$ | $7\frac{1}{2}\%$ | $8\%$ |
| 2 | 1.0147 8151 | 1.0159 9419 | 1.0172 0402 | 1.0184 1103 | 1.0196 1524 |
| 3 | 1.0197 4104 | 1.0213 6348 | 1.0229 8254 | 1.0245 9826 | 1.0262 1065 |
| 4 | 1.0222 2688 | 1.0240 5523 | 1.0258 8002 | 1.0277 0129 | 1.0295 1904 |
| 6 | 1.0247 1676 | 1.0267 5172 | 1.0287 8298 | 1.0308 1059 | 1.0328 3456 |
| 12 | 1.0272 1070 | 1.0294 5294 | 1.0316 9143 | 1.0339 2617 | 1.0361 5721 |
| 13 | 1.0274 0270 | 1.0296 6093 | 1.0319 1538 | 1.0341 6609 | 1.0364 1309 |
| 26 | 1.0285 5526 | 1.0309 0941 | 1.0332 5978 | 1.0356 0640 | 1.0379 4927 |
| 52 | 1.0291 3186 | 1.0315 3404 | 1.0339 3242 | 1.0363 2705 | 1.0387 1794 |

## TABLE XI. American Experience Table of Mortality

| Age $x$ | Number living $l_x$ | Number of deaths $d_x$ | Yearly probability of dying $q_x$ | Yearly probability of living $p_x$ | Complete expectation of life $\mathring{e}_x$ | Age $x$ | Number living $l_x$ | Number of deaths $d_x$ | Yearly probability of dying $q_x$ | Yearly probability of living $p_x$ | Complete expectation of life $\mathring{e}_x$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 100,000 | 749 | 0.007 490 | 0.992 510 | 48.72 | 53 | 66,797 | 1,091 | 0.016 333 | 0.983 667 | 18.79 |
| 11 | 99,251 | 746 | 0.007 516 | 0.992 484 | 48.08 | 54 | 65,706 | 1,143 | 0.017 396 | 0.982 604 | 18.09 |
| 12 | 98,505 | 743 | 0.007 543 | 0.992 457 | 47.45 | 55 | 64,563 | 1,199 | 0.018 571 | 0.981 429 | 17.40 |
| 13 | 97,762 | 740 | 0.007 569 | 0.992 431 | 46.80 | 56 | 63,364 | 1,260 | 0.019 885 | 0.980 115 | 16 72 |
| 14 | 97,022 | 737 | 0.007 596 | 0.992 404 | 46.16 | 57 | 62,104 | 1,325 | 0.021 335 | 0.978 665 | 16.05 |
| 15 | 96,285 | 735 | 0.007 634 | 0.992 366 | 45.50 | 58 | 60,779 | 1,394 | 0.022 936 | 0.977 064 | 15.39 |
| 16 | 95,550 | 732 | 0.007 661 | 0.992 339 | 44.85 | 59 | 59,385 | 1,468 | 0.024 720 | 0.975 280 | 14.74 |
| 17 | 94,818 | 729 | 0.007 688 | 0.992 312 | 44.19 | 60 | 57,917 | 1,546 | 0.026 693 | 0.973 307 | 14.10 |
| 18 | 94,089 | 727 | 0.007 727 | 0.992 273 | 43.53 | 61 | 56,371 | 1,628 | 0.028 880 | 0.971 120 | 13.47 |
| 19 | 93,362 | 725 | 0.007 765 | 0.992 235 | 42.87 | 62 | 54,743 | 1,713 | 0.031 292 | 0.968 708 | 12.86 |
| 20 | 92,637 | 723 | 0.007 805 | 0.992 195 | 42.20 | 63 | 53,030 | 1,800 | 0.033 943 | 0.966 057 | 12.26 |
| 21 | 91,914 | 722 | 0.007 855 | 0.992 145 | 41.53 | 64 | 51,230 | 1.889 | 0.036 873 | 0.963 127 | 11.67 |
| 22 | 91,192 | 721 | 0.007 906 | 0.992 094 | 40.85 | 65 | 49,341 | 1,980 | 0.040 129 | 0.959 871 | 11.10 |
| 23 | 90,471 | 720 | 0.007 958 | 0.992 042 | 40.17 | 66 | 47,361 | 2,070 | 0.043 707 | 0.956 293 | 10.54 |
| 24 | 89,751 | 719 | 0.008 011 | 0.991 989 | 39.49 | 67 | 45,291 | 2,158 | 0.047 647 | 0.952 353 | 10.00 |
| 25 | 89,032 | 718 | 0.008 065 | 0.991 935 | 38.81 | 68 | 43,133 | 2,243 | 0.052 002 | 0.947 998 | 9.47 |
| 26 | 88,314 | 718 | 0.008 130 | 0.991 870 | 38.12 | 69 | 40,890 | 2,321 | 0.056 762 | 0.943 238 | 8.97 |
| 27 | 87,596 | 718 | 0.008 197 | 0.991 803 | 37.43 | 70 | 38,569 | 2,391 | 0.061 993 | 0.938 007 | 8.48 |
| 28 | 86,878 | 718 | 0.008 264 | 0.991 736 | 36.73 | 71 | 36,178 | 2,448 | 0.067 665 | 0.932 335 | 8.00 |
| 29 | 86,160 | 719 | 0.008 345 | 0.991 655 | 36.03 | 72 | 33,730 | 2,487 | 0.073 733 | 0.926 267 | 7.55 |
| 30 | 85,441 | 720 | 0.008 427 | 0.991 573 | 35.33 | 73 | 31,243 | 2,505 | 0.080 178 | 0.919 822 | 7.11 |
| 31 | 84,721 | 721 | 0.008 510 | 0.991 490 | 34.63 | 74 | 28,738 | 2,501 | 0.087 028 | 0.912 972 | 6.68 |
| 32 | 84,000 | 723 | 0.008 607 | 0.991 393 | 33.92 | 75 | 26,237 | 2,476 | 0.094 371 | 0.905 629 | 6.27 |
| 33 | 83,277 | 726 | 0.008 718 | 0.991 282 | 33.21 | 76 | 23,761 | 2,431 | 0.102 311 | 0.897 689 | 5.88 |
| 34 | 82,551 | 729 | 0.008 831 | 0.991 169 | 32.50 | 77 | 21,330 | 2,369 | 0.111 064 | 0.888 936 | 5.49 |
| 35 | 81,822 | 732 | 0.008 946 | 0.991 054 | 31.78 | 78 | 18,961 | 2,291 | 0.120 827 | 0.879 173 | 5.11 |
| 36 | 81,090 | 737 | 0.009 089 | 0.990 911 | 31.07 | 79 | 16,670 | 2,196 | 0.131 734 | 0.868 266 | 4.74 |
| 37 | 80,353 | 742 | 0.009 234 | 0.990 776 | 30.35 | 80 | 14,474 | 2,091 | 0.144 466 | 0.855 534 | 4.39 |
| 38 | 79,611 | 749 | 0.009 408 | 0.990 592 | 29.62 | 81 | 12,383 | 1,964 | 0.158 605 | 0.841 395 | 4.05 |
| 39 | 78,862 | 756 | 0.009 586 | 0.990 414 | 28 90 | 82 | 10,419 | 1,816 | 0.174 297 | 0.825 703 | 3.71 |
| 40 | 78,106 | 765 | 0.009 794 | 0.990 206 | 28.18 | 83 | 8,603 | 1,648 | 0.191 561 | 0.808 439 | 3.39 |
| 41 | 77,341 | 774 | 0.010 008 | 0.989 992 | 27.45 | 84 | 6,955 | 1,470 | 0.211 359 | 0.788 641 | 3.08 |
| 42 | 76,567 | 785 | 0.010 252 | 0.989 748 | 26.72 | 85 | 5,485 | 1,292 | 0.235 552 | 0.764 448 | 2.77 |
| 43 | 75,782 | 797 | 0.010 517 | 0.989 483 | 26.00 | 86 | 4,193 | 1,114 | 0.265 681 | 0.734 319 | 2.47 |
| 44 | 74,985 | 812 | 0.010 829 | 0.989 171 | 25.27 | 87 | 3,079 | 933 | 0.303 020 | 0.696 980 | 2.18 |
| 45 | 74,173 | 828 | 0.011 163 | 0.988 837 | 24.54 | 88 | 2,146 | 744 | 0.346 692 | 0.653 308 | 1.91 |
| 46 | 73,345 | 848 | 0.011 562 | 0.988 438 | 23.81 | 89 | 1,402 | 555 | 0.395 863 | 0.604 137 | 1.66 |
| 47 | 72,497 | 870 | 0.012 000 | 0.988 000 | 23.08 | 90 | 847 | 385 | 0.454 545 | 0.545 455 | 1.42 |
| 48 | 71,627 | 896 | 0.012 509 | 0.987 491 | 22.36 | 91 | 462 | 246 | 0.532 468 | 0.467 534 | 1.19 |
| 49 | 70,731 | 927 | 0.013 106 | 0.986 894 | 21.63 | 92 | 216 | 137 | 0.634 259 | 0.365 741 | .98 |
| 50 | 69,804 | 962 | 0.013 781 | 0.986 219 | 20.91 | 93 | 79 | 58 | 0.734 177 | 0.265 823 | .80 |
| 51 | 68,842 | 1,001 | 0.014 541 | 0.985 459 | 20.20 | 94 | 21 | 18 | 0.857 143 | 0.142 857 | .64 |
| 52 | 67,841 | 1,044 | 0.015 389 | 0.984 611 | 19.49 | 95 | 3 | 3 | 1.000 000 | 0.000 000 | .50 |

**TABLE XII. Commutation Columns, American Experience Table, $3\frac{1}{2}\%$**

| Age $x$ | $D_x$ | $N_x$ | $C_x$ | $M_x$ | $a_x = \frac{N_x}{D_x} = 1 + a_x$ | $A_x = \frac{M_x}{D_x}$ |
|---|---|---|---|---|---|---|
| 10 | 70 891.9 | 1 575 535.3 | 513.02 | 17 612.91 | 22.2245 | 0.24845 |
| 11 | 67 981.5 | 1 504 643.4 | 493.69 | 17 099.89 | 22.1331 | 0.25154 |
| 12 | 65 189.0 | 1 436 661.9 | 475.08 | 16 606.20 | 22.0384 | 0.25474 |
| 13 | 62 509.4 | 1 371 472.9 | 457.16 | 16 131.12 | 21.9403 | 0.25806 |
| 14 | 59 938.4 | 1 308 963.5 | 439.91 | 15 673.96 | 21.8385 | 0.26151 |
| 15 | 54 471.6 | 1 249 025.0 | 423.88 | 15 234.05 | 21.7329 | 0.26508 |
| 16 | 55 104.2 | 1 191 553.4 | 407.87 | 14 810.17 | 21.6236 | 0.26877 |
| 17 | 52 832.9 | 1 136 449.2 | 392.47 | 14 402.30 | 21.5102 | 0.27261 |
| 18 | 50 653.9 | 1 083 616.2 | 378.15 | 14 009.83 | 21.3926 | 0.27659 |
| 19 | 48 562.8 | 1 032 962.4 | 364.36 | 13 631.68 | 21.2707 | 0.28071 |
| 20 | 46 556.2 | 984 399.6 | 351.07 | 13 267.32 | 21.1443 | 0.28497 |
| 21 | 44 630.8 | 937 843.4 | 338.73 | 12 916.25 | 21.0134 | 0.28940 |
| 22 | 42 782.8 | 893 212.6 | 326.82 | 12 577.53 | 20.8779 | 0.29399 |
| 23 | 41 009.2 | 850 429.9 | 315.33 | 12 250.71 | 20.7375 | 0.29873 |
| 24 | 39 307.1 | 809 420.6 | 304.24 | 11 935.38 | 20.5922 | 0.30365 |
| 25 | 37 673.6 | 770 113.6 | 293.55 | 11 631.14 | 20.4417 | 0.30873 |
| 26 | 36 106.1 | 732 439.9 | 283.62 | 11 337.59 | 20.2858 | 0.31401 |
| 27 | 34 601.5 | 696 333.8 | 274.03 | 11 053.97 | 20.1244 | 0.31947 |
| 28 | 33 157.4 | 661 732.4 | 264.76 | 10 779.94 | 19.9573 | 0.32512 |
| 29 | 31 771.3 | 628 575.0 | 256.16 | 10 515.18 | 19.7843 | 0.33097 |
| 30 | 30 440.8 | 596 803.6 | 247.85 | 10 259.02 | 19.6054 | 0.33702 |
| 31 | 29 163.5 | 566 362.9 | 239.797 | 10 011.17 | 19.4202 | 0.34328 |
| 32 | 27 937.5 | 537 199.3 | 232.381 | 9 771.375 | 19.2286 | 0.34976 |
| 33 | 26 760.5 | 509 261.8 | 225.406 | 9 539.044 | 19.0304 | 0.35646 |
| 34 | 25 630.1 | 482 501.3 | 218.683 | 9 313.638 | 18.8256 | 0.36339 |
| 35 | 24 544.7 | 456 871.2 | 212.157 | 9 094.955 | 18.6138 | 0.37055 |
| 36 | 23 502.5 | 432 326.5 | 206.383 | 8 882.798 | 18.3949 | 0.37795 |
| 37 | 22 501.4 | 408 824.0 | 200.757 | 8 676.415 | 18.1688 | 0.38560 |
| 38 | 21 539.7 | 386 322.6 | 195.798 | 8 475.658 | 17.9354 | 0.39349 |
| 39 | 20 615.5 | 364 782.9 | 190.945 | 8 279.860 | 17.6946 | 0.40163 |
| 40 | 19 727.4 | 344 167.4 | 186.684 | 8 088.915 | 17.4461 | 0.41003 |
| 41 | 18 873.6 | 324 440.0 | 182.493 | 7 902.231 | 17.1901 | 0.41869 |
| 42 | 18 052.9 | 305 566.3 | 178.828 | 7 719.738 | 16.9262 | 0.42762 |
| 43 | 17 263.6 | 287 513.4 | 175.421 | 7 540.910 | 16.6543 | 0.43681 |
| 44 | 16 504.4 | 270 249.8 | 172.680 | 7 365.489 | 16.3744 | 0.44628 |
| 45 | 15 773.6 | 253 745.5 | 170.127 | 7 192.809 | 16.0867 | 0.45600 |
| 46 | 15 070.0 | 237 971.9 | 168.345 | 7 022.682 | 15.7911 | 0.46600 |
| 47 | 14 392.1 | 222 901.9 | 166.872 | 6 854.337 | 15.4878 | 0.47626 |
| 48 | 13 738.5 | 208 509.8 | 166.047 | 6 687.466 | 15.1770 | 0.48677 |
| 49 | 13 107.9 | 194 771.3 | 165.983 | 6 521.419 | 14.8591 | 0.49752 |
| 50 | 12 498.6 | 181 663.4 | 166.424 | 6 355.436 | 14.5346 | 0.50849 |
| 51 | 11 909.6 | 169 164.7 | 167.316 | 6 189.012 | 14.2041 | 0.51967 |
| 52 | 11 339.5 | 157 255.2 | 168.601 | 6 021.696 | 13.8679 | 0.53104 |

**TABLE XII. Commutation Columns, American Experience Table, $3\frac{1}{2}\%$**

| Age $x$ | $D_x$ | $N_x$ | $C_x$ | $M_x$ | $a_x = \frac{N_x}{D_x} = 1 + a_x$ | $A_x = \frac{M_x}{D_x}$ |
|---|---|---|---|---|---|---|
| 53 | 10 787.4 | 145 915.7 | 170.234 | 5 853.095 | 13.5264 | 0.54258 |
| 54 | 10 252.4 | 135 128.2 | 172.317 | 5 682.861 | 13.1801 | 0.55430 |
| 55 | 9 733.40 | 124 875.8 | 174.646 | 5 510.544 | 12.8296 | 0.56615 |
| 56 | 9 229.60 | 115 142.4 | 177.325 | 5 335.898 | 12.4753 | 0.57813 |
| 57 | 8 740.17 | 105 912.8 | 180.168 | 5 158.573 | 12.1179 | 0.59022 |
| 58 | 8 264.44 | 97 172.64 | 183.139 | 4 978.405 | 11.7579 | 0.60239 |
| 59 | 7 801.83 | 88 908.20 | 186.340 | 4 795.266 | 11.3958 | 0.61463 |
| 60 | 7 351.65 | 81 106.38 | 189.604 | 4 608.926 | 11.0324 | 0.62692 |
| 61 | 6 913.44 | 73 754.73 | 192.909 | 4 419.322 | 10.6683 | 0.63924 |
| 62 | 6 486.75 | 66 841.28 | 196 117 | 4 226.413 | 10.3043 | 0.65155 |
| 63 | 6 071.27 | 60 354.54 | 199.109 | 4 030.296 | 9.9410 | 0.66383 |
| 64 | 5 666.85 | 54 283.27 | 201.887 | 3 831.187 | 9.5791 | 0.67607 |
| 65 | 5 273.33 | 48 616.41 | 204.457 | 3 629.300 | 9.2193 | 0.68824 |
| 66 | 4 890.55 | 43 343.08 | 206.522 | 3 424.843 | 8.8626 | 0.70030 |
| 67 | 4 518.65 | 38 452.53 | 208.022 | 3 218.321 | 8.5097 | 0.71223 |
| 68 | 4 157.82 | 33 933.88 | 208.903 | 3 010.299 | 8.1615 | 0.72401 |
| 69 | 3 808.32 | 29 776.06 | 208.858 | 2 801.396 | 7 8187 | 0.73560 |
| 70 | 3 470.67 | 25 967.74 | 207.881 | 2 592.538 | 7.4820 | 0.74698 |
| 71 | 3 145.43 | 22 497.07 | 205.639 | 2 384.657 | 7.1523 | 0.75813 |
| 72 | 2 833.42 | 19 351.64 | 201.851 | 2 179.018 | 6.8298 | 0.76904 |
| 73 | 2 535.75 | 16 518.22 | 196.436 | 1 977.167 | 6.5141 | 0.77972 |
| 74 | 2 253.57 | 13 982.47 | 189.491 | 1 780.731 | 6.2046 | 0.79018 |
| 75 | 1 987.87 | 11 728.90 | 181.253 | 1 591.240 | 5.9002 | 0.80048 |
| 76 | 1 739.39 | 9 741.028 | 171.940 | 1 409.988 | 5.6002 | 0.81062 |
| 77 | 1 508.63 | 8 001.633 | 161.889 | 1 238.047 | 5.3039 | 0.82064 |
| 78 | 1 295.73 | 6 492.999 | 151.264 6 | 1 076.158 | 5.0111 | 0.83054 |
| 79 | 1 100.65 | 5 197.271 | 140.089 1 | 924.893 7 | 4.7220 | 0.84032 |
| 80 | 923.338 | 4 096.624 | 128.880 1 | 784.804 6 | 4.4368 | 0.84997 |
| 81 | 763.234 | 3 173.286 | 116.958 8 | 655.924 5 | 4.1577 | 0.85940 |
| 82 | 620.465 | 2 410.052 | 104.488 | 538.965 7 | 3.8843 | 0.86865 |
| 83 | 494.995 | 1 789.587 | 91.615 2 | 434.477 6 | 3.6154 | 0.87774 |
| 84 | 386.641 | 1 294.592 | 78.956 5 | 342.862 4 | 3.3483 | 0.88677 |
| 85 | 294.610 | 907.951 3 | 67.049 0 | 263.905 9 | 3.0819 | 0.89578 |
| 86 | 217.598 | 613.341 7 | 55.856 6 | 196.856 9 | 2.8187 | 0.90468 |
| 87 | 154.383 | 395.743 8 | 45.199 2 | 141.000 3 | 2 5634 | 0.91332 |
| 88 | 103.963 | 241.360 9 | 34.824 26 | 95.801 07 | 2.3216 | 0.92149 |
| 89 | 65.623 1 | 137.397 8 | 25.099 29 | 60.976 82 | 2.0937 | 0.92920 |
| 90 | 38.304 7 | 71.774 70 | 16.822 44 | 35.877 52 | 1.8738 | 0.93664 |
| 91 | 20.186 9 | 33.470 01 | 10.385 393 | 19.055 09 | 1.6580 | 0.94393 |
| 92 | 9.118 89 | 13.283 09 | 5.588 150 | 8.669 695 | 1.4567 | 0.95074 |
| 93 | 3.222 36 | 4.164 21 | 2.285 484 | 3.081 545 | 1.2923 | 0.95630 |
| 94 | 0.827 611 | 0.941 84 | 0.685 393 | .795 762 | 1.1380 | 0.96152 |
| 95 | 0.114 232 | 0.114 23 | 0.110 369 | .110 369 | 1.0000 | 0.96618 |

## TABLE XIII. Valuation Columns, American Experience Table, $3\frac{1}{2}\%$

$$u_x = \frac{D_x}{D_{x+1}}, \quad k_x = \frac{C_x}{D_{x+1}}$$

| Age $x$ | $u_x$ | $k_x$ | Age $x$ | $u_x$ | $k_x$ |
|---|---|---|---|---|---|
| 10 | 1.042 811 | 0.007 546 | 53 | 1.052 185 | 0.016 604 |
| 11 | 1.042 838 | 0.007 573 | 54 | 1.053 323 | 0.017 704 |
| 12 | 1.042 866 | 0.007 600 | 55 | 1.054 585 | 0.018 922 |
| 13 | 1.042 894 | 0.007 627 | 56 | 1.055 999 | 0.020 289 |
| 14 | 1.042 922 | 0.007 654 | 57 | 1.057 563 | 0.021 800 |
| 15 | 1.042 962 | 0.007 692 | 58 | 1.059 296 | 0.023 474 |
| 16 | 1.042 990 | 0.007 720 | 59 | 1.061 234 | 0.025 347 |
| 17 | 1.043 019 | 0.007 748 | 60 | 1.063 385 | 0.027 425 |
| 18 | 1.043 059 | 0.007 787 | 61 | 1.065 780 | 0.029 739 |
| 19 | 1.043 100 | 0.007 826 | 62 | 1.068 433 | 0.032 303 |
| 20 | 1.043 141 | 0.007 866 | 63 | 1.071 365 | 0.035 136 |
| 21 | 1.043 195 | 0.007 917 | 64 | 1.074 625 | 0.038 285 |
| 22 | 1.043 248 | 0.007 969 | 65 | 1.078 270 | 0.041 807 |
| 23 | 1.043 303 | 0.008 022 | 66 | 1.082 304 | 0.045 704 |
| 24 | 1.043 358 | 0.008 076 | 67 | 1.086 782 | 0.050 031 |
| 25 | 1.043 415 | 0.008 130 | 68 | 1.091 774 | 0.054 855 |
| 66 | 1.043 484 | 0.008 197 | 69 | 1.097 284 | 0.060 178 |
| 27 | 1.043 554 | 0.008 264 | 70 | 1.103 403 | 0.066 090 |
| 28 | 1.043 625 | 0.008 333 | 71 | 1.110 117 | 0.072 576 |
| 29 | 1.043 710 | 0.008 415 | 72 | 1.117 388 | 0.079 602 |
| 30 | 1.043 796 | 0.008 498 | 73 | 1.125 218 | 0.087 167 |
| 31 | 1.043 884 | 0.008 583 | 74 | 1.133 660 | 0.095 323 |
| 32 | 1.043 986 | 0.008 682 | 75 | 1.142 852 | 0.104 204 |
| 33 | 1.044 102 | 0.008 795 | 76 | 1.152 960 | 0.113 971 |
| 34 | 1.044 221 | 0.008 910 | 77 | 1.164 314 | 0.124 941 |
| 35 | 1.044 343 | 0.009 027 | 78 | 1.177 243 | 0.137 433 |
| 36 | 1.044 493 | 0.009 172 | 79 | 1.192 031 | 0.151 720 |
| 37 | 1.044 647 | 0.009 320 | 80 | 1.209 771 | 0.168 861 |
| 38 | 1.044 830 | 0.009 498 | 81 | 1.230 099 | 0.188 502 |
| 39 | 1.045 018 | 0.009 679 | 82 | 1.253 477 | 0.211 089 |
| 40 | 1.045 238 | 0.009 891 | 83 | 1.280 245 | 0.236 952 |
| 41 | 1.045 463 | 0.010 109 | 84 | 1.312 384 | 0.268 004 |
| 42 | 1.045 721 | 0.010 359 | 85 | 1.353 917 | 0.308 133 |
| 43 | 1.046 001 | 0.010 629 | 86 | 1.409 469 | 0.361 806 |
| 44 | 1.046 331 | 0.010 947 | 87 | 1.484 979 | 0.434 762 |
| 45 | 1.046 684 | 0.011 289 | 88 | 1.584 244 | 0.530 671 |
| 46 | 1.047 106 | 0.011 697 | 89 | 1.713 188 | 0.655 254 |
| 47 | 1.047 571 | 0.012 146 | 90 | 1.897 500 | 0.833 333 |
| 48 | 1.048 111 | 0.012 668 | 91 | 2.213 750 | 1.138 889 |
| 49 | 1.048 745 | 0.013 280 | 92 | 2.829 873 | 1.734 177 |
| 50 | 1.049 463 | 0.013 974 | 93 | 3.893 571 | 2.761 905 |
| 51 | 1.050 272 | 0.014 755 | 94 | 7.245 000 | 6.000 000 |
| 52 | 1.051 177 | 0.015 629 | 95 | | |

**TABLE XIV. Commutation Columns, Two Lives, Equal Ages. Hunter's Makehamized American Experience Table of Mortality, $3\frac{1}{2}\%$**

| Age $x$ | $l_x$ | $\mu_x$ | $D_{xx}$ | $N_{xx}$ | $M_{xx}$ |
|---|---|---|---|---|---|
| **10** | 100 081 | 0.007 68 | 71 006.79 | 1 351 270.60 | 25 312.24 |
| **11** | 99 315 | 0.007 67 | 67 559.44 | 1 280 263.81 | 24 266.06 |
| **12** | 98 553 | 0.007 70 | 64 276.96 | 1 212 704.37 | 23 268.17 |
| **13** | 97 796 | 0.007 72 | 61 152.96 | 1 148 427.41 | 22 317.76 |
| **14** | 97 044 | 0.007 73 | 58 179.86 | 1 087 274.45 | 21 412.61 |
| **15** | 96 296 | 0.007 75 | 55 349.23 | 1 029 094.59 | 20 549.39 |
| **16** | 95 552 | 0.007 76 | 52 654.32 | 973 745.36 | 19 726.17 |
| **17** | 94 812 | 0.007 78 | 50 088.83 | 921 091.04 | 18 941.24 |
| **18** | 94 076 | 0.007 81 | 47 646.53 | 871 002.21 | 18 192.74 |
| **19** | 93 343 | 0.007 83 | 45 320.75 | 823 355.68 | 17 478.18 |
| **20** | 92 614 | 0.007 86 | 43 106.86 | 778 034.93 | 16 796.86 |
| **21** | 91 888 | 0.007 88 | 40 998.73 | 734 928.07 | 16 146.43 |
| **22** | 91 165 | 0.007 92 | 38 991.41 | 693 929.34 | 15 525.52 |
| **23** | 90 444 | 0.007 95 | 37 079.33 | 654 937.93 | 14 931.97 |
| **24** | 89 726 | 0.007 99 | 35 258.85 | 617 858.60 | 14 365.36 |
| **25** | 89 010 | 0.008 04 | 33 525.00 | 582 599.75 | 13 823.82 |
| **26** | 88 295 | 0.008 09 | 31 873.04 | 549 074.75 | 13 305.54 |
| **27** | 87 581 | 0.008 14 | 30 299.13 | 517 201.71 | 12 809.45 |
| **28** | 86 868 | 0.008 21 | 28 799.80 | 486 902.58 | 12 334.72 |
| **29** | 86 156 | 0.008 27 | 27 371.63 | 458 102.78 | 11 880.44 |
| **30** | 85 442 | 0.008 35 | 26 009.48 | 430 731.15 | 11 443.89 |
| **31** | 84 728 | 0.008 43 | 24 711.70 | 404 721.67 | 11 025.65 |
| **32** | 84 013 | 0.008 53 | 23 474.81 | 380 009.97 | 10 624.41 |
| **33** | 83 295 | 0.008 63 | 22 294.96 | 356 535.16 | 10 238.37 |
| **34** | 82 575 | 0.008 75 | 21 170.22 | 334 240.20 | 9 867.57 |
| **35** | 81 850 | 0.008 88 | 20 096.73 | 313 069.98 | 9 509.97 |
| **36** | 81 121 | 0.009 02 | 19 072.80 | 292 973.25 | 9 165.63 |
| **37** | 80 387 | 0.009 18 | 18 095.86 | 273 900.45 | 8 833.66 |
| **38** | 79 646 | 0.009 35 | 17 163.06 | 255 804.59 | 8 512.79 |
| **39** | 78 896 | 0.009 55 | 16 271.86 | 238 641.53 | 8 201.98 |
| **40** | 78 138 | 0.009 77 | 15 420.90 | 222 369.67 | 7 901.27 |
| **41** | 77 369 | 0.010 01 | 14 607.60 | 206 948.77 | 7 609.44 |
| **42** | 76 589 | 0.010 28 | 13 830.50 | 192 341.17 | 7 326.31 |
| **43** | 75 794 | 0.010 58 | 13 086.84 | 178 510.67 | 7 050.34 |
| **44** | 74 985 | 0.010 91 | 12 375.79 | 165 423.83 | 6 781.83 |
| **45** | 74 158 | 0.011 28 | 11 694.99 | 153 048.04 | 6 519.53 |
| **46** | 73 311 | 0.011 69 | 11 042.88 | 141 353.05 | 6 262.90 |
| **47** | 72 443 | 0.012 15 | 10 418.30 | 130 310.17 | 6 011.75 |
| **48** | 71 551 | 0.012 65 | 9 819.595 | 119 891.87 | 5 765.35 |
| **49** | 70 631 | 0.013 21 | 9 245.132 | 110 072.28 | 5 522.95 |
| **50** | 69 683 | 0.013 84 | 8 694.313 | 100 827.14 | 5 284.76 |
| **51** | 68 702 | 0.014 53 | 8 165.446 | 92 132.83 | 5 049.60 |
| **52** | 67 685 | 0.015 31 | 7 657.482 | 83 967.384 | 4 818.06 |
| **53** | 66 628 | 0.016 17 | 7 169.279 | 76 309.902 | 4 588.80 |
| **54** | 65 529 | 0.017 12 | 6 700.222 | 69 140.623 | 4 362.18 |

**TABLE XIV. Commutation Columns, Two Lives, Equal Ages. Hunter's Makehamized American Experience Table of Mortality, $3\frac{1}{2}\%$**

| Age $x$ | $l_x$ | $\mu_x$ | $D_{xx}$ | $N_{xx}$ | $M_{xx}$ |
|---|---|---|---|---|---|
| **55** | 64 383 | 0.018 18 | 6249.176 | 62 440.401 | 4 137.71 |
| **56** | 63 187 | 0.019 36 | 5815.616 | 56 191.225 | 3 915.47 |
| **57** | 61 936 | 0.020 66 | 5 398.652 | 50 375.609 | 3 695.17 |
| **58** | 60 626 | 0.022 12 | 4 997.777 | 44 976.957 | 3 476.86 |
| **59** | 59 253 | 0.023 73 | 4 612.539 | 39 979.180 | 3 260.63 |
| **60** | 57 812 | 0.025 53 | 4 242.422 | 35 366.641 | 3 046.49 |
| **61** | 56 300 | 0.027 52 | 3 887.371 | 31 124.219 | 2 834.90 |
| **62** | 54 711 | 0.029 74 | 3 546.903 | 27 236.848 | 2 625.89 |
| **63** | 53 044 | 0.032 20 | 3 221.281 | 23 689.945 | 2 420.21 |
| **64** | 51 294 | 0.034 94 | 2 910.389 | 20 468.664 | 2 218.25 |
| **65** | 49 459 | 0.037 98 | 2 614.369 | 17 558.275 | 2 020.65 |
| **66** | 47 536 | 0.041 36 | 2 333.359 | 14 943.906 | 1 828.05 |
| **67** | 45 526 | 0.045 12 | 2 067.829 | 12 610.547 | 1 641.42 |
| **68** | 43 429 | 0.049 29 | 1 818.085 | 10 542.718 | 1 461.60 |
| **69** | 41 246 | 0.053 93 | 1 584.460 | 8 724.633 | 1 289.45 |
| **70** | 38 982 | 0.059 08 | 1 367.424 | 7 140.173 | 1 125.99 |
| **71** | 36 642 | 0.064 81 | 1 167.328 | 5 772.749 | 972.13 |
| **72** | 34 235 | 0.071 17 | 984.544 9 | 4 605.420 8 | 828.82 |
| **73** | 31 773 | 0.078 24 | 819.349 4 | 3 620.875 9 | 696.96 |
| **74** | 29 269 | 0.086 10 | 671.786 8 | 2 801.526 5 | 577.05 |
| **75** | 26 739 | 0.094 83 | 541.707 3 | 2 129.739 7 | 469.69 |
| **76** | 24 204 | 0.104 53 | 428.853 7 | 1 588.032 4 | 375.151 |
| **77** | 21 687 | 0.115 31 | 332.652 1 | 1 159.178 7 | 293.452 |
| **78** | 19 211 | 0.127 29 | 252.206 2 | 826.526 6 | 224.255 |
| **79** | 16 805 | 0.140 60 | 186.462 7 | 574.320 4 | 167.041 |
| **80** | 14 495 | 0.155 40 | 134.032 3 | 387.857 7 | 120.916 |
| **81** | 12 309 | 0.171 83 | 93.385 61 | 253.825 4 | 84.801 6 |
| **82** | 10 273 | 0.190 10 | 62.846 87 | 160.439 8 | 57.420 6 |
| **83** | 8 410 | 0.210 40 | 40.695 54 | 97.592 91 | 37.394 6 |
| **84** | 6 739 | 0.232 95 | 25.246 62 | 56.897 37 | 23.322 6 |
| **85** | 5 274 | 0.258 01 | 14.940 03 | 31.650 75 | 13.869 7 |
| **86** | 4 019 | 0.285 86 | 8.382 428 | 16.710 72 | 7.817 29 |
| **87** | 2 974 | 0.316 81 | 4.434 808 | 8.328 292 | 4.153 09 |
| **88** | 2 130 | 0.351 20 | 2.197 902 | 3.893 484 | 2.066 19 |
| **89** | 1 471 | 0.389 41 | 1.012 828 | 1.695 582 | .955 492 |
| **90** | 976 | 0.431 87 | .430 792 9 | .682 754 | .407 702 |
| **91** | 619 | 0.479 05 | .167 420 9 | .251 962 | .158 902 |
| **92** | 374 | 0.531 49 | .059 051 5 | .084 541 2 | .056 191 8 |
| **93** | 214 | 0.589 75 | .018 679 9 | .025 489 7 | .017 817 8 |
| **94** | 115 | 0.654 49 | .005 211 9 | .006 809 84 | .004 981 78 |
| **95** | 58 | 0.726 43 | .001 280 9 | .001 597 94 | .001 226 58 |
| **96** | 27 | 0.806 37 | .000 268 20 | .000 317 038 | .000 257 479 |
| **97** | 11 | 0.895 21 | .000 043 011 | .000 048 837 9 | .000 041 359 2 |
| **98** | 4 | 0.993 92 | .000 005 495 0 | .000 005 826 82 | .000 005 293 17 |
| **99** | 1 | | .000 000 331 82 | .000 000 331 82 | .000 000 320 60 |

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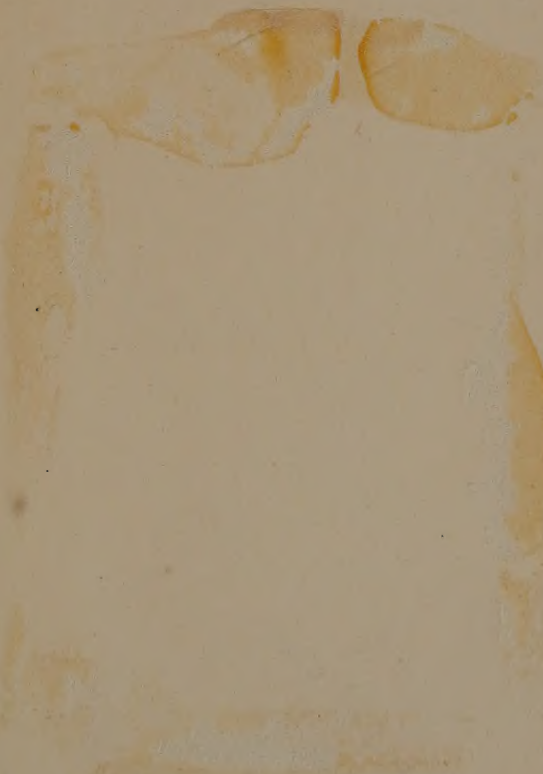
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